Forbidden complexes for the 3-sphere

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1 Introduction

A simplicial complex is said to be *critical* (or *forbidden*) for the 3-sphere S^3 if it cannot be embedded in S^3 but after removing any one point, it can be embedded. We show that if a multibranched surface cannot be embedded in S^3, it contains a critical complex which is a union of a multibranched surface and a (possibly empty) graph. We exhibit all critical complexes which are contained in $K_5 \times S^1$ and $K_{3,3} \times S^1$ families. We further classify all critical complexes for S^3 which can be decomposed into $G \times S^1$ and H, where G and H are graphs, and $G \cap H$ consists of vertices of H.

In spite of the above property, there exist complexes which cannot be embedded in S^3, but they do not contain any critical complexes. From the property of those examples, we define an equivalence relation on all simplicial complexes $\mathcal{C}$ and a partially ordered set of complexes $(\mathcal{C}/\sim; \subsetneqq)$, and refine the definition of critical. According to the refined definition of critical, we show that if a complex X cannot be embedded in S^3, then there exists $[X'] \subsetneqq [X]$ such that $[X']$ is critical for $[S^3]$.

Throughout this article, we work in the piecewise linear category, consisting of simplicial complexes and piecewise-linear maps. The polyhedron $|X|$ is expressed directly using X.

2 Critical complexes

For two simplicial complexes X and Y, X is said to be *critical* for Y if X cannot be embedded in Y, but for any point $p \in X$, $X - p$ can be embedded in Y.

Example.

1. $S^1 \bigsqcup \{\text{a point}\}$ is critical for S^1.
2. $S^1 \bigsqcup S^1$ is critical for a bouquet of 2 circles.
3. K_5 and $K_{3,3}$ are critical for S^2.
4. S^2 is critical for the torus $S^1 \times S^1$.
5. S^n is critical for a closed n-manifold except for S^n.

Henceforce, we assume the connectivity of simplicial complexes. Let $\Gamma(Y)$ denote the set of critical complexes for Y.

2.1 Critical complexes for closed manifolds

Lemma. If $X \in \Gamma(Y)$, then $\dim X \leq \dim Y$.

Proof. Suppose that $\dim X > \dim Y$. Let B^{n+1} be an open $(n+1)$-ball in X, where $n = \dim Y$. Then for a point $p \in B^{n+1}$, $X - p$ cannot be embedded in Y since $X - p$ contains an open $(n+1)$-ball in $B^{n+1} - p$. □

Proposition. $\Gamma(S^2) = \{K_5, K_{3,3}\}$.

Proof. It is straightforward to check that $\Gamma(S^2) \ni K_5, K_{3,3}$.

Conversely, let $X \in \Gamma(S^2)$. By the above lemma, $\dim X \leq 2$.

First, suppose that $\dim X = 2$. Then X contains an open disk D. Since X is critical for S^2, for any point $p \in D$, $X - p$ can be embedded in S^2 and hence $X - D$ can be embedded in S^2. Since X is assumed to be connected, $X - D$ is also connected. This implies that the boundary component of $X - D$ corresponding to ∂D bounds a disk E in $S^2 - (X - D)$. Therefore, by filling with E, we have an embedding of X in S^2. This contradicts the criticality of X and we have $\dim X = 1$.

Next, since X cannot be embedded in S^2, by the Kuratowski's theorem, X contains K_5 or $K_{3,3}$. If X contains K_5 and $X - K_5 \neq \emptyset$, then for a point $p \in X - K_5$, $X - p$ cannot be embedded in S^2. Hence $X = K_5$. The same holds true for $K_{3,3}$. Thus X is K_5 or $K_{3,3}$. □

Let F_g be a closed orientable surface of genus $g > 0$, and $\Omega(F_g)$ be the set of forbidden graphs for F_g.

Theorem. $\Gamma(F_g) = \{F_0, \ldots, F_{g-1}\} \cup \Omega(F_g)$.

The next theorem gives a characterization of critical complexes with the same dimension.

Theorem. Let M be a closed n-manifold and $X \in \Gamma(M)$ be a critical complex for M with $\dim X = n$. Then X is a closed n-manifold which is homeomorphic to a connected proper summand of M including S^n, namely, $M = X \# M'$ for some closed n-manifold M' which is not homeomorphic to S^n.

