Fox's $\mathbb{Z}$-colorings of braids and related topics

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1 Introduction

This article is an announcement of a forthcoming paper [4]. We refer the reader to [4] for more details.

Fox's coloring is one of the fundamental tools in knot theory. For example, the $\mathbb{Z}/p\mathbb{Z}$-coloring number not only distinguishes various pairs of knots but also gives a lower bound for the unknotting number of a knot [5], and the $\mathbb{Z}$-coloring is useful for classifying rational tangles [3].

The $\mathbb{Z}/p\mathbb{Z}$-colorings of m-braids relate to the Hurwitz action of the m-braid group on $(\mathbb{Z}/p\mathbb{Z})^m$, and Berger [1] determines the orbits of this action. In this article we consider $\mathbb{Z}$-colorings of m-braids corresponding to the Hurwitz action of the m-braid group on $\mathbb{Z}^m$. For two elements $v, w \in \mathbb{Z}^m$, we write $v \sim w$ if there is an m-braid admitting a $\mathbb{Z}$-coloring such that the top and bottom points receive v and w, respectively. See Figure 1.1. Equivalently, $v \sim w$ holds if they belong to the same orbit. The main aim of this article is to characterize this equivalence relation $\sim$ on $\mathbb{Z}^m$ by introducing several invariants.

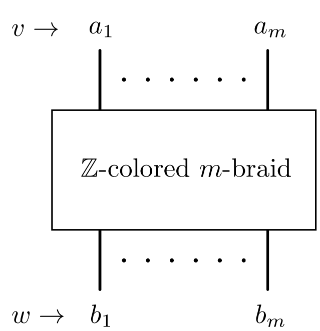

Figure 1.1: $v = (a_1, \ldots, a_m) \sim w = (b_1, \ldots, b_m) \in \mathbb{Z}^m$

An (m, m)-*tangle* is an m-braid without monotone property. It is permitted that an (m, m)-tangle has a finite number of loops. Considering $\mathbb{Z}$-colorings of (m, m)-tangles, we define two coarser equivalence relations on $\mathbb{Z}^m$ than $\sim$ defined above. We write $v \overset{\mathcal{T}_0}{\sim} w$ (or $v \overset{\mathcal{T}}{\sim} w$) if there is an (m, m)-tangle without loops (or possibly with some loops) admitting a $\mathbb{Z}$-coloring such that the top and bottom points receive v and w, respectively.

A *pure m-braid* (or an *m-string link*) is an m-braid (or an (m,m)-tangle) such that the ith top point connects to the ith bottom point by a string for any $i = 1, \dots, m$. Then we introduce three equivalence relations on $\mathbb{Z}^m$ such that $v \overset{\mathcal{P}}{\sim} w$ ($v \overset{\mathcal{L}_0}{\sim} w$, or $v \overset{\mathcal{L}}{\sim} w$) if there is a pure m-braid (an m-string link without loops, or an m-string link possibly with some loops) admitting a $\mathbb{Z}$-coloring such that the top and bottom points receive v and w, respectively.

By definition, the six equivalence relations on $\mathbb{Z}^m$ have a relationship as shown in Table 1.1. For example, if $v \overset{\mathcal{P}}{\sim} w$ holds, that is, v and w are connected by a $\mathbb{Z}$-colored pure m-braid, then we have $v \sim w$ and $v \overset{\mathcal{L}_0}{\sim} w$ by regarding the pure m-braid as just an m-braid and an m-string link without loops.

Table 1.1: A relationship among the six equivalence relations

$$\begin{array}{ccccc} v \sim w & \Rightarrow & v \overset{\mathcal{T}_0}{\sim} w & \Rightarrow & v \overset{\mathcal{T}}{\sim} w \\ \Uparrow & & \Uparrow & & \Uparrow \\ v \overset{\mathcal{P}}{\sim} w & \Rightarrow & v \overset{\mathcal{L}_0}{\sim} w & \Rightarrow & v \overset{\mathcal{L}}{\sim} w \end{array}$$

Moreover, by permitting virtual crossings, we consider six equivalence relations $\overset{v}{\sim}$ (connected by a $\mathbb{Z}$-colored virtual m-braid), $\overset{v\mathcal{T}_0}{\sim}$ (connected by a $\mathbb{Z}$-colored virtual (m,m)-tangle without loops), $\overset{v\mathcal{T}}{\sim}$ (connected by a $\mathbb{Z}$-colored virtual (m,m)-tangle possibly with loops), $\overset{v\mathcal{P}}{\sim}$ (connected by a $\mathbb{Z}$-colored virtual pure m-braid), $\overset{v\mathcal{L}_0}{\sim}$ (connected by a $\mathbb{Z}$-colored virtual m-string link without loops) and $\overset{v\mathcal{L}}{\sim}$ (connected by a $\mathbb{Z}$-colored virtual m-string link possibly with loops).

For an element $v = (a_1, \dots, a_m) \in \mathbb{Z}^m$, we set

- $\Delta(v) = \sum_{i=1}^{m} (-1)^{i-1} a_i \in \mathbb{Z}$,
- $d(v) = \gcd\{a_2 - a_1, \dots, a_m - a_1\} \geq 0$,
- $d_2(v) = 2^s$ for $d(v) = 2^s t > 0$ with $s \geq 0$ and t odd, and $d_2(v) = 0$ for $d(v) = 0$,
- $M(v)_N = \{a_1, \dots, a_m\} \pmod N$ as a multi-set, and
- $\overrightarrow{M}(v)_N = (a_1, \dots, a_m) \pmod N$ as an ordered set.

Theorem 1.1. *The twelve equivalence relations on* $\mathbb{Z}^m$ $(m \geq 2)$ *as above are characterized as shown in* Table 1.2*.*

For example, $v \sim w$ if and only if $\Delta(v) = \Delta(w)$, $d(v) = d(w)$ and $M(v)_{2d(v)} = M(w)_{2d(w)}$. Also, $v \overset{\mathcal{L}}{\sim} w$ if and only if $\Delta(v) = \Delta(w)$ and $\overrightarrow{M}(v)_2 = \overrightarrow{M}(w)_2$.

