Asymptotic stability for linear differential equations with two kinds of time delays

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1 Introduction

In this study, we discuss the asymptotic stability for the zero solution of a scalar linear differential equation with two kinds of time delays

$$x'(t) = -ax(t) - bx(t-\tau) - c\int_{t-\tau}^{t} x(s)ds, \tag{E}$$

where a, b, $c \in \mathbb{R}$ and $\tau > 0$. Our study is motivated by the following stability results for (E) in the special cases where $c = 0$, $b = 0$, and $a = 0$.

When $c = 0$, equation (E) becomes

$$x'(t) = -ax(t) - bx(t-\tau). \tag{E_1}$$

In 1950, Hayes [4] obtained the following stability criterion for (E_1).

Theorem A. *The zero solution of* (E_1) *is asymptotically stable if and only if*

$$a > -\frac{1}{\tau}, \quad a + b > 0, \quad \text{and} \quad b < \omega \sin \omega\tau - a\cos\omega\tau$$

where ω *is the solution in* $(0, \pi/\tau)$ *of* $\omega\cos\omega\tau = -a\sin\omega\tau$.

The stability region for (E_1), the set of all (a, b) in which the zero solution of (E_1) is asymptotically stable, is presented by the region in Figure 1. The upper boundary of the stability region of (E_1) is given parametrically by the equation

$$a = -\frac{\omega}{\tan\omega\tau}, \quad b = \frac{\omega}{\sin\omega\tau}, \quad 0 < \omega < \frac{\pi}{\tau}.$$

A natural question now arises: how does the asymptotic stability of (E_1) with fixed a and b depend on the delay τ? In 1982, Cooke and Grossman [1] obtained another type of stability criterion for (E_1).

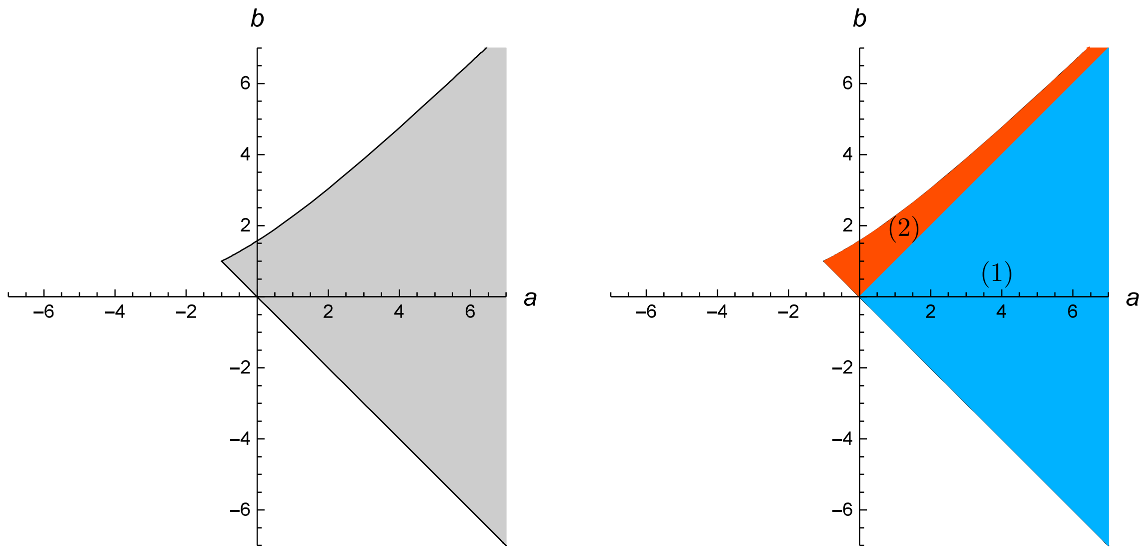

Figure 1. Stability region for (E_1) with $\tau = 1$.

Theorem B. *The zero solution of* (E_1) *is asymptotically stable if and only if either*

$$a + b > 0, \quad a - b \geq 0, \quad \text{and} \quad \tau \text{ is arbitrary,} \tag{1}$$

or

$$a + b > 0, \quad a - b < 0, \quad \text{and} \quad 0 < \tau < \frac{1}{\sqrt{b^2 - a^2}} \arccos\left(-\frac{a}{b}\right) \tag{2}$$

is satisfied.

In Figure 1, we notice that conditions (1) and (2) correspond to the blue and red regions, respectively.

