Classical invariants and rack coloring invariants of Legendrian knots

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1 Introduction

Contact geometry is an odd dimensional counterpart of symplectic geometry. Legendrian submanifolds play an important role in contact geometry in a way similar to that Lagrangian submanifolds do in symplectic geometry. Legendrian knots are objects which are located at the intersection of contact geometry and 3-dimensional topology.

We consider the classification of Legendrian knots up to Legendrian isotopy, which is more strict than ambient isotopy. Actually each knot type has infinitely many Legendrian isotopy classes. It is known that there exist local moves for diagrams which characterize the relation between diagrams of mutually Legendrian isotopic two Legendrian knots [17]. The local moves are called the Legendrian Reidemeister moves. We can study Legendrian knots by means of diagrams in combinatorial ways due to the existence of the Legendrian Reidemeister moves.

In this article, we focus on invariants of Legendrian knots which are constructed from diagrams. Among such invariants, the Thurston-Bennequin number and the rotation number are the most fundamental ones. They are called the classical invariants of Legendrian knots. The classical invariants are known to be powerful invariants, while it is easy to compute them from diagrams.

Racks and quandles are algebraic structures without associativity. Quandles were introduced by Joyce [11] and Matveev [16] independently. Racks, which are generalizations of quandles, were introduced by Fenn and Rourke [8]. Since the axioms of quandles correspond to the Reidemeister moves, quandle colorings of knot diagrams bring knot invariants such as coloring numbers and fundamental quandles. Those knot invariants are known to be useful to distinguish knots in many cases. Moreover, quandles produce invariants of several generalizations of knots such as surface-knots and handlebody-knots.

We present invariants of Legendrian knots using racks in this article. There are several preceding works on invariants of Legendrian knots using racks. These invariants are analogues of quandle coloring numbers of diagrams of topological knots. A quandle coloring is a map from the set of the arcs of a knot diagram to a quandle satisfying a certain relation at each crossing of the knot diagram. Front diagrams of Legendrian knots have not only crossings but also cusps. Hence besides relations at crossings, relations at cusps are necessary to be set in colorings of front diagrams of Legendrian knots. Kulkarni and

Prathamesh [15] introduced invariants of Legendrian knots by using racks. They set relations of cusps by using the rack operation. Ceniceros, Elhamdadi and Nelson [2] defined a Legendrian rack whose axioms correspond to the Legendrian Reidemeister moves. They set relations of cusps by introducing a map on the rack with several conditions, which assure invariance under the Legendrian Reidemeister moves. The invariants of Legendrian knots using Legendrian rack colorings in [2] contain the invariants in [15] as special cases.

In this article, we introduce a bi-Legendrian rack, which is a generalization of a Legendrian rack in [2]. The axioms of bi-Legendrian racks also correspond to the Legendrian Reidemeister moves. Therefore a bi-Legendrian rack coloring number of a diagram is an invariant of Legendrian knots. The difference between bi-Legendrian rack colorings and Legendrian rack colorings is to distinguish downward cusps and upward cusps. Due to this small difference bi-Legendrian racks give rise to properly stronger invariants. In [15] and [2], each example of pairs of Legendrian knots they distinguish by their invariants has different Thurston-Benneqiun numbers. In this article, we state that bi-Legendrian rack coloring numbers can distinguish simultaneously all Legendrian unknots with the same Thurston-Bennequin number. Notice that Legendrian rack coloring numbers in [2] cannot distinguish the family of Legendrian unknots. We also consider pairs of Legendrian knots which cannot be distinguished by bi-Legendrian rack coloring numbers. We give a sufficient condition for pairs of Legendrian knots not to be distinguished by bi-Legendrian quandle coloring numbers. Moreover, we present a pair of Legendrian knots which cannot be distinguished by bi-Legendrian rack coloring numbers.

In [12], Karmakar, Saraf and Singh introduced the fundamental bi-Legendrian rack of a Legendrian knot, which they call the generalized Legendrian rack of a Legendrian knot. The fundamental bi-Legendrian rack of a Legendrian knot is an analogy of the fundamental quandle of a topological knot. A presentation of the fundamental bi-Legendrian rack of a Legendrian knot is obtained from the front diagram. Arcs of the front diagram correspond to the generators, while crossings and cusps give the relations. A bi-Legendrian rack coloring is regarded as a homomorphism from the fundamental bi-Legendrian rack to the given rack. Hence the fundamental bi-Legendrian rack of a Legendrian knot is the universal invariant for bi-Legendrian rack coloring numbers in the same way as the fundamental quandle of a topological knot is universal for quandle coloring numbers.