2 Preliminaries

For an integer $m \geq 2$, let $\mathcal{B}_m$ be the m-braid group with the standard generators $\sigma_1, \dots, \sigma_{m-1}$. The set $\mathbb{Z}^m = \{(a_1, \dots, a_m) \mid a_1, \dots, a_m \in \mathbb{Z}\}$ has the Hurwitz action

Table 1.2: Results

	classical case						virtual case					
	$\sim$	$\overset{\mathcal{T}_0}{\sim}$	$\overset{\mathcal{T}}{\sim}$	$\overset{\mathcal{P}}{\sim}$	$\overset{\mathcal{L}_0}{\sim}$	$\overset{\mathcal{L}}{\sim}$	$\overset{v}{\sim}$	$\overset{v\mathcal{T}_0}{\sim}$	$\overset{v\mathcal{T}}{\sim}$	$\overset{v\mathcal{P}}{\sim}$	$\overset{v\mathcal{L}_0}{\sim}$	$\overset{v\mathcal{L}}{\sim}$
$\Delta(v)=\Delta(w)$	○	○	○	○	○	○						
$d(v)=d(w)$	○			○			○			○		
$d_2(v)=d_2(w)$		○			○			○			○	
$M(v)_{2d}=M(w)_{2d}$	○						○					
$M(v)_{2d_2}=M(w)_{2d_2}$		○						○				
$M(v)_2=M(w)_2$			○						○			
$\overrightarrow{M}(v)_{2d}=\overrightarrow{M}(w)_{2d}$				○						○		
$\overrightarrow{M}(v)_{2d_2}=\overrightarrow{M}(w)_{2d_2}$					○						○	
$\overrightarrow{M}(v)_2=\overrightarrow{M}(w)_2$						○						○

of $\mathcal{B}_m$ from the right defined by

$$v\cdot\sigma_i=(a_1,\ldots,a_{i-1},a_{i+1},2a_{i+1}-a_i,a_{i+2},\ldots,a_m)\text{ and}$$
$$v\cdot\sigma_i^{-1}=(a_1,\ldots,a_{i-1},2a_i-a_{i+1},a_i,a_{i+2},\ldots,a_m)$$

for an element $v=(a_1,\ldots,a_m)\in\mathbb{Z}^m$. See the left of Figure 2.1. We say that two elements $v,w\in\mathbb{Z}^m$ are (*Hurwitz*) *equivalent* if there is an m-braid $\beta\in\mathcal{B}_m$ such that $v\cdot\beta=w$, and denote it by $v\sim w$. The right figure shows that

$$(1,-5,4)\cdot(\sigma_1^{-1}\sigma_2^2)=(7,7,10)\in\mathbb{Z}^3,$$

and hence $(1,-5,4)\sim(7,7,10)$.

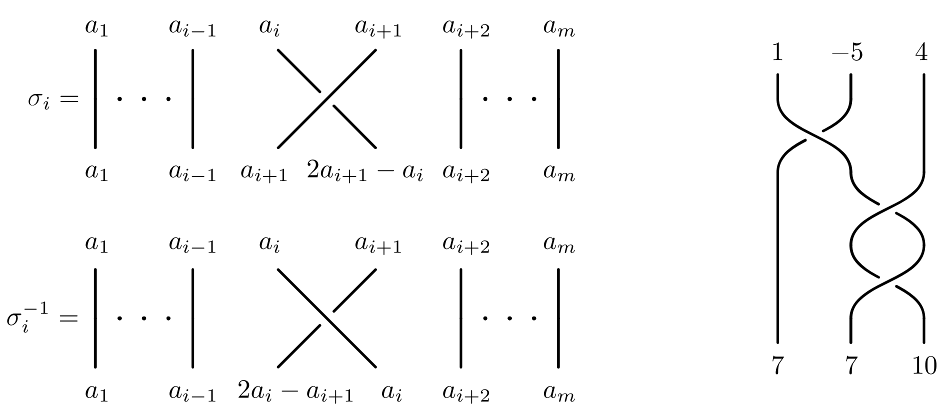

Figure 2.1: The Hurwitz action of $\mathcal{B}_m$ on $\mathbb{Z}^m$

An element $v\in\mathbb{Z}^m$ is called *trivial* if $v=a\cdot\mathbf{1}=(a,\ldots,a)$ for some $a\in\mathbb{Z}$, where $\mathbf{1}=(1,\ldots,1)$. By definition, if $v\sim w$ and v is trivial, then we have $v=w$; in other words, the orbit of a trivial element v consists of v only.

For an element $v = (a_1, \dots, a_m) \in \mathbb{Z}^m$, we will consider two integers defined by

$$\Delta(v) = \sum_{i=1}^{m} (-1)^{i-1} a_i \text{ and}$$
$$d(v) = \gcd\{a_i - a_j \mid 1 \le i \ne j \le m\} = \gcd\{a_i - a_1 \mid 2 \le i \le m\} \ge 0.$$

We remark that $v \in \mathbb{Z}^m$ is trivial if and only if $d(v) = 0$. In what follows, "$a \equiv b \pmod 0$" means "$a = b \in \mathbb{Z}$" for convenience. For example, the element $v = (1, -5, 4) \in \mathbb{Z}^3$ has

$$\Delta(v) = 1 - (-5) + 4 = 10 \text{ and } d(v) = \gcd\{-6, 3\} = 3.$$

Lemma 2.1. *For an element $v = (a_1, \dots, a_m) \in \mathbb{Z}^m$, we have the following.*

(i) $a_1 \equiv \cdots \equiv a_m \pmod{d(v)}$.

(ii) *If m is odd, then $\Delta(v) \equiv a_1 \pmod{d(v)}$.*

(iii) *If m is even, then $\Delta(v) \equiv 0 \pmod{d(v)}$.*

Let $\mathcal{S}_m$ be the symmetric group on $\{1, \dots, m\}$. For an m-braid $\beta \in \mathcal{B}_m$, we denote by $\pi_\beta \in \mathcal{S}_m$ the permutation associated with β; that is, β connects each ith top point to the $\pi_\beta(i)$th bottom point ($i = 1, \dots, m$). For a multi-subset X of $\mathbb{Z}$ and an integer $N \ge 2$, we denote by X_N the multi-subset of $\mathbb{Z}/N\mathbb{Z}$ consisting of the congruence classes of all integers in X modulo N. We also use the symbol $X_0 = X$ for convenience. For $v = (a_1, \dots, a_m) \in \mathbb{Z}^m$, we set $M(v) = \{a_1, \dots, a_m\}$ as a multi-subset of $\mathbb{Z}$.

Now we consider the multi-set $M(v)_{2d(v)} \subset \mathbb{Z}/2d(v)\mathbb{Z}$. If $v = a \cdot \mathbf{1} = (a, \dots, a) \in \mathbb{Z}^m$ is trivial, then we have $d(v) = 0$ and

$$M(v)_0 = \{\underbrace{a, \dots, a}_{m}\}_0 \subset \mathbb{Z}.$$

If $v \in \mathbb{Z}^m$ is nontrivial, then it follows from Lemma 2.1(i) that

$$M(v)_{2d(v)} = \{\underbrace{r, \dots, r}_{p}, \underbrace{r + d(v), \dots, r + d(v)}_{m-p}\}_{2d(v)}$$

for some $0 \le r < d(v)$ and $1 \le p \le m - 1$. For example, the element $v = (1, -5, 4)$ has

$$d(v) = 3 \text{ and } M(v)_6 = \{1, 1, 4\}_6.$$

Lemma 2.2. *Let $v = (a_1, \dots, a_m)$ and $w = (b_1, \dots, b_m) \in \mathbb{Z}^m$ be elements satisfying $v \cdot \beta = w$ for an m-braid $\beta \in \mathcal{B}_m$. Then we have the following.*

(i) $\Delta(v) = \Delta(w)$.