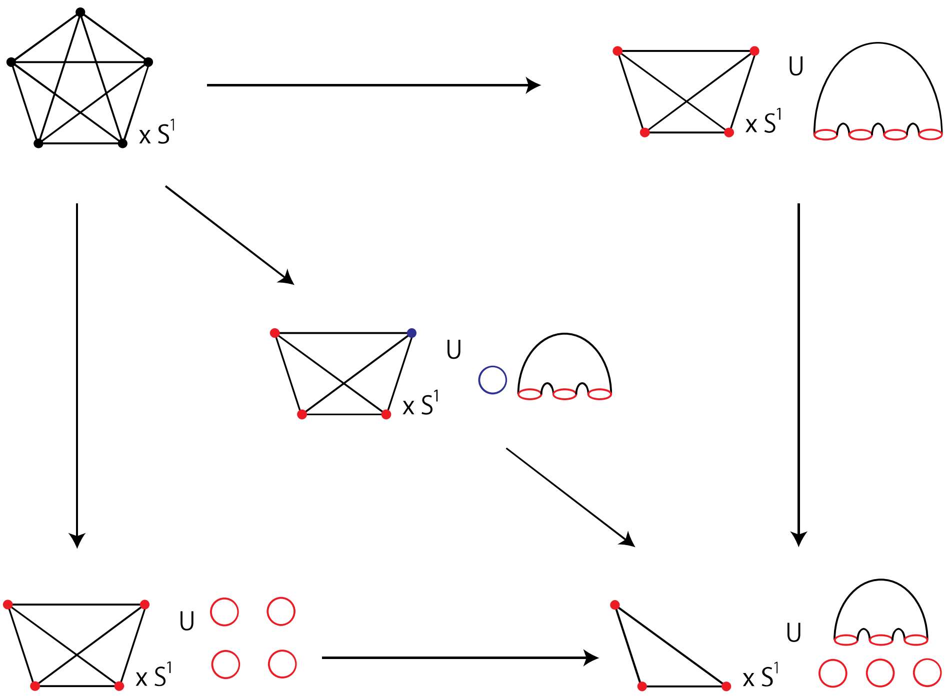
2.2 Critical multibranched surfaces

We say that a 2-dimensional simplicial complex is a *multibranched surface* if removing all points whose open neighborhoods are homeomorphic to the 2-dimensional Euclidean space yields a 1-dimensional complex homeomorphic to a disjoint union of simple closed curves.

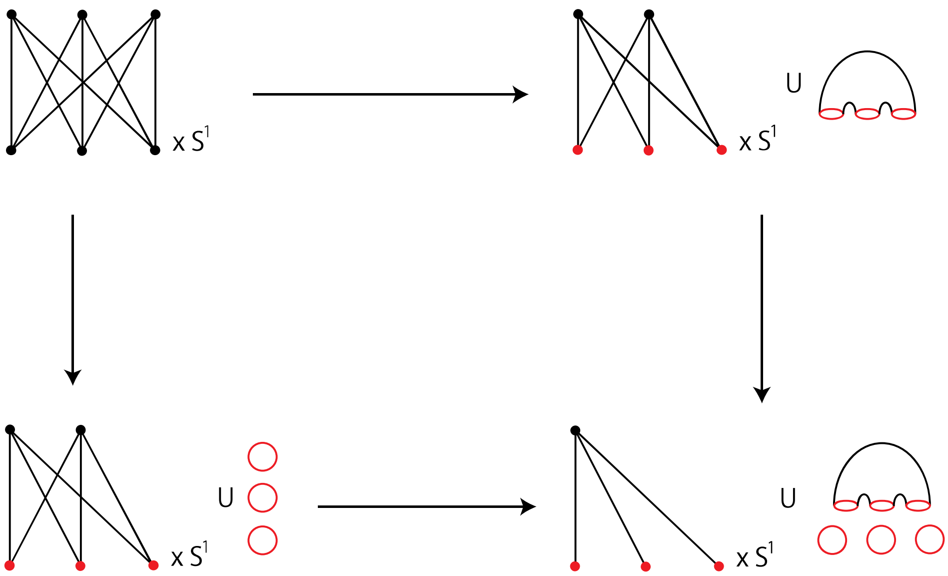
Eto–Matsuzaki–the author proved that some family of multibranched surfaces belong to $\Gamma(S^3)$ ([1], [3]).