When $b = 0$, equation (E) becomes

$$x'(t) = -ax(t) - c\int_{t-\tau}^{t} x(s)ds. \tag{E$_2$}$$

In 2004, Sakata and Hara [6] provided the following stability result for (E_2).

Theorem C. *The zero solution of* (E_2) *is asymptotically stable if and only if*

$$a + c\tau > 0, \quad \text{and} \quad c < \varphi(a),$$

where the curve $c = \varphi(a)$ *is given parametrically by the equation*

$$a = -\frac{\omega \sin \omega\tau}{1 - \cos \omega\tau}, \quad c = \frac{\omega^2}{1 - \cos \omega\tau}, \quad 0 < \omega < \frac{2\pi}{\tau}.$$

The stability region for (E_2), the set of all (a, c) in which the zero solution of (E_2) is asymptotically stable, is presented by the region in Figure 2.

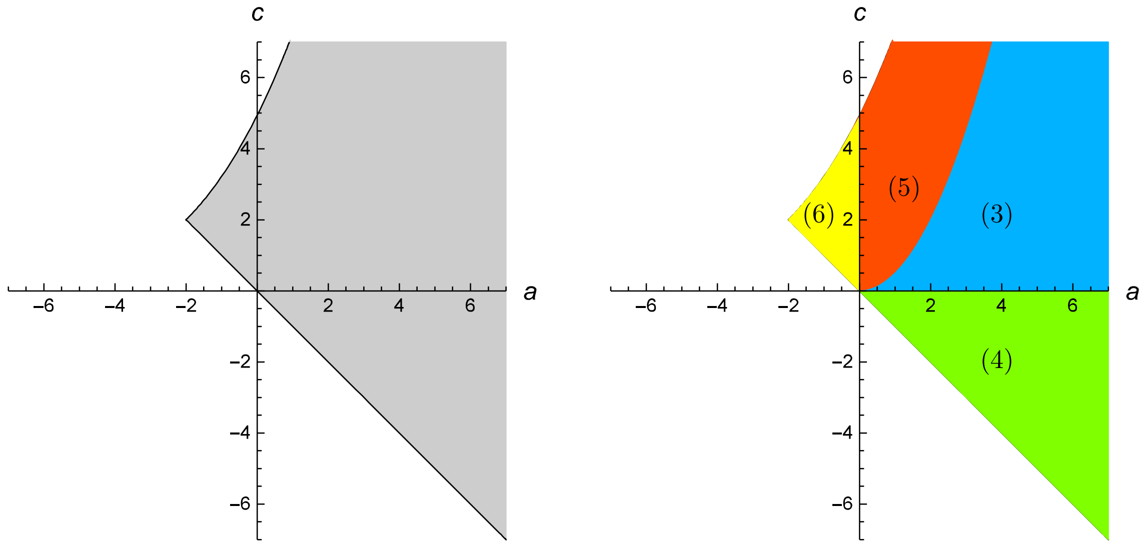

Figure 2. Stability region for (E_2) with $\tau = 1$.

Funakubo et al. [2] and Hara and Sakata [3] investigated the delay-dependent stability criterion for (E_2) with fixed a and c. By virtue of their work, we have the following result.

Theorem D. *The zero solution of* (E_2) *is asymptotically stable if and only if any one of the following four conditions is satisfied.*

$$a > 0, \quad c \geq 0, \quad 2c - a^2 \leq 0, \quad \textit{and} \quad \tau \ \textit{is arbitrary}, \tag{3}$$

$$a > 0, \quad c < 0, \quad \textit{and} \quad 0 < \tau < -\frac{a}{c}, \tag{4}$$

$$a > 0, \quad 2c - a^2 > 0, \quad \textit{and} \quad 0 < \tau < \frac{1}{\sqrt{2c - a^2}} \left(2\pi - \arccos\left(\frac{a^2 - c}{c} \right) \right), \tag{5}$$

$$a \leq 0, \quad 2c - a^2 > 0, \quad \textit{and} \quad -\frac{a}{c} < \tau < \frac{1}{\sqrt{2c - a^2}} \arccos\left(\frac{a^2 - c}{c} \right). \tag{6}$$

In Figure 2, we notice that conditions (3), (4), (5), and (6) correspond to the blue, green, red, and yellow regions, respectively.