(ii) $d(v) = d(w)$.

(iii) $b_{\pi_\beta(k)} \equiv a_k \pmod{2d(v)}$ *for any $k = 1, \dots, m$.*

(iv) $M(v)_{2d(v)} = M(w)_{2d(v)}$.

For example, since $w = (7, 7, 10)$ is equivalent to $v = (1, -5, 4)$, we have

$$\Delta(w) = \Delta(v) = 10,\ d(w) = d(v) = 3 \text{ and } M(w)_6 = M(v)_6 = \{1, 1, 4\}_6$$

by Lemma 2.2(i), (ii) and (iv).

3 The case $m \geq 3$ odd

Throughout this section we consider the equivalence relation $\sim$ on $\mathbb{Z}^m$ for $m = 2k-1 \geq 3$. The following theorem provides a classification of $\mathbb{Z}^{2k-1}$ under $\sim$.

Theorem 3.1. *For two elements $v, w \in \mathbb{Z}^{2k-1}$, the following are equivalent.*

(i) $v \sim w$.

(ii) $\Delta(v) = \Delta(w)$, $d(v) = d(w)$ *and* $M(v)_{2d(v)} = M(w)_{2d(w)}$.

For example, $v = (2, -4, 11, 8, -1)$ and $w = (5, 5, 2, 2, 8) \in \mathbb{Z}^5$ satisfy

$$\Delta(v) = \Delta(w) = 8,\ d(v) = d(w) = 3 \text{ and } M(v)_6 = M(w)_6 = \{2, 2, 2, 5, 5\}_6.$$

Therefore we have $v \sim w$ by Theorem 3.1.

In the case $m = 2k-1 = 3$, the multi-set $M(v)_{2d(v)}$ can be uniquely determined by $\Delta(v)$ and $d(v)$. In fact, we have

$$M(v)_{2d(v)} = \{\Delta(v), \Delta(v) + d(v), \Delta(v) + d(v)\}_{2d(v)},$$

which includes the case where v is trivial with $d(v) = 0$. Therefore we see that $v \sim w \in \mathbb{Z}^3$ if and only if $\Delta(v) = \Delta(w)$ and $d(v) = d(w)$ only.

To prove Theorem 3.1, we prepare Lemmas 3.2–3.6 and Proposition 3.7.

For an element $v = (a_1, a_2, a_3) \in \mathbb{Z}^3$, we set

$$|v| = \max\{a_1, a_2, a_3\} - \min\{a_1, a_2, a_3\} \geq 0.$$

Lemma 3.2. *Let $v = (a_1, a_2, a_3)$ be an element in $\mathbb{Z}^3$. If a_1, a_2 and a_3 are mutually distinct, then v is equivalent to some $w \in \mathbb{Z}^3$ with $|w| < |v|$.*

Lemma 3.3. *Any element $v \in \mathbb{Z}^3$ is equivalent to (x, y, y) for some $x \leq y$.*

Lemma 3.4. *Let $v \in \mathbb{Z}^{2k-1}$ be an element of the form*

$$v = (\underbrace{x, \ldots, x}_{2p-1}, \underbrace{y, \ldots, y}_{2q}, \underbrace{z, \ldots, z}_{2r})$$

for some $x, y, z, p, q, r \in \mathbb{Z}$ with $p, q, r \geq 1$ and $p + q + r = k$. If $x < y < z$ holds, then v is equivalent to

$$(\underbrace{x, \ldots, x}_{2p-1}, \underbrace{y', \ldots, y'}_{2q'}, \underbrace{z', \ldots, z'}_{2r'})$$

for some $y', z', q', r' \in \mathbb{Z}$ with $x \leq y' \leq z' < z$ and $\{q', r'\} = \{q, r\}$.

Lemma 3.5. *Let $v \in \mathbb{Z}^{2k-1}$ be an element of the form*

$$v = (\underbrace{x, \ldots, x}_{2p-1}, \underbrace{y, \ldots, y}_{2q}, \underbrace{z, \ldots, z}_{2r})$$

for some $x, y, z, p, q, r \in \mathbb{Z}$ with $p, q, r \geq 1$ and $p + q + r = k$. If $x < y < z$ holds, then v is equivalent to

$$(\underbrace{x, \ldots, x}_{2p'-1}, \underbrace{y', \ldots, y'}_{2k-2p'})$$

for some $y', p' \in \mathbb{Z}$ with $x < y'$ and $p' \in \{p, p+q, p+r\}$.

Lemma 3.6. *Any element $v \in \mathbb{Z}^{2k-1}$ is equivalent to*

$$(\underbrace{x, \dots, x}_{2p-1}, \underbrace{y, \dots, y}_{2k-2p})$$

for some $x, y, p \in \mathbb{Z}$ with $x < y$ and $1 \leq p \leq k$.

By Lemma 2.1(i) and (ii), any element $v = (a_1, \dots, a_{2k-1}) \in \mathbb{Z}^{2k-1}$ satisfies

$$a_1 \equiv \cdots \equiv a_{2k-1} \equiv \Delta(v) \pmod{d(v)}.$$

Therefore we have $a_i \equiv \Delta(v)$ or $\Delta(v) + d(v) \pmod{2d(v)}$. Moreover, the number of a_i's congruent to $\Delta(v)$ modulo $2d(v)$ is odd.

Proposition 3.7. *Let v be an element in $\mathbb{Z}^{2k-1}$. Set $\Delta = \Delta(v)$, $d = d(v)$ and*

$$M(v)_{2d} = \{\underbrace{\Delta, \dots, \Delta}_{2p-1}, \underbrace{\Delta + d, \dots, \Delta + d}_{2k-2p}\}_{2d}$$

for some $p \in \mathbb{Z}$ with $1 \leq p \leq k$. Then we have

$$v \sim (\underbrace{\Delta, \dots, \Delta}_{2p-1}, \underbrace{\Delta + d, \dots, \Delta + d}_{2k-2p}).$$

4 The case $m \geq 4$ even

Throughout this section we consider the equivalence relation $\sim$ on $\mathbb{Z}^m$ for $m = 2k \geq 4$. The following theorem provides a classification of $\mathbb{Z}^{2k}$ under $\sim$.

Theorem 4.1. *For two elements $v, w \in \mathbb{Z}^{2k}$, the following are equivalent.*

(i) *$v \sim w$.*

(ii) *$\Delta(v) = \Delta(w)$, $d(v) = d(w)$ and $M(v)_{2d(v)} = M(w)_{2d(w)}$.*

For example, $v = (-4, 11, 8, -1)$ and $w = (5, 5, 2, 8) \in \mathbb{Z}^4$ satisfy

$$\Delta(v) = \Delta(w) = -6,\ d(v) = d(w) = 3 \text{ and } M(v)_6 = M(w)_6 = \{2, 2, 5, 5\}_6.$$

Therefore we have $v \sim w$ by Theorem 4.1.