Theorem. If a multibranched surface X cannot be embedded in S^3, then there exists a critical subcomplexes $M \cup G \subset X$ of X, where M is a multibranched surface and H is a (possibly empty) graph.

Let Y_n, P_n, D_n denote $K_{1,n} \times S^1$, an n-punctured sphere, n disks respectively. Suppose that a multibranched surface X contains Y_n as a sub-multibranched surface. We replace Y_n with $P_i \cup D_j$ $(n = i + j)$, where ∂P_i and ∂D_j are attached to branches corresponding to branches of degree 1 in Y_n by degree 1 maps. Make this replacement as recursive as possible into $K_5 \times S^1$ and $K_{3,3} \times S^1$ and get the $K_5 \times S^1$ family (1) – (5) and $K_{3,3} \times S^1$ family (6) – (9).



$K_5 \times S^1$ family



$K_{3,3} \times S^1$ family

Theorem. All members of $K_5 \times S^1$ and $K_{3,3} \times S^1$ families cannot be embedded in S^3, and they contain critical subcomplexes.

(1) $K_5 \times S^1 \supset (K_4 \times S^1) \cup K_{1,4}$

(2) $(K_4 \times S^1) \cup P_4 \supset (K_4 \times S^1) \cup K_{1,4}$

(3) $(K_4 \times S^1) \cup P_3 \cup D_1 = (K_4 \times S^1) \cup P_3 \cup D_1$

(4) $(K_4 \times S^1) \cup D_4 \supset (K_4 - K_3) \times S^1 \cup D_4 \cup K_3$

(5) $(K_3 \times S^1) \cup P_3 \cup D_3 = (K_3 \times S^1) \cup P_3 \cup D_3$

(6) $K_{3,3} \times S^1 \supset (K_{2,3} \times S^1) \cup K_{1,3}$

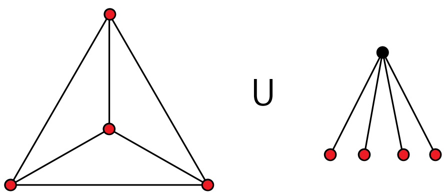
(7) $(K_{2,3} \times S^1) \cup P_3 \supset (K_{2,3} \times S^1) \cup K_{1,3}$

(8) $(K_{2,3} \times S^1) \cup D_3 \supset (K_{1,3} \times S^1) \cup D_3 \cup K_{1,3}$

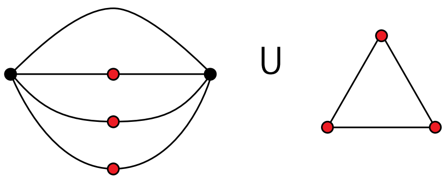
(9) $(K_{1,3} \times S^1) \cup P_3 \cup D_3 \supset (K_{1,3} \times S^1) \cup D_3 \cup K_{1,3}$

We classifiy these critical complexes $M \cup G$ $(G \neq \emptyset)$

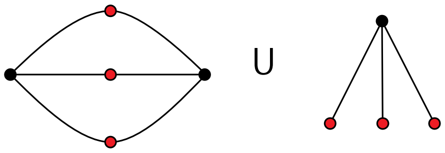
(I) K_4-type — In the above list, (1), (2) are of K_4-type.



(II) Θ_4-type — In the above list, (4) are of Θ_4-type.



(III) $K_{2,3}$-type — In the above list, (6), (7), (8), (9) are of $K_{2,3}$-type.