When $a = 0$, equation (E) becomes

$$x'(t) = -bx(t - \tau) - c \int_{t-\tau}^{t} x(s) ds. \tag{E$_3$}$$

In 2004, Sakata and Hara [6] gave the following stability result for (E_3).

Theorem E. *The zero solution of* (E_3) *is asymptotically stable if and only if*

$$b + c\tau > 0, \quad \textit{and} \quad c < \psi(b),$$

where the curve $c = \psi(b)$ *is given parametrically by the equation*

$$b = \frac{\omega \sin \omega\tau}{1 - \cos \omega\tau}, \quad c = -\frac{\omega^2 \cos \omega\tau}{1 - \cos \omega\tau}, \quad 0 < \omega < \frac{2\pi}{\tau}.$$

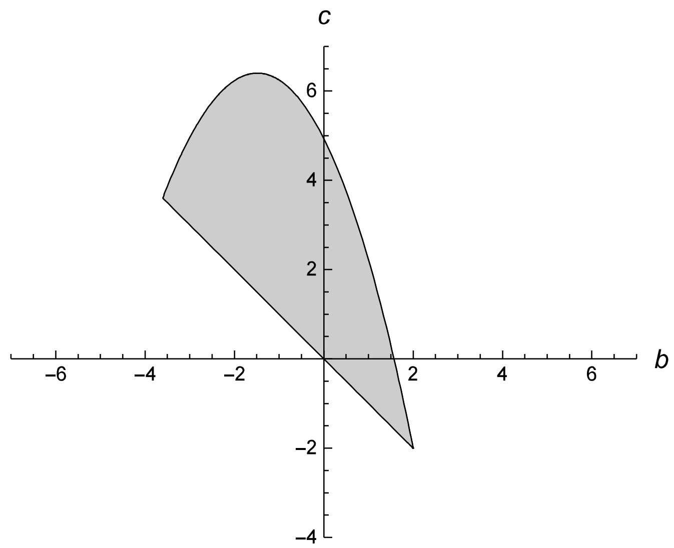

Figure 3. Stability region for (E_3) with $\tau = 1$.

The stability region for (E_3) is presented by the region in Figure 3. To our best knowledge, no delay-dependent stability criterion for (E_3) with fixed b and c has been obtained. The purpose of this study is to establish the delay-dependent stability criterion for (E_3) and to extend Theorems B and D to equation (E).

2 Main Results

Throughout this study, let ω_0 denote a constant defined as $\omega_0 = \sqrt{b^2 + 2c - a^2}$. Also, let τ^*, τ_n, and σ_n denote the critical values of τ defined as

$$\begin{aligned}
\tau^* &= -\frac{a+b}{c}, \\
\tau_n &= \frac{1}{\omega_0}\left(\arccos\left(\frac{(b-a)^2 - \omega_0^2}{(b-a)^2 + \omega_0^2}\right) + 2n\pi\right), \quad n \in \mathbb{Z}^+ := \{0, 1, 2, \dots\}, \\
\sigma_n &= \frac{1}{\omega_0}\left(2(n+1)\pi - \arccos\left(\frac{(b-a)^2 - \omega_0^2}{(b-a)^2 + \omega_0^2}\right)\right), \quad n \in \mathbb{Z}^+.
\end{aligned}$$

Our main results are stated below:

Theorem 1. *Let $a > 0$. Then, the zero solution of* (E) *is asymptotically stable if and only if $a + b + c\tau > 0$ and any one of the following five conditions holds:*

(i) $b \geq a$, $b^2 + 2c - a^2 > 0$, *and* $0 < \tau < \tau_0$,

(ii) $b > -a$, $b^2 + 2c - a^2 \leq 0$, $c < 0$, *and* $0 < \tau < \tau^*$,

(iii) $b^2 + 2c - a^2 \leq 0$, $c \geq 0$, *and* τ *is arbitrary,*

(iv) $|b| < a$, $b^2 + 2c - a^2 > 0$, *and* $0 < \tau < \sigma_0$,

(v) $b \leq -a$, $c > 0$, *and* $\tau^* < \tau < \sigma_0$.