To prove Theorem 4.1, we prepare Lemma 4.2 and Proposition 4.3.

Lemma 4.2. *Let $v \in \mathbb{Z}^{2k}$ be a nontrivial element of the form*

$$v = (\underbrace{x, \dots, x}_{p}, x + \lambda d, \underbrace{x + d, \dots, x + d}_{2k-p-1})$$

for some $x, \lambda, d, p \in \mathbb{Z}$ with $d > 0$ and $1 \leq p \leq 2k - 2$. Then we have

$$v \sim v + 2nd \cdot \mathbf{1}$$

for any $n \in \mathbb{Z}$.

Proposition 4.3. *Let v be a nontrivial element in $\mathbb{Z}^{2k}$. Set $d = d(v)$ and*

$$M(v)_{2d} = \{\underbrace{r, \dots, r}_{p}, \underbrace{r+d, \dots, r+d}_{2k-p}\}_{2d}$$

for some $r, p \in \mathbb{Z}$ with $0 \leq r < d$ and $1 \leq p \leq 2k-1$.

(i) *If $p \geq 2$ holds, then there is an even integer λ such that*

$$v \sim (\underbrace{r, \dots, r}_{p-1}, r+\lambda d, \underbrace{r+d, \dots, r+d}_{2k-p}).$$

(ii) *If $2k-p \geq 2$ holds, then there is an odd integer λ such that*

$$v \sim (\underbrace{r, \dots, r}_{p}, r+\lambda d, \underbrace{r+d, \dots, r+d}_{2k-p-1}).$$

5 Tangles

In this section we define two coarser equivalence relations $\overset{\mathcal{T}_0}{\sim}$ and $\overset{\mathcal{T}}{\sim}$ on $\mathbb{Z}^m$ than $\sim$, and characterize them. Throughout this section we assume $k \geq 2$ and $m \geq 3$.

5.1 Tangles

By a *k-string tangle T*, we mean a tangle diagram of k strings possibly with some loops. A map $C : \{\text{arcs of } T\} \to \mathbb{Z}$ is called a *$\mathbb{Z}$-coloring* of T if the equation $a + c = 2b$ holds at each crossing of T, where a and c are the integers assigned to the under-arcs at the crossing, and b is the integer assigned to the over-arc. The integer assigned to an arc is called the *color* of the arc. Let $a_1, \dots, a_{2k}$ be the colors of the $2k$ endpoints of T listed in counterclockwise order. Then we say that the endpoints receive $v = (a_1, \dots, a_{2k}) \in \mathbb{Z}^{2k}$. For example, we consider a 3-string tangle T with a single loop admitting a $\mathbb{Z}$-coloring as shown in Figure 5.1. Then the 6 endpoints of T receive $(1, 5, 19, 20, 6, 1) \in \mathbb{Z}^6$.

Proposition 5.1. *For an element $v \in \mathbb{Z}^{2k}$, the following are equivalent.*

(i) *There is a k-string tangle T without loops admitting a $\mathbb{Z}$-coloring such that the $2k$ endpoints of T receive v.*

(ii) *There is a k-string tangle T' possibly with some loops admitting a $\mathbb{Z}$-coloring such that the $2k$ endpoints of T' receive v.*

(iii) $\Delta(v) = 0$.

5.2 (m, m)-tangles without loops

By an *(m, m)-tangle*, we mean an m-string tangle diagram possibly with some loops such that each string connects between top and bottom points. We denote by $\mathcal{T}_m$ the set of (m, m)-tangles, and by $\mathcal{T}_{m,0} \subset \mathcal{T}_m$ that of (m, m)-tangles without loops.

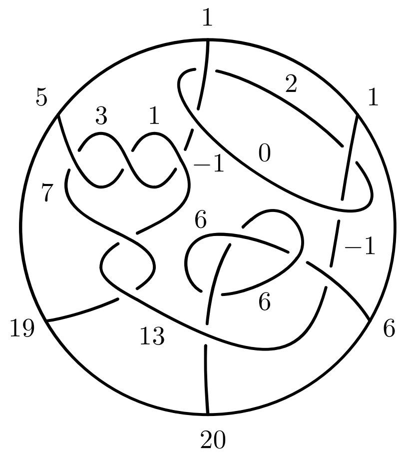

Figure 5.1: A 3-string tangle with a single loop admitting a $\mathbb{Z}$-coloring

For two elements $v, w \in \mathbb{Z}^m$, we write $v \overset{\mathcal{T}_0}{\sim} w$ if there is an (m,m)-tangle without loops admitting a $\mathbb{Z}$-coloring such that the top and bottom points receive v and w, respectively. Figure 5.2 shows an example of a $(3,3)$-tangle without loops admitting a $\mathbb{Z}$-coloring which gives $(-1,2,2) \overset{\mathcal{T}_0}{\sim} (0,3,2)$. We remark that $(-1,2,2) \not\sim (0,3,2)$ by Lemma 2.2(ii); in fact, we have

$$d((-1,2,2)) = 3 \neq 1 = d((0,3,2)).$$

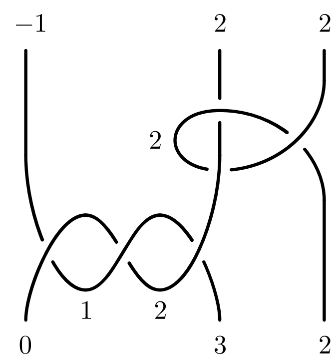

Figure 5.2: A $(3,3)$-tangle without loops admitting a $\mathbb{Z}$-coloring

The following theorem provides a classification of $\mathbb{Z}^m$ under $\overset{\mathcal{T}_0}{\sim}$.

Theorem 5.2. *For two elements $v, w \in \mathbb{Z}^m$, the following are equivalent.*

(i) $v \overset{\mathcal{T}_0}{\sim} w$.

(ii) $\Delta(v) = \Delta(w)$, $d_2(v) = d_2(w)$ *and* $M(v)_{2d_2(v)} = M(w)_{2d_2(w)}$.

To prove this theorem, we prepare Lemmas 5.3–5.5, Propositions 5.6 and 5.7.

Lemma 5.3. *Let $T \in \mathcal{T}_{m,0}$ be an (m,m)-tangle without loops admitting a $\mathbb{Z}$-coloring such that the top m points of T receive $v = (a_1, \ldots, a_m) \in \mathbb{Z}^m$, and*

$$a_i = c_{i1}, c_{i2}, \ldots, c_{in(i)}$$

the colors of all the arcs on each ith string of T from top to bottom ($i = 1, \ldots, m$). If $c_{ij} - a_1$ is divisible by an integer $N \geq 1$ for any i, j, then we have the following.