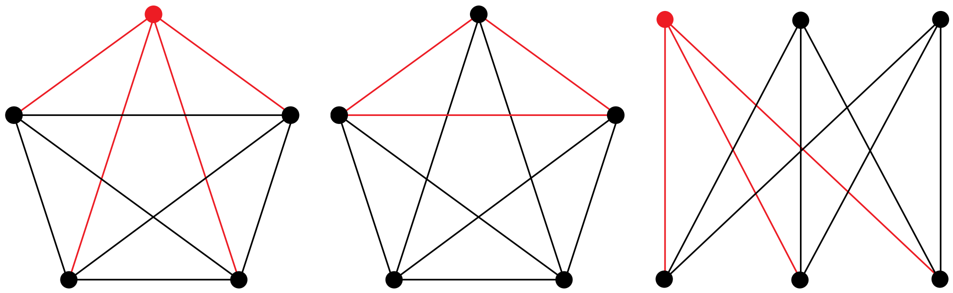


2.3 Critical complexes which have a form $(G \times S^1) \cup H$

Let X be a simplicial complex such that the 2-dimensional part X_2 of X is a product $G \times S^1$ for a graph G. Then X can be expressed as $X = (G \times S^1) \cup H$, where H is the 1-dimensional part X_1 of X. We define a *reduction* of $X = (G \times S^1) \cup H$ to $\hat{X} = G \cup H$ as follows. We regard S^1 as the quotient space $[0,1]/\{0\} \sim \{1\}$. By a map $f : (G \times S^1) \cup H \to (G \times \{0\}) \cup H$, we obtain a reduction $\hat{X} = G \cup H$ of $X = (G \times S^1) \cup H$.

Theorem. Let $X = (G \times S^1) \cup H$ be a critical complex, where G and H are graphs. Then a reduction $\hat{X} = G \cup H$ has a minor $G' \cup H'$ which is one of the following.

1. $G' \cup H'$ is K_5, where $H' = K_{1,4}$.
2. $G' \cup H'$ is K_5, where $H' = K_3$.
3. $G' \cup H'$ is $K_{3,3}$, where $H' = K_{1,3}$.



(1) (2) (3)

The characterization (1), (2) and (3) in this theorem coincide with three types (I), (II) and (III) for $M \cup G$ $(G \neq \emptyset)$.

We say that an embedding $f : G \times S^1 \to S^3$ is *standard* if $f(G \times S^1)$ is contained in a standard solid torus $D^2 \times S^1$ in S^3 so that $p^{-1}(p(f(G \times S^1))) = f(G \times S^1)$, where $p : D^2 \times S^1 \to D^2$ is the projection.

A *circular permutation system* for a multibranched surface is a choice of a circular ordering of the sectors attached to each branch.

Lemma. Let $e \in E(G)$ be an edge and $p \in \mathrm{int}(e \times S^1)$ be a point. Suppose that there exists an embedding $f : X - p \to S^3$. Then there exists an embedding $f' : X - p \to S^3$ with the same circular permutation system as f such that $f'((G \times S^1) - p)$ is contained in a standard embedding $f_0 : G \times S^1 \to S^3$.

Lemma. If $X = (G \times S^1) \cup H$ is critical, then a reduction $\hat{X} = G \cup H$ is also critical for S^2.

Proof. First suppose that $\hat{X}$ can be embedded in S^2. Then $\hat{X}$ is contained in a disk $D^2 \subset S^2$ and by embedding $D^2 \times S^1$ in S^3, $X = (G \times S^1) \cup H$ can be embedded in S^3. This contradicts the criticality of X.

Next we will show that for any edge e in $G \cup H$, $(G \cup H) - e$ can be embedded in S^2.

Let $e \in E(G)$ be an edge and $p \in \mathrm{int}(e \times S^1)$ be a point. Then there exists an embedding $f : X - p \to S^3$. By the above Lemma, there exists an embedding $f' : X - p \to S^3$ with the same circular permutation system as f such that $f'((G \times S^1) - p)$ is contained in a standard embedding $f_0 : G \times S^1 \to S^3$. This shows that a reduction $\hat{X} = (G - e) \cup H$ can be embedded in S^2. We omit the case for $e \in E(H)$. $\square$

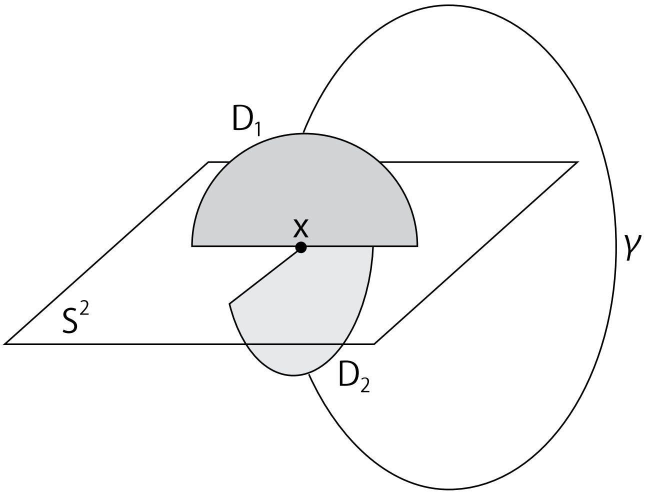
Proof of Theorem. By Lemma, a reduction $\hat{X} = G \cup H$ is critical for S^2. Hence by Kuratowski's and Wagner's Theorem, $\hat{X}$ has a minor of K_5 or $K_{3,3}$. It is straightforward to check that if $\hat{X} = G \cup H$ has a minor K_5, then we have the conclusions (1) or (2), and if $\hat{X} = G \cup H$ has a minor $K_{3,3}$, then we have the conclusion (3). $\square$

