Grid homology and the connected sum of knots

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1 Grid homology and connected sums

Grid homology is a combinatorial reconstruction of knot Floer homology for knots in S^3 developed by Manolescu, Ozsváth, Szabó, and Thurston [14]. There are several versions of grid homology like knot Floer homology. The hat and minus versions have been mainly considered. For a knot $K \in S^3$, the minus version of grid homology $GH^-(K)$ is a bigraded $\mathbb{F}[U]$-module and the hat version $\widehat{GH}(K)$ is a bigraded $\mathbb{F}$-vector space. Of course, the invariance of these versions can be proved by knot Floer homology [13, Theorem 3.3], but a combinatorial proof of the invariance is also given [14, Theorem 1.2]. See [17] for details.

In this talk, we treat the minus version of grid homology for knots. The main purpose of this talk is to give a combinatorial proof of a Künneth formula for knot Floer homology of connected sums proved in [16],

$$HFK^-(K_1) \otimes_{\mathbb{F}[U]} HFK^-(K_2) \cong HFK^-(K_1 \# K_2). \tag{1}$$

The connected sum operation has not rarely been treated in grid homology, despite being a basic operation for knots. The Legendrian and transverse grid invariants are combinatorial invariants originally defined using grid homology, but due to the lack of a method to study connected sums in grid homology, their additivity was proved using knot Floer homology [21].

Recently, the author [9] gave a combinatorial proof of the Künneth formula of the hat version. In this talk, we will prove the Künneth formula (1) of the minus version using grid homology. Furthermore, we quickly give a combinatorial proof of the additivity of the tau invariant and the Legendrian and transverse grid invariants.

By definition, for an $n \times n$ grid diagram $\mathbb{G}$, the grid chain complex $GC^-(\mathbb{G})$ is a complex over $\mathbb{F}[U_1, \dots, U_n]$. We regard $GC^-(\mathbb{G})$ as a complex over $\mathbb{F}[U]$, where the action by U is multiplication by U_1.

Theorem 1.1. *Let $\mathbb{G}_1$, $\mathbb{G}_2$, and $\mathbb{G}_\#$ be grid diagrams representing K_1, K_2, and $K_1 \# K_2$ respectively as in Figure 1. There are a subcomplex C of $GC^-(\mathbb{G}_\#)$ and two quasi-isomorphisms $C \to GC^-(\mathbb{G}_\#)$ and $C \to GC^-(\mathbb{G}_1) \otimes_{\mathbb{F}[U]} GC^-(\mathbb{G}_2)$.*

The definition of the tau invariant in grid homology is the same as in knot Floer homology.

Definition 1.2. For a knot K, $\tau(K)$ is -1 times the maximal integer i for which there is a homogeneous, non-torsion element in $GH^-(K)$ with Alexander grading equal to i.

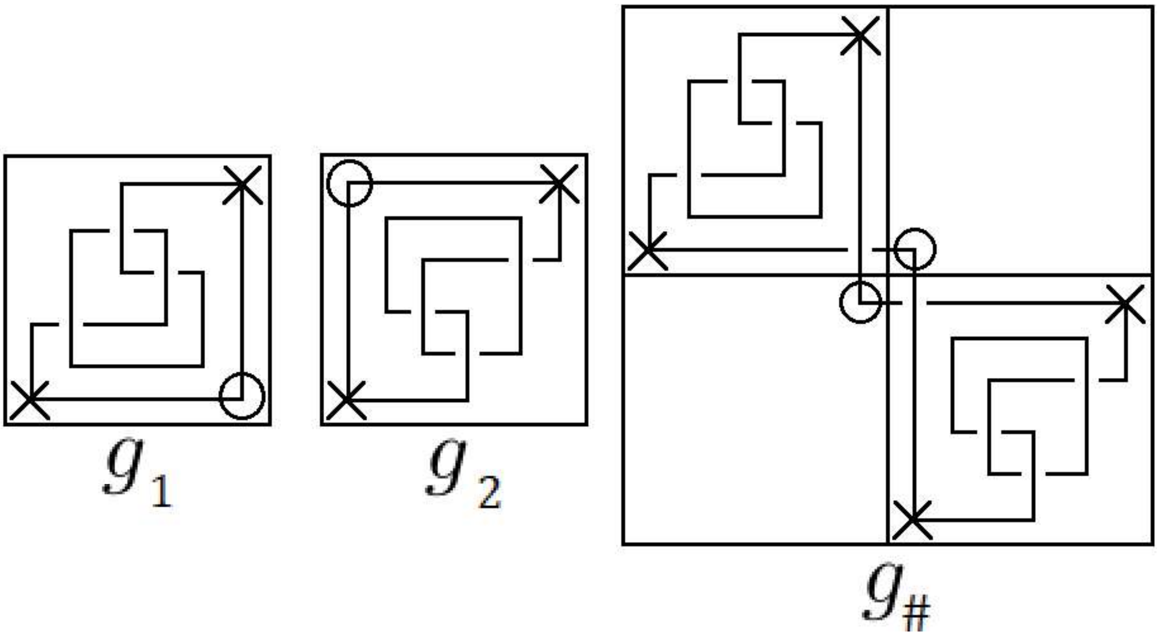

Figure 1: Grid diagrams representing K_1, K_2, and $K_1\#K_2$.