(i) *$c_{ij} - a_i$ is divisible by $2N$ for any i, j.*

(ii) *If $a_i - a_1$ is divisible by $2N$ for any i, then so is $c_{ij} - a_1$ for any i, j.*

For $T \in \mathcal{T}_{m,0}$, we denote by $\pi_T \in \mathcal{S}_m$ the permutation on $\{1, \ldots, m\}$ associated with T; that is, each ith string of T connects the ith top point to the $\pi_T(i)$th bottom point ($i = 1, \ldots, m$). For $v = (a_1, \ldots, a_m) \in \mathbb{Z}^m$, we define an integer $d_2(v) \geq 0$ as follows: If v is nontrivial with $d(v) = 2^s t$ for $s \geq 0$ and t odd, then we set $d_2(v) = 2^s$. If v is trivial, that is, $d(v) = 0$, then we set $d_2(v) = 0$.

Lemma 5.4. *Let $T \in \mathcal{T}_{m,0}$ be an (m, m)-tangle without loops admitting a $\mathbb{Z}$-coloring such that the top and bottom points receive $v = (a_1, \ldots, a_m)$ and $w = (b_1, \ldots, b_m) \in \mathbb{Z}^m$, respectively. Then we have the following.*

(i) *$\Delta(v) = \Delta(w)$.*

(ii) *$d_2(v) = d_2(w)$.*

(iii) *$b_{\pi_T(i)} \equiv a_i \pmod{2d_2(v)}$ for any $i = 1, \ldots, m$.*

(iv) *$M(v)_{2d_2(v)} = M(w)_{2d_2(w)}$.*

Lemma 5.5. *For any integers $x, s, t \in \mathbb{Z}$ with $s \geq 0$ and t odd, we have*

$$(x, x + 2^s t) \overset{\mathcal{T}_0}{\sim} (x + 2^s, x + 2^s(t + 1)).$$

For any element $v \in \mathbb{Z}^m$, we have

$$a_1 \equiv \cdots \equiv a_m \pmod{d_2(v)}$$

by Lemma 2.1(i) and the fact that $d_2(v)$ divides $d(v)$. In the case where m is odd, it follows from Lemma 2.1(ii) that

$$a_i \equiv \Delta(v) \text{ or } \Delta(v) + d_2(v) \pmod{2d_2(v)},$$

and the number of a_i's congruent to $\Delta(v)$ modulo $2d_2(v)$ is odd.

Proposition 5.6. *Let v be a nontrivial element in $\mathbb{Z}^{2k-1}$. Set $\Delta = \Delta(v)$, $d_2 = d_2(v)$ and*

$$M(v)_{2d_2} = \{\underbrace{\Delta, \ldots, \Delta}_{2p-1}, \underbrace{\Delta + d_2, \ldots, \Delta + d_2}_{2k-2p}\}_{2d_2}$$

for some $p \in \mathbb{Z}$ with $1 \leq p \leq k - 1$. Then we have

$$v \overset{\mathcal{T}_0}{\sim} (\underbrace{\Delta, \ldots, \Delta}_{2p-1}, \underbrace{\Delta + d_2, \ldots, \Delta + d_2}_{2k-2p}).$$

We consider the case where $m = 2k$ is even. For a nontrivial element $v = (a_1, \ldots, a_{2k}) \in \mathbb{Z}^{2k}$, there is an integer r with $0 \leq r < d_2(v)$ such that $a_i \equiv r \pmod{d_2(v)}$ for any $i = 1, \ldots, 2k$. Then it holds that

$$a_i \equiv r \text{ or } r + d_2(v) \pmod{2d_2(v)}.$$

Proposition 5.7. *Let v be a nontrivial element in $\mathbb{Z}^{2k}$. Set $d_2 = d_2(v)$ and*

$$M(v)_{2d_2} = \{\underbrace{r, \dots, r}_{p}, \underbrace{r + d_2, \dots, r + d_2}_{2k-p}\}_{2d_2}$$

for some $r, p \in \mathbb{Z}$ with $0 \le r < d_2$ and $1 \le p \le 2k - 1$.

(i) *If $p \ge 2$ holds, then there is an even integer λ such that*

$$v \overset{\mathcal{T}_0}{\sim} (\underbrace{r, \dots, r}_{p-1}, r + \lambda d_2, \underbrace{r + d_2, \dots, r + d_2}_{2k-p}).$$

(ii) *If $2k - p \ge 2$ holds, then there is an odd integer λ such that*

$$v \overset{\mathcal{T}_0}{\sim} (\underbrace{r, \dots, r}_{p}, r + \lambda d_2, \underbrace{r + d_2, \dots, r + d_2}_{2k-p-1}).$$

5.3 (m, m)-tangles possibly with loops

For two elements $v, w \in \mathbb{Z}^m$, we write $v \overset{\mathcal{T}}{\sim} w$ if there is an (m, m)-tangle possibly with some loops admitting a $\mathbb{Z}$-coloring such that the top and bottom points receive v and w, respectively. We remark that $v \overset{\mathcal{T}_0}{\sim} w$ implies $v \overset{\mathcal{T}}{\sim} w$ by definition.

Figure 5.3 shows an example of a $(3, 3)$-tangle with a single loop admitting a $\mathbb{Z}$-coloring which gives $(6, 10, 4) \overset{\mathcal{T}}{\sim} (0, 0, 0)$. We remark that $(6, 10, 4) \overset{\mathcal{T}_0}{\not\sim} (0, 0, 0)$ by Lemma 5.4(ii); in fact, we have

$$d_2((6, 10, 4)) = 2 \neq 0 = d_2((0, 0, 0)).$$

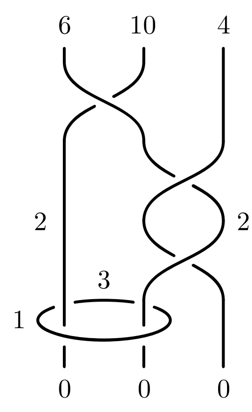

Figure 5.3: A $(3, 3)$-tangle with a single loop admitting a $\mathbb{Z}$-coloring

The following theorem provides a classification of $\mathbb{Z}^m$ under $\overset{\mathcal{T}}{\sim}$.

Theorem 5.8. *For two elements $v, w \in \mathbb{Z}^m$, the following are equivalent.*

(i) $v \overset{\mathcal{T}}{\sim} w$.

(ii) $\Delta(v) = \Delta(w)$ *and* $M(v)_2 = M(w)_2$.

To prove this theorem, we prepare Lemmas 5.9–5.11 and Proposition 5.12.

Lemma 5.9. *Let $T \in \mathcal{T}_m$ be an (m,m)-tangle possibly with some loops admitting a $\mathbb{Z}$-coloring such that the top m points of T receive $v = (a_1, \dots, a_m) \in \mathbb{Z}^m$, and*

$$a_i = c_{i1}, c_{i2}, \dots, c_{in(i)}$$

the colors of all the arcs on each ith string of T from top to bottom ($i = 1, \dots, m$). Then $c_{ij} \equiv a_i \pmod 2$ for any i, j.