2 Legendrian knots

In this section, we review basics on Legendrian knots briefly. For a more comprehensive introduction to contact topology and Legendrian knots the reader is referred to [10] [6]. Legendrian knots are defined when an ambient 3-manifold is equipped with a contact structure. Hence we begin with the definition of a contact structure.

Definition 1. Let M be a 3-manifold. A *contact structure* ξ on M is a plane field on M satisfying, when $\xi = \ker \alpha$ for a local 1-form α on M, $\alpha \wedge d\alpha$ is nowhere vanishing. The pair (M, ξ) is called a *contact* 3*-manifold*.

Roughly speaking, a contact structure is a "twisted" plane field on M.

Example 1. Let (x, y, z) be the standard coordinate on $\mathbb{R}^3$. Let

$$\alpha_{std} = dz + xdy$$

and

$$\xi_{std} = \ker \alpha_{std} = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y} - x\frac{\partial}{\partial z} \rangle_{\mathbb{R}}.$$

Then ξ_{std} is a contact structure on $\mathbb{R}^3$. ξ_{std} is called the *standard contact structure* on $\mathbb{R}^3$.

Definition 2. Let (M, ξ) be a contact 3-manifold. A smooth knot K in (M, ξ) is called *Legendrian* if $T_pK \subset \xi_p$ for any $p \in K$.

Definition 3. Let K_0 and K_1 be Legendrian knots in a contact 3-manifold (M, ξ). K_0 is said to be *Legendrian isotopic* to K_1 if there exists an isotopy through Legendrian knots in (M, ξ) from K_0 to K_1.

Remark 1. By the isotopy extension theorem, this definition is equivalent to the following apparently stronger condition:
K_0 is Legendrian isotopic to K_1 if and only if there exists an isotopy φ_t of M $(t \in [0, 1])$ which fixes ξ such that φ_0 is id_M, $\varphi_1(K_0)$ is K_1 and $\varphi_t(K_0)$ is Legendrian for any $t \in [0, 1]$.

On account of Remark 1, Legendrian isotopy is more strict equivalence relation than smooth ambient isotopy. We would like to consider the classification of Legendrian knots in (M, ξ) up to Legendrian isotopy.

From now on, we only consider Legendrian knots in $(\mathbb{R}^3, \xi_{std})$. We often use diagrams in order to study Legendrian knots in the same way as the case of topological knots. The projection

$$\mathbb{R}^3 \ni (x, y, z) \mapsto (y, z) \in \mathbb{R}^2$$

is called the front projection. Front projection diagrams of Legendrian knots have the following features. Let $\gamma(t) = (x(t), y(t), z(t))$ $(t \in [0, 1])$ be a parametrization of a Legendrian knot K. Since the tangent vector $\gamma'(t) = (x'(t), y'(t), z'(t))$ of K satisfies $\alpha_{std}(\gamma'(t)) = 0$,

$$z'(t) + x(t)y'(t) = 0 \tag{1}$$

holds. As $y'(t) = 0$ implies $z'(t) = 0$ due to (1), a point $(y(t), z(t))$ on $\mathbb{R}^2$ satisfying $y'(t) = 0$ is a singular point, which is called a cusp. See the right hand side of Figure 1. Because of (1), the x-coordinate of a point in K is recovered from the front projection as $x(t) = -\frac{dz}{dy}(t)$. Hence at each crossing of the front projection diagram, the slope of the overcrossing is smaller than that of the undercrossing. Therefore the type of crossings shown in the left hand side of Figure 1 only appears in front projection diagrams of Legendrian knots.

The following theorem implies that each knot type has a Legendrian representative.

Theorem 2.1. Let K be a knot in a contact 3-manifold (M, ξ). Then K can be C^0-approximated by a Legendrian knot ambient isotopic to K.

Here we just mention the outline of the proof. For a curve γ in $(\mathbb{R}^3, \xi_{std})$, a C^0-close Legendrian approximation of γ is obtained by adding small zig-zags to the front projection of γ. See Figure 2. For a knot K in arbitrary contact 3-manifold, we cut K into sufficiently small pieces and then we carry out the procedure for such small pieces. This gives a C^0-close Legendrian approximation of K because of Darboux's theorem.

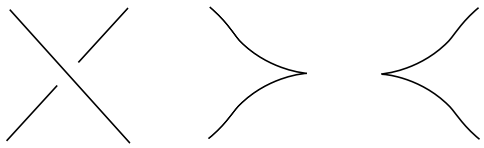

Figure 1: Crossings and cusps appeared in front diagrams of Legendrian knots.