Corollary 1.3.

$$\tau(K_1\#K_2) = \tau(K_1) + \tau(K_2)$$

for any two knots K_1 and K_2.

The Legendrian grid invariants $\lambda^+, \lambda^- \in GH^-(\mathcal{K})$ are defined to be the homology classes determined by the canonical generators $\mathbf{x}^\pm$ of the grid chain complex. To prove Theorem 1.1, we constructed quasi-isomorphisms for grid chain complexes. These quasi-isomorphisms send the canonical generators $\mathbf{x}^\pm$ to themselves, so the additivity of the Legendrian grid invariants can be quickly shown.

Theorem 1.4. *Let $\mathbb{G}_1$, $\mathbb{G}_2$, and $\mathbb{G}_\#$ be grid diagrams representing Legendrian knots $\mathcal{K}_1$, $\mathcal{K}_2$, and $\mathcal{K}_1\#\mathcal{K}_2$ respectively. Then there is an isomorphism*

$$GH^-(\mathbb{G}_1) \otimes GH^-(\mathbb{G}_2) \to GH^-(\mathbb{G}_\#),$$

which maps $\lambda^+(\mathcal{K}_1) \otimes \lambda^+(\mathcal{K}_2)$ to $\lambda^+(\mathcal{K}_1\#\mathcal{K}_2)$ and $\lambda^-(\mathcal{K}_1) \otimes \lambda^-(\mathcal{K}_2)$ to $\lambda^-(\mathcal{K}_1\#\mathcal{K}_2)$.

The grid transverse invariant is defined to be the homology class $\lambda^+ \in GH^-(\mathbb{G})$, so the additivity of the transverse grid invariants is also proved.

Corollary 1.5. *Let $\mathbb{G}_1$, $\mathbb{G}_2$, and $\mathbb{G}_\#$ be two good grid diagrams representing transverse knots $\mathcal{T}_1$, $\mathcal{T}_2$, and $\mathcal{T}_1\#\mathcal{T}_2$ respectively. Then there is an isomorphism*

$$GH^-(\mathbb{G}_1) \otimes GH^-(\mathbb{G}_2) \to GH^-(\mathbb{G}_\#)$$

which maps $\theta^-(\mathcal{T}_1) \otimes \theta^-(\mathcal{T}_2)$ to $\theta^-(\mathcal{T}_1\#\mathcal{T}_2)$.

2 Problems in grid homology

Compared with knot Floer homology, grid homology has many properties that are expected but still not proven in the framework of grid homology.

There are many operations on knots and links and some of them such as connected sum, mutation, and Murasugi sum are studied using knot Floer homology. In [10], the connected sum operation in grid homology is combinatorially described.

Question 2.1. Characterize operations on knots and links using grid homology.

The original grid homology is an invariant of knots in S^3. Its definition comes from a special case of a genus one multi-pointed Heegaard diagram. Since a grid diagram is on the torus, it is essential that the genus of the Heegaard diagram is one. A typical example of three-manifolds with Heegaard genus one is lens spaces. Baker, Grigsby, and Hedden [1] defined grid homology for lens spaces. Tripp [20] gave a combinatorial proof that grid homology for lens spaces is a link invariant. On the other hand, grid homology for general three-manifolds has not been defined.

Note that Manolescu, Ozsváth, and Thurston [15] gave combinatorial descriptions of the Heegaard Floer homology (not knot Floer homology) for arbitrary three-manifolds using the link surgery formula and grid homology.

Question 2.2. Define grid homology for knots/links in an arbitrary oriented, closed three-manifold.

The tau, epsilon, and Upsilon invariants in knot Floer homology are reconstructed using grid homology. These invariants are concordance invariants and well-studied [7].

- Sarkar [18] gave a combinatorial description of the tau invariant using grid homology. Using grid homology, Sarkar showed that the tau invariant is a concordance invariant and gave a combinatorial proof of the Milnor conjecture. The idea in this paper is in effect the only way to show the concordance invariance of invariants in grid homology. Using grid homology, the tau invariant for knots with at most 11 crossings was determined [2].

- Dey and Doğa [5] gave a combinatorial description of the epsilon invariant using grid homology and combinatorially proved that the epsilon invariant is a concordance invariant. They obtained an alternative derivation of the epsilon invariant of cables of negative torus knots and computed the invariant for closures of positive braids.

- Földvári [6] defined the Upsilon invariant using grid homology and proved some properties known in knot Floer homology. The author [8] gave another formulation of the grid Upsilon invariant and proved that it is a concordance invariant. Using this formulation, the Upsilon invariant in grid homology can be proved to coincide with one in knot Floer homology. Using grid homology, the Upsilon invariant for prime knots with at most 11 crossings was computed [19].