3 Refined critical complexes

3.1 Complexes which do not contain critical complexes

Suppose that a complex X cannot be embedded in S^3. Then we expect that there is a subspace $X' \subset X$ which is critical. However, there are many complexes which cannot be embedded in S^3, but do not contain any critical complexes.

Example. Let X be a complex consisting of S^2, D_1, D_2 and γ.



X cannot be embedded in S^3, but X does not contain any critical subcomplex as shown below.

$\because$ Suppose that $X' \subset X$ is critical. Since X' cannot be embedded in S^3, X' must contain the whole of S^2 and γ. For any small neighborhood $N(x; X')$, $N(x; X')$ must contain two subdisks $D_1' \subset D_1$ and $D_2' \subset D_2$, and X' must have a path connecting D_1' and D_2' containing γ. Thus, X' must contain a subcomplex which is homeomorphic to X. However, for any point $p \in \mathrm{int} D_1'$, $X' - p$ cannot be embedded in S^3 since it contains a subcomplex which is homeomorphic to X. $\square$

Theorem. The cone over K_5 cannot be embedded in S^3. But, it does not contain any critical complex.

3.2 Partially ordered set of complexes

From the above example and theorem, we derive the following refined definition of critical. For two connected simplicial complexes X and Y, X is said to be *refined critical* for Y if X cannot be embedded in Y, but for any proper subspace X' of X, which does not contain a subspace homeomorphic to X, X' can be embedded in Y.

This refined definition of critical leads us the following equivalence relation. Let $\mathcal{C}$ denote the set of all connected simplicial complexes. $X,\ Y \in \mathcal{C}$ are *equivalent*, denoted by $X \sim Y$, if X can be embedded in Y and Y can be embedded in X. We denote by $[X] \subseteqq [Y]$ if X can be embedded in Y. Then $(\mathcal{C}/\sim, \subseteqq)$ is a partially ordered set. For $[X],\ [Y] \in \mathcal{C}/\sim$, $[X]$ is said to be *critical* for $[Y]$ if $[X] \nsubseteqq [Y]$, but for any $[X'] \subsetneqq [X]$, $[X'] \subseteqq [Y]$. Put

$$\Gamma([Y]) = \{[X] \in \mathcal{C}/\sim \mid [X] \text{ is critical for } [Y]\}$$

Example. Let E_1 and E_2 denote the example $X = S^2 \cup D_1 \cup D_2 \cup \gamma$ and the cone over K_5. Then, we have

$$\Gamma([S^3]) \ni [E_1],\ [E_2].$$

Proposition. If $X \in \Gamma(Y)$, then $[X] \in \Gamma([Y])$.

We denote the quotient space obtained from an n-ball B^n and the closed interval $[0,1]$ by identifying p and $\{0\}$ by $B^{n\perp}$.

Proposition. $\Gamma([S^1]) = \{B^{1\perp}\}$.

Proposition ([2]). $\Gamma([S^2]) = \{[K_5], [K_{3,3}], [B^{2\perp}]\}$.

We generalize Mardešić–Segal's Theorem.

Theorem. $\Gamma([F_g]) = \{[F_0], \ldots, [F_{g-1}], [B^{2\perp}]\} \cup \{[G] \mid G \in \Omega(F_g)\}$.

As we have seen the above example and theorem, those examples do not satisfy the natural property. However, by considering the equivalence relation above, we obtain the next natural property.

Theorem. Suppose that a 2-dimensional complex X cannot be embedded in a closed n-manifold M $(n \leq 3)$. Then there exists an element $[X'] \subsetneqq [X]$ such that $[X']$ is critical for $[M]$.

For a typical example, a torus T cannot be embedded in a 2-sphere S^2. By applying this existence theorem, there exist $[K_5], [K_{3,3}] \subsetneqq [T]$ such that $[K_5], [K_{3,3}]$ are critical for $[S^2]$.

References

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