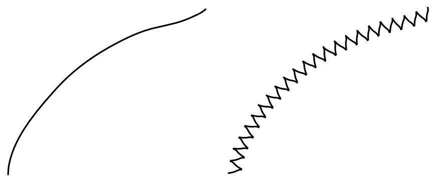

Figure 2: C^0-close Legendrian approximation of a curve.

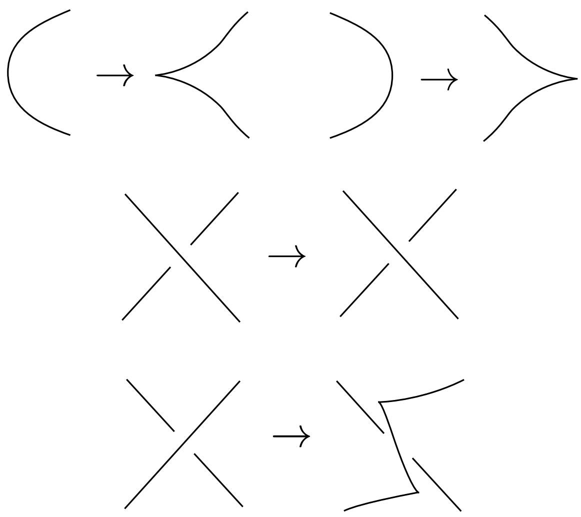

Figure 3: How to obtain a front diagram of a Legendrian knot from a given knot diagram.

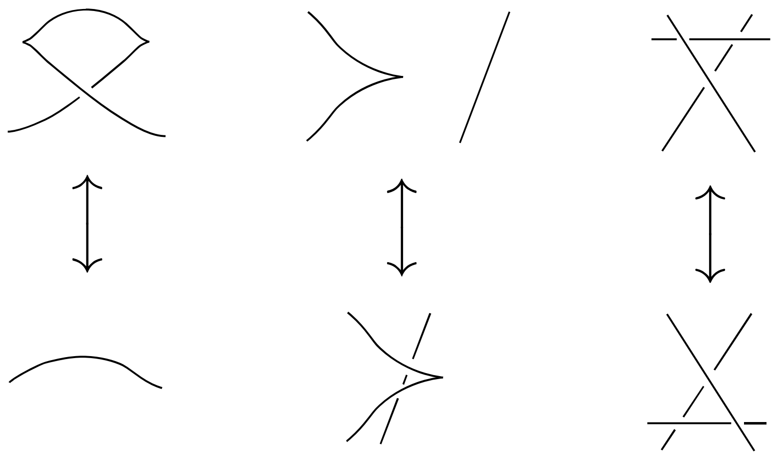

Figure 4: Legendrian Reidemeister moves.

Remark 2. One can also get a Legendrian representative in $(\mathbb{R}^3, \xi_{std})$ of a given knot type in $\mathbb{R}^3$ in the following way. Take any diagram of the given knot. Convert each vertical tangency in the diagram to a cusp and add two cusps near each crossing which is not the type in Figure 1. See Figure 3. Note that the obtained Legendrian representative is not C^0-close to the given knot.

The following theorem is Legendrian analogy of Reidemeister's theorem.

Theorem 2.2 ([17])**.** Let K_0 and K_1 be Legendrian knots in $(\mathbb{R}^3, \xi_{std})$ and D_i the front projection diagram of K_i $(i = 0, 1)$. Then K_0 and K_1 are Legendrian isotopic if and only if D_0 and D_1 are related by a finite sequence of the three types of local moves shown in Figure 4.

The local moves shown in Figure 4 are called the Legendrian Reidemeister moves.

3 Classical invariants of Legendrian knots

In this section, we explain the classical invariants of Legendrian knots.
The most fundamental invariant of Legendrian knots is, of course, the (topological) knot type. The next fundamental invariants of Legendrian knots are the Thurston-Bennequin number and the rotation number. They are called the classical invariants of Legendrian knots. Both of the classical invariants are integer-valued invariants. Take a Seifert surface F for a Legendrian knot K. The Thurston-Bennequin number $tb(K) \in \mathbb{Z}$ of K is the twisting number of the contact plane ξ relative to F along K. If we take a vector field ν along K transverse to ξ and define a parallel knot K' by pushing K along ν, then $tb(K)$ is equal to the linking number $lk(K, K')$ of K and K'.