Theorem 2. *Let $a \leq 0$. Then, the zero solution of* (E) *is asymptotically stable if and only if $a + b + c\tau > 0$ and any one of the following four conditions holds:*

$$\begin{aligned}
&\text{(i)} \quad b > -a, \quad b^2 + 2c - a^2 > 0, \quad \textit{and} \quad 0 < \tau < \tau_0,\\
&\text{(ii)} \quad b > -a, \quad b^2 + 2c - a^2 \leq 0, \quad \textit{and} \quad 0 < \tau < \tau^*,\\
&\text{(iii)} \quad |b| \leq -a, \quad b^2 + 2c - a^2 > 0, \quad \textit{and} \quad \tau^* < \tau < \tau_0,\\
&\text{(iv)} \quad b < a, \quad c > 0, \quad \textit{and} \quad \tau^* < \tau < \sigma_0.
\end{aligned}$$

Remark 1. *For $c = 0$, the combined result of Theorems 1 and 2 coincides with Theorem B.*

Remark 2. *For $b = 0$, the combined result of Theorems 1 and 2 coincides with Theorem D.*

In addition, let $a = 0$ in Theorem 2. Then, we obtain the following delay-dependent stability criterion for (E_3) that pairs with Theorem E.

Corollary 1. *The zero solution of* (E_3) *is asymptotically stable if and only if $b + c\tau > 0$ and any one of the following three conditions holds:*

$$b > 0, \quad b^2 + 2c > 0, \quad \textit{and} \quad 0 < \tau < \frac{1}{\sqrt{b^2 + 2c}} \arccos\left(-\frac{c}{b^2 + c}\right), \tag{7}$$

$$b > 0, \quad b^2 + 2c \leq 0, \quad \textit{and} \quad 0 < \tau < -\frac{b}{c}, \tag{8}$$

$$b \leq 0, \quad c > 0, \quad \textit{and} \quad -\frac{b}{c} < \tau < \frac{1}{\sqrt{b^2 + 2c}}\left(2\pi - \arccos\left(-\frac{c}{b^2 + c}\right)\right). \tag{9}$$

In Figure 3, we notice that conditions (7), (8), and (9) correspond to the blue, green, and red regions, respectively.

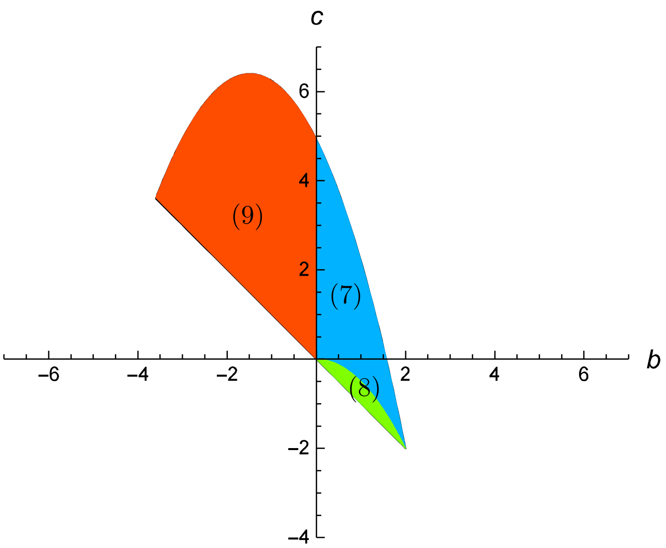

Figure 4. Stability region for (E_3) with $\tau = 1$.

Moreover, let $a = b$ in Theorems 1 and 2. Then, we obtain the following result for

$$x'(t) = -a(x(t) + x(t-\tau)) - c\int_{t-\tau}^{t} x(s)ds. \tag{E$_4$}$$

Corollary 2. *The zero solution of* (E_4) *is asymptotically stable if and only if* $2a + c\tau > 0$ *and any one of the following four conditions holds:*

$$\begin{aligned}
&a > 0, \quad c > 0, \quad \text{and} \quad 0 < \tau < \frac{\pi}{\sqrt{2c}}, \\
&a > 0, \quad c = 0, \quad \text{and} \quad \tau \text{ is arbitrary}, \\
&a > 0, \quad c < 0, \quad \text{and} \quad 0 < \tau < -\frac{2a}{c}, \\
&a \le 0, \quad c > 0, \quad \text{and} \quad -\frac{2a}{c} < \tau < \frac{\pi}{\sqrt{2c}}.
\end{aligned}$$

Theorems 1 and 2 are proved using the fact that the zero solution of (E) is asymptotically stable if and only if all the roots of the associated characteristic equation

$$\lambda + a + be^{-\lambda\tau} + c\int_{-\tau}^{0} e^{\lambda s}ds = 0$$

have negative real parts; see reference [5] for proof details.

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