Question 2.3. Give applications of the above combinatorial descriptions of the tau, epsilon, and Upsilon invariants.

In Floer theory, it is an interesting problem whether Floer homology can be lifted to a Floer spectrum or pro-spectrum, in the sense of stable homotopy theory. Stable homotopy refinements of Seiberg-Witten Floer homology have been constructed [12]. Also, there is a lift of Khovanov homology to a stable homotopy type [11].

Recently, Manolescu and Sarkar constructed a stable homotopy refinement of knot Floer homology using grid homology [3]. So far, the only given stable homotopy refinement of knot Floer homology is their construction. For a grid diagram $\mathbb{G}$, they defined *knot Floer spectrum* $\mathcal{X}^+(\mathbb{G})$. Its homology is the grid homology GH^+, the plus version of grid homology. The construction of $\mathcal{X}^+(\mathbb{G})$ is based on a framed flow category introduced by Cohen, Jones, and Segal [4].

Question 2.4. Compute knot Floer spectra of various knots.

Question 2.5. Define an invariant that captures the information of knot Floer spectra.

Question 2.6. Find a pair of knots that we can be distinguished by knot Floer spectra but not by knot Floer homology.

References

[1] Kenneth L. Baker, J. Elisenda Grigsby, and Matthew Hedden. Grid diagrams for lens spaces and combinatorial knot Floer homology. *Int. Math. Res. Not. IMRN*, (10):Art. ID rnm024, 39, 2008.

[2] John A. Baldwin and William D. Gillam. Computations of Heegaard-Floer knot homology. *J. Knot Theory Ramifications*, 21(8):1250075, 65, 2012.

[3] Sucharit Sarkar Ciprian Manolescu. A knot floer stable homotopy type. arXiv:2108.13566, 2021.

[4] R. L. Cohen, J. D. S. Jones, and G. B. Segal. Floer's infinite-dimensional Morse theory and homotopy theory. Number 883, pages 68–96. 1994. Geometric aspects of infinite integrable systems (Japanese) (Kyoto, 1993).

[5] Subhankar Dey and Hakan Doğa. A combinatorial description of the knot concordance invariant epsilon. *J. Knot Theory Ramifications*, 30(6):Paper No. 2150036, 26, 2021.

[6] Viktória Földvári. The knot invariant Υ using grid homologies. *J. Knot Theory Ramifications*, 30(7):Paper No. 2150051, 26, 2021.

[7] Jennifer Hom. A survey on Heegaard Floer homology and concordance. *J. Knot Theory Ramifications*, 26(2):1740015, 24, 2017.

[8] Hajime Kubota. Concordance invariant Υ for balanced spatial graphs using grid homology. *J. Knot Theory Ramifications*, 32(13):Paper No. 2350088, 30, 2023.

[9] Hajime Kubota. Grid homology for spatial graphs and a künneth formula of connected sum. arXiv:2206.15048v2, 2023.

[10] Hajime Kubota. Quasi-isomorphism of grid chain complexes for a connected sum of knots. arXiv:2312.02610, 2024.

[11] Robert Lipshitz and Sucharit Sarkar. A Khovanov stable homotopy type. *J. Amer. Math. Soc.*, 27(4):983–1042, 2014.

[12] Ciprian Manolescu. Seiberg-Witten-Floer stable homotopy type of three-manifolds with $b_1 = 0$. *Geom. Topol.*, 7:889–932, 2003.

[13] Ciprian Manolescu, Peter Ozsváth, and Sucharit Sarkar. A combinatorial description of knot Floer homology. *Ann. of Math. (2)*, 169(2):633–660, 2009.

[14] Ciprian Manolescu, Peter Ozsváth, Zoltán Szabó, and Dylan Thurston. On combinatorial link Floer homology. *Geom. Topol.*, 11:2339–2412, 2007.

[15] Ciprian Manolescu, Peter Ozsvath, and Dylan Thurston. Grid diagrams and heegaard floer invariants. arXiv:2206.15048v2, 2020.

[16] Peter Ozsváth and Zoltán Szabó. Holomorphic disks and knot invariants. *Adv. Math.*, 186(1):58–116, 2004.

[17] Peter S. Ozsváth, András I. Stipsicz, and Zoltán Szabó. *Grid homology for knots and links*, volume 208 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2015.

[18] Sucharit Sarkar. Grid diagrams and the Ozsváth-Szabó tau-invariant. *Math. Res. Lett.*, 18(6):1239–1257, 2011.

[19] Kouki Sato Taketo Sano. An algorithm for computing the υ-invariant and the d-invariants of dehn surgeries. arXiv:2002.09210, 2020.

[20] Samuel Willis Tripp. *On Grid Homology for Lens Space Links*. ProQuest LLC, Ann Arbor, MI, 2022. Thesis (Ph.D.)–Dartmouth College.

[21] Vera Vértesi. Transversely nonsimple knots. *Algebr. Geom. Topol.*, 8(3):1481–1498, 2008.

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