While the Thurston-Bennequin number is an invariant of unoriented Legendrian knots, the rotation number is an invariant of oriented Legendrian knots. Since F is a surface with boundary, the plane bundle $\xi|_F$ over F is the trivial bundle. We take any trivialization of

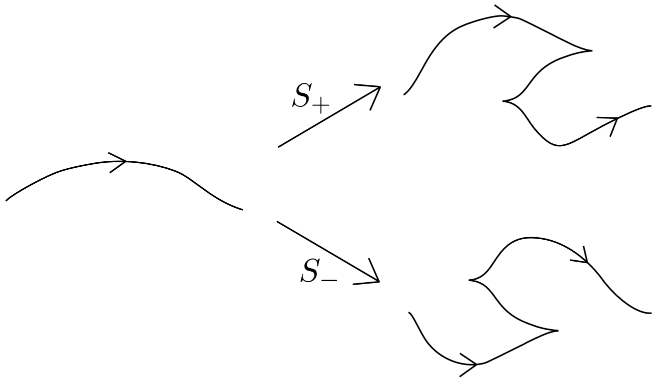

Figure 5: Stabilizations.

$\xi|_F$. By this trivialization, we can identify $\xi|_F$ with $F \times \mathbb{R}^2$. Then the rotation number $rot(K) \in \mathbb{Z}$ of an oriented Legendrian knot K is the rotation number of the tangent vector of K on the contact plane ξ along K. Note that the value of $rot(K)$ is independent of a choice of a trivialization of $\xi|_F$. Denote the Legendrian knot with the reverse orientation to K by $-K$, $rot(-K) = -rot(K)$ holds.

The classical invariants of a Legendrian knot K are computed from the front projection diagram of K as follows:

$$tb(K) = w(D) - \frac{1}{2}c(D),$$
$$rot(K) = \frac{1}{2}(dc(D) - uc(D)),$$

where $w(D)$ is the writhe of D, $c(D)$ is the number of the cusps of D, $dc(D)$ is the number of the downward cusps of D and $uc(D)$ is the number of the upward cusps of D.

A stabilization of a Legendrian knot K is an operation which changes the Legendrian isotopy class and does not change the knot type. A positive (or negative) stabilization is represented by adding two downward (or upward) cusps to a trivial arc for the front projection diagram. See Figure 5. We denote a positive (or negative) stabilization by S_+ (or S_-). A positive sabilization and a negative stabilization are commutative, i.e., $S_+S_- = S_-S_+$. A stabilization changes the classical invariants in the following way.

$$tb(S_\pm(K)) = tb(K) - 1,$$
$$rot(S_\pm(K)) = rot(K) \pm 1.$$

For any knot type $\mathcal{K}$, one can get a Legendrian knot with arbitrary small Thurston-Bennequin number whose knot type is $\mathcal{K}$ by performing many stabilizations. This implies that any knot type has infinitely many Legendrian isotopy classes.

The following theorem is shown by Fuchs and Tabachnikov [9].

Theorem 3.1 ([9])**.** Let K_0 and K_1 be Legendrian knots in $(\mathbb{R}^3,\ \xi_{std})$. Then K_0 and K_1 have the same knot type if and only if $S_+^{p_0}S_-^{n_0}(K_0)$ and $S_+^{p_1}S_-^{n_1}(K_1)$ are Legendrian isotopic for some non-negative integers p_0, n_0, p_1 and n_1.

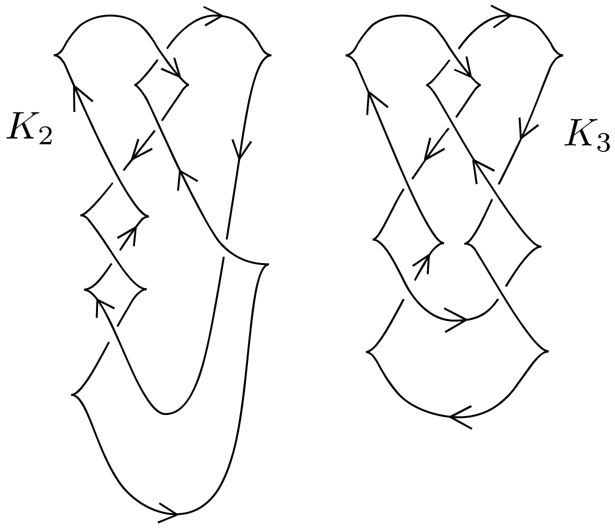

Figure 6: the Chekanov knots.

This theorem states that two Legendrian knots in $(\mathbb{R}^3, \xi_{std})$ with the same knot type are stably Legendrian isotopic.

The pair of the classical invariants are known to be rather strong. In order to explain this, we introduce the following notion.

Definition 4. A knot type $\mathcal{K}$ is called *Legendrian simple* if for any two Legendrian knots K_0 and K_1 of the knot type $\mathcal{K}$, if $tb(K_0) = tb(K_1)$ and $rot(K_0) = rot(K_1)$, then K_0 and K_1 are Legendrian isotopic.