Lemma 5.10. *If two elements $v, w \in \mathbb{Z}^m$ satisfy $v \overset{\mathcal{T}}{\sim} w$, then we have the following.*

(i) $\Delta(v) = \Delta(w)$.

(ii) $M(v)_2 = M(w)_2$.

Lemma 5.11. *If two integers $x, y \in \mathbb{Z}$ satisfy $x \equiv y \pmod 2$, then we have*

$$(\underbrace{x, \dots, x}_{2p}) \overset{\mathcal{T}}{\sim} (\underbrace{y, \dots, y}_{2p}).$$

Proposition 5.12. *Let v be an element in $\mathbb{Z}^m$. Set*

$$M(v)_2 = \{\underbrace{0, \dots, 0}_{p}, \underbrace{1, \dots, 1}_{m-p}\}_2$$

for some $p \in \mathbb{Z}$ with $0 \le p \le m$.

(i) *If $p \ge 1$ holds, then there is an even integer λ such that*

$$v \overset{\mathcal{T}}{\sim} (\underbrace{0, \dots, 0}_{p-1}, \lambda, \underbrace{1, \dots, 1}_{m-p}).$$

(ii) *If $m - p \ge 1$ holds, then there is an odd integer λ such that*

$$v \overset{\mathcal{T}}{\sim} (\underbrace{0, \dots, 0}_{p}, \lambda, \underbrace{1, \dots, 1}_{m-p-1}).$$

6 Pure braids and string links

In this section we define three finer equivalence relations $\overset{\mathcal{P}}{\sim}$, $\overset{\mathcal{L}_0}{\approx}$ and $\overset{\mathcal{L}}{\sim}$ on $\mathbb{Z}^m$ than $\sim$, $\overset{\mathcal{T}_0}{\approx}$ and $\overset{\mathcal{T}}{\sim}$, respectively, and characterize them. Throughout this section we assume $m \ge 3$.

6.1 Pure braids

A *pure m-braid* is an m-braid $\beta \in \mathcal{B}_m$ such that the permutation $\pi_\beta \in \mathcal{S}_m$ associated with β is the identity e; that is, the ith top point connects to the ith bottom point by a string of β for any $i = 1, \ldots, m$. Let $\mathcal{P}_m$ denote the pure m-braid group, which is the subgroup of $\mathcal{B}_m$ consisting of pure m-braids. We denote by $\overset{\mathcal{P}}{\sim}$ the equivalence relation on $\mathbb{Z}^m$ induced from the Hurwitz action of $\mathcal{P}_m \subset \mathcal{B}_m$.

For an element $v = (a_1, \ldots, a_m) \in \mathbb{Z}^m$, we set $\overrightarrow{M}(v) = (a_1, \ldots, a_m)$ as an ordered set of $\mathbb{Z}$. The following theorem provides a classification of $\mathbb{Z}^m$ under $\overset{\mathcal{P}}{\sim}$.

Theorem 6.1. *For two elements $v, w \in \mathbb{Z}^m$, the following are equivalent.*

(i) $v \overset{\mathcal{P}}{\sim} w$.

(ii) $\Delta(v) = \Delta(w)$, $d(v) = d(w)$ *and* $\overrightarrow{M}(v)_{2d(v)} = \overrightarrow{M}(w)_{2d(w)}$.

For example, we consider $v = (1, -5, 4)$, $w = (10, 7, 7)$ and $u = (7, 7, 10) \in \mathbb{Z}^3$. They satisfy

$$\begin{aligned}&\Delta(v) = \Delta(w) = \Delta(u) = 10,\ d(v) = d(w) = d(u) = 3,\\ &M(v)_6 = M(w)_6 = M(u)_6 = \{1, 1, 4\}_6 \text{ and}\\ &\overrightarrow{M}(v)_6 = \overrightarrow{M}(u)_6 = (1, 1, 4)_6 \neq (4, 1, 1)_6 = \overrightarrow{M}(w)_6.\end{aligned}$$

Then we have

$$v \sim w,\ v \overset{\mathcal{P}}{\not\sim} w \text{ and } v \overset{\mathcal{P}}{\sim} u$$

by Theorem 6.1. See Figure 6.1.

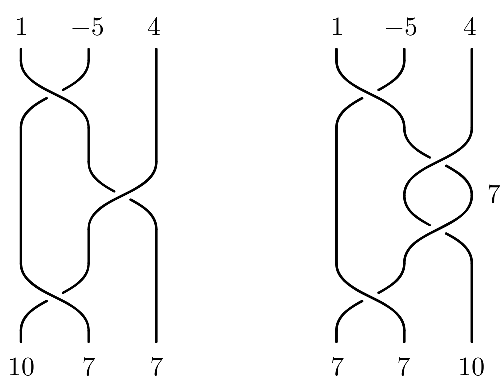

Figure 6.1: $(1, -5, 4) \sim (10, 7, 7)$ and $(1, -5, 4) \overset{\mathcal{P}}{\sim} (7, 7, 10)$

To prove Theorem 6.1, we prepare Lemma 6.2.

For an integer i with $1 \leq i \leq m$, we consider two subgroups of $\mathcal{S}_m$ defined by

$$\begin{aligned}&\mathcal{S}_{1,i} = \{\pi \in \mathcal{S}_m \mid \pi(j) = j \text{ for } i+1 \leq j \leq m\} \text{ and}\\ &\mathcal{S}_{i,m} = \{\pi \in \mathcal{S}_m \mid \pi(j) = j \text{ for } 1 \leq j \leq i-1\}.\end{aligned}$$

Lemma 6.2. *Let $v \in \mathbb{Z}^m$ be an element of the form*

$$v = (\underbrace{x, \dots, x}_{p}, x + \lambda d, \underbrace{x + d, \dots, x + d}_{m-p-1}).$$

for some $x, \lambda, d, p \in \mathbb{Z}$ with $d > 0$ and $1 \leq p \leq m - 2$.

(i) *If λ is even, then for any $\pi \in \mathcal{S}_{1,p+1}$ and $\pi' \in \mathcal{S}_{p+2,m}$, there is an m-braid $\beta \in \mathcal{B}_m$ such that $v \cdot \beta = v$ and $\pi_\beta = \pi\pi'$.*

(ii) *If λ is odd, then for any $\pi \in \mathcal{S}_{1,p}$ and $\pi' \in \mathcal{S}_{p+1,m}$, there is an m-braid $\beta \in \mathcal{B}_m$ such that $v \cdot \beta = v$ and $\pi_\beta = \pi\pi'$.*

For example, we consider the element $v = (0, 0, 2, 1, 1, 1) \in \mathbb{Z}^6$ as in Lemma 6.2 for $x = 0$, $\lambda = 2$, $d = 1$ and $p = 2$. For the permutations $\pi = (2\ 3) \in \mathcal{S}_{1,3}$ and $\pi' = e \in \mathcal{S}_{4,6}$, the 6-braid $\beta = (\sigma_3\sigma_2^{-1})\sigma_3(\sigma_2\sigma_3^{-1}) \in \mathcal{B}_6$ satisfies $v \cdot \beta = v$ and $\pi_\beta = \pi\pi' = (2\ 3)$. See Figure 6.2.

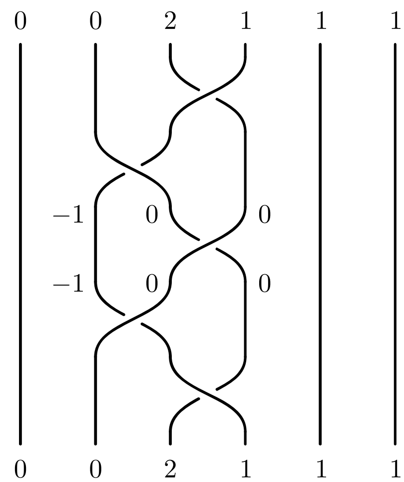

Figure 6.2: The 6-braid $\beta = \sigma_3\sigma_2^{-1}\sigma_3\sigma_2\sigma_3^{-1}$ with $\pi_\beta = (2\ 3)$

6.2 String links

An *m-string link* is an (m, m)-tangle T possibly with some loops such that the permutation $\pi_T \in \mathcal{S}_m$ associated with T is the identity; that is, the ith top point connects to the ith bottom point by a string for any $i = 1, \dots, m$. We denote by $\mathcal{L}_m$ the set of m-string links, and by $\mathcal{L}_{m,0} \subset \mathcal{L}_m$ that of m-string links without loops.

For two elements $v, w \in \mathbb{Z}^m$, we write $v \overset{\mathcal{L}_0}{\sim} w$ if there is an m-string link without loops admitting a $\mathbb{Z}$-coloring such that the top and bottom points receive v and w, respectively. By Lemmas 5.4, 6.2, Propositions 5.6 and 5.7, we have a classification of $\mathbb{Z}^m$ under $\overset{\mathcal{L}_0}{\sim}$ as follows.

Theorem 6.3. *For two elements $v, w \in \mathbb{Z}^m$, the following are equivalent.*

(i) $v \overset{\mathcal{L}_0}{\sim} w$.

(ii) $\Delta(v) = \Delta(w)$, $d_2(v) = d_2(w)$ *and* $\overrightarrow{M}(v)_{2d_2(v)} = \overrightarrow{M}(w)_{2d_2(w)}$.

For two elements $v, w \in \mathbb{Z}^m$, we write $v \overset{\mathcal{L}}{\sim} w$ if there is an m-string link possibly with some loops admitting a $\mathbb{Z}$-coloring such that the top and bottom points receive v and w, respectively. By Lemmas 5.9, 5.10 and Proposition 5.12, we have a classification of $\mathbb{Z}^m$ under $\overset{\mathcal{L}}{\sim}$ as follows.

Theorem 6.4. *For two elements $v, w \in \mathbb{Z}^m$, the following are equivalent.*

(i) $v \overset{\mathcal{L}}{\sim} w$.

(ii) $\Delta(v) = \Delta(w)$ *and* $\overrightarrow{M}(v)_2 = \overrightarrow{M}(w)_2$.

7 Virtual versions

In this section we define six equivalence relations $\overset{v}{\sim}$, $\overset{v\mathcal{T}_0}{\sim}$, $\overset{v\mathcal{T}}{\sim}$, $\overset{v\mathcal{P}}{\sim}$, $\overset{v\mathcal{L}_0}{\sim}$ and $\overset{v\mathcal{L}}{\sim}$ on $\mathbb{Z}^m$ as virtual versions of $\sim$, $\overset{\mathcal{T}_0}{\sim}$, $\overset{\mathcal{T}}{\sim}$, $\overset{\mathcal{P}}{\sim}$, $\overset{\mathcal{L}_0}{\sim}$ and $\overset{\mathcal{L}}{\sim}$, respectively, and characterize them for $m \geq 3$.

Let $\mathcal{VB}_m$ $(m \geq 2)$ be the *virtual m-braid group* with the standard generators

$$\sigma_1, \ldots, \sigma_{m-1} \text{ and } \tau_1, \ldots, \tau_{m-1}$$

such that σ_i corresponds to a classical crossing (see the left of Figure 2.1 again), and τ_i corresponds to a virtual crossing between the ith and $(i+1)$st strings (cf. [2]).

The set $\mathbb{Z}^m$ has the Hurwitz action of $\mathcal{VB}_m$ such that $v \cdot \sigma_i$ is defined as the same as in Section 2 and

$$v \cdot \tau_i = (a_1, \ldots, a_{i-1}, a_{i+1}, a_i, a_{i+2}, \ldots, a_m),$$

where $v = (a_1, \ldots, a_m) \in \mathbb{Z}^m$. See the left of Figure 7.1. We denote by $v \overset{v}{\sim} w$ if there is a virtual m-braid $\beta \in \mathcal{VB}_m$ with $v \cdot \beta = w$. The right figure shows that

$$(1, -5, 4) \cdot (\tau_1 \sigma_2 \sigma_1^{-1} \tau_2) = (-2, 1, 1) \in \mathbb{Z}^3,$$

and hence $(1, -5, 4) \overset{v}{\sim} (-2, 1, 1)$.

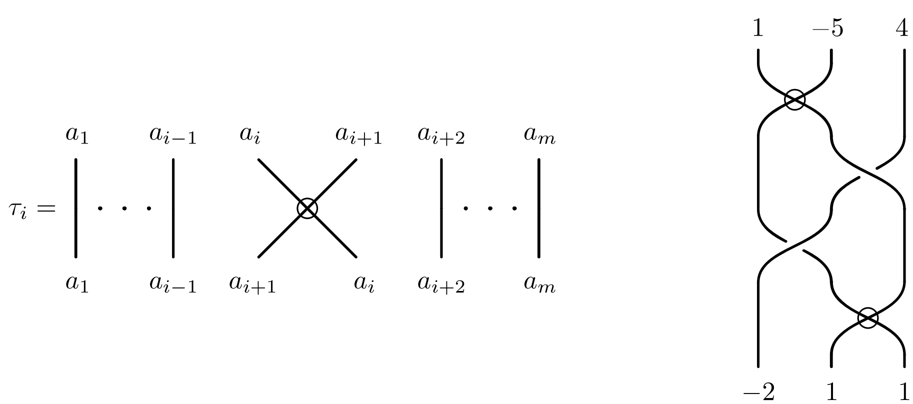

Figure 7.1: The Hurwitz action of $\mathcal{VB}_m$ on $\mathbb{Z}^m$

Theorem 7.1. *For two elements $v, w \in \mathbb{Z}^m$, the following are equivalent.*

(i) $v \overset{v}{\sim} w$.