In other words, Legendrian simplicity of $\mathcal{K}$ means that the pair of the classical invariants completely classifies the Legendrian isotopy classes of $\mathcal{K}$.

Several knot types are known to be Legendrian simple. The following theorems give examples of Legendrian simple knot types.

Theorem 3.2 ([5])**.** The unknot is Legendrian simple.

Theorem 3.3 ([7])**.** Each torus knot is Legendrian simple.

Theorem 3.4 ([7])**.** The figure eight knot is Legendrian simple.

On the other hand, there are many knot types which are not Legendrian simple. See [4]. The first example of knot types which are shown to be not Legendrian simple is the knot $m(5_2)$, where $m(\mathcal{K})$ means the mirror image of the knot type $\mathcal{K}$. K_2 and K_3 in Figure 6 have the same knot type $m(5_2)$, $tb(K_2) = tb(K_3) = 1$ and $rot(K_2) = rot(K_3) = 0$. However, Chekanov [3] proved K_2 and K_3 are not Legendrian isotopic by using Legendrian contact homology. K_2 and K_3 are called the Chekanov knots.

Theorem 3.5 (Bennequin's inequality [1])**.** Let K be a Legendrian knot $(\mathbb{R}^3, \xi_{std})$ and F a Seifert surface for K. Then the following inequality holds:

$$tb(K) + |rot(K)| \leq -\chi(F).$$

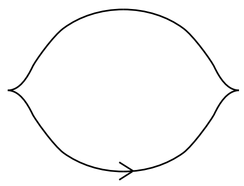

Figure 7: A Legendrian unknot with $tb = -1$.

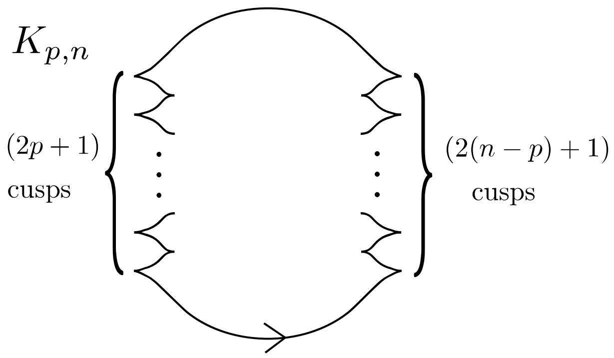

Figure 8: A Legendrian unknot $K_{p,n}$ with $tb(K_{p,n}) = -1-n$, $rot(K_{p,n}) = 2p - n$.

The Bennequin's inequality implies that the Thurston-Bennequin number of a Legendrian knot in $(\mathbb{R}^3, \xi_{std})$ has an upper bound. For a knot type $\mathcal{K}$, the Thurston-Bennequin number of any Legendrian knot with the knot type $\mathcal{K}$ is bounded above by $2g(\mathcal{K}) - 1$, where $g(\mathcal{K})$ is the genus of $\mathcal{K}$.

At the end of this section, we give the classification result of Legendrian unknots in $(\mathbb{R}^3, \xi_{std})$ explicitly. From the Bennequin's inequality, a Legendrian unknot K satisfies

$$tb(K) + |rot(K)| \leq -1.$$

Due to this inequality, $tb(K) = -1 - n$ for some non-negative integer n and $-n \leq rot(K) \leq n$. In addition, from the fact ([5] Proposition 1.7) that $tb(K) + rot(K)$ is always odd, if $tb(K) = -1 - n$, then $rot(K) = 2p - n$ for some integer p with $0 \leq p \leq n$.

For any non-negative integers n and p with $p \leq n$, a Legendrian unknot K with $tb(K) = -1 - n$ and $rot(K) = 2p - n$ can be realized by p times positive stabilizations and $(n-p)$ times negative stabilizations from the Legendrian unknot with $tb = -1$ shown in Figure 7. We denote the Legendrian unknot obtained by such stabilizations by $K_{p,n}$. See Figure 8.

Since the unknot is Legendrian simple ([5] Theorem 1.5), each Legendrian unknot is Legendrian isotopic to $K_{p,n}$ for some non-negative integers n and p with $p \leq n$.

4 Rack coloring invariants of Legendrian knots

In this section, we introduce rack coloring invariants of Legendrian knots. We define a bi-Legendrian rack and an invariant of Legendrian knots derived from a bi-Legendrian rack. Furthermore, we present distinguishing results of Legendrian knots by using bi-Legendrian rack coloring numbers. We also mention the fundamental bi-Legendrian rack of a Legendrian knot, which is introduced by Karmakar, Saraf and