(ii) $d(v) = d(w)$ *and* $M(v)_{2d(v)} = M(w)_{2d(w)}$.

For two elements $v, w \in \mathbb{Z}^m$, we write $v \overset{v\mathcal{T}_0}{\sim} w$ (or $v \overset{v\mathcal{T}}{\sim} w$) if there is a virtual (m, m)-tangle without loops (or possibly with some loops) admitting a $\mathbb{Z}$-coloring such that the top and bottom points receive v and w, respectively.

Theorem 7.2. *For two elements $v, w \in \mathbb{Z}^m$, the following are equivalent.*

(i) $v \overset{v\mathcal{T}_0}{\sim} w$.

(ii) $d_2(v) = d_2(w)$ *and* $M(v)_{2d_2(v)} = M_2(w)_{2d_2(w)}$.

Theorem 7.3. *For two elements $v, w \in \mathbb{Z}^m$, the following are equivalent.*

(i) $v \overset{v\mathcal{T}}{\sim} w$.

(ii) $M(v)_2 = M(w)_2$.

The *virtual pure m-braid group* is defined by

$$\mathcal{VP}_m = \{\beta \in \mathcal{VB}_m \mid \pi_\beta = e\}.$$

Then $\mathcal{VP}_m$ acts on $\mathbb{Z}^m$ as well as $\mathcal{VB}_m$ does. For two elements $v, w \in \mathbb{Z}^m$, we write $v \overset{v\mathcal{P}}{\sim} w$ if there is a virtual pure m-braid $\beta \in \mathcal{VP}_m$ with $v \cdot \beta = w$.

Theorem 7.4. *For two elements $v, w \in \mathbb{Z}^m$, the following are equivalent.*

(i) $v \overset{v\mathcal{P}}{\sim} w$.

(ii) $d(v) = d(w)$ *and* $\overrightarrow{M}(v)_{2d(v)} = \overrightarrow{M}(w)_{2d(w)}$.

For two elements $v, w \in \mathbb{Z}^m$, we write $v \overset{v\mathcal{L}_0}{\sim} w$ (or $v \overset{v\mathcal{L}}{\sim} w$) if there is a virtual m-string link without loops (or possibly with some loops) admitting a $\mathbb{Z}$-coloring such that the top and bottom points receive v and w, respectively.

Theorem 7.5. *For two elements $v, w \in \mathbb{Z}^m$, the following are equivalent.*

(i) $v \overset{v\mathcal{L}_0}{\sim} w$.

(ii) $d_2(v) = d_2(w)$ *and* $\overrightarrow{M}(v)_{2d_2(v)} = \overrightarrow{M}(w)_{2d_2(w)}$.

Theorem 7.6. *For two elements $v, w \in \mathbb{Z}^m$, the following are equivalent.*

(i) $v \overset{v\mathcal{L}}{\sim} w$.

(ii) $\overrightarrow{M}(v)_2 = \overrightarrow{M}(w)_2$.

Proposition 7.7. *For an element $v \in \mathbb{Z}^{2k}$, the following are equivalent.*

(i) *There is a virtual k-string tangle without loops T admitting a $\mathbb{Z}$-coloring such that the $2k$ endpoints of T receive v.*

(ii) $\Delta(v) \equiv 0 \pmod{2d_2(v)}$.

Proposition 7.8. *For an element $v \in \mathbb{Z}^{2k}$, the following are equivalent.*

(i) *There is a virtual k-string tangle possibly with some loops T admitting a $\mathbb{Z}$-coloring such that the $2k$ endpoints of T receive v.*

(ii) $\Delta(v) \equiv 0 \pmod{2}$.

8 The case $m = 2$

In the previous sections we provide the characterization results of the twelve equivalence relations on $\mathbb{Z}^m$ for $m \geq 3$. This section extends them to the case $m = 2$.

Theorem 8.1. *For two elements $v, w \in \mathbb{Z}^2$, we have the following.*

(i) $v \sim w$ *if and only if* $\Delta(v) = \Delta(w)$ *and* $M(v)_{2d(v)} = M(w)_{2d(w)}$.

(ii) $v \overset{\mathcal{T}_0}{\sim} w$ *if and only if* $\Delta(v) = \Delta(w)$ *and* $M(v)_{2d_2(v)} = M(w)_{2d_2(w)}$.

(iii) $v \overset{\mathcal{T}}{\sim} w$ *if and only if* $\Delta(v) = \Delta(w)$ *and* $M(v)_2 = M(w)_2$.

Theorem 8.2. *For two elements $v, w \in \mathbb{Z}^2$, we have the following.*

(i) $v \overset{\mathcal{P}}{\sim} w$ *if and only if* $\Delta(v) = \Delta(w)$ *and* $\overrightarrow{M}(v)_{2d(v)} = \overrightarrow{M}(w)_{2d(w)}$.

(ii) $v \overset{\mathcal{L}_0}{\sim} w$ *if and only if* $\Delta(v) = \Delta(w)$ *and* $\overrightarrow{M}(v)_{2d_2(v)} = \overrightarrow{M}(w)_{2d_2(w)}$.

(iii) $v \overset{\mathcal{L}}{\sim} w$ *if and only if* $\Delta(v) = \Delta(w)$ *and* $\overrightarrow{M}(v)_2 = \overrightarrow{M}(w)_2$.

Theorem 8.3. *For two elements $v, w \in \mathbb{Z}^2$, we have the following.*

(i) $v \overset{v}{\sim} w$ *if and only if* $d(v) = d(w)$ *and* $M(v)_{2d(v)} = M(w)_{2d(w)}$.

(ii) $v \overset{v\mathcal{T}_0}{\sim} w$ *if and only if* $d_2(v) = d_2(w)$ *and* $M(v)_{2d_2(v)} = M(w)_{2d_2(w)}$.

(iii) $v \overset{v\mathcal{T}}{\sim} w$ *if and only if* $M(v)_2 = M(w)_2$.

Theorem 8.4. *For two elements $v, w \in \mathbb{Z}^2$, we have the following.*

(i) $v \overset{v\mathcal{P}}{\sim} w$ *if and only if* $d(v) = d(w)$ *and* $\overrightarrow{M}(v)_{2d(v)} = \overrightarrow{M}(w)_{2d(w)}$.

(ii) $v \overset{v\mathcal{L}_0}{\sim} w$ *if and only if* $d_2(v) = d_2(w)$ *and* $\overrightarrow{M}(v)_{2d_2(v)} = \overrightarrow{M}(w)_{2d_2(w)}$.

(iii) $v \overset{v\mathcal{L}}{\sim} w$ *if and only if* $\overrightarrow{M}(v)_2 = \overrightarrow{M}(w)_2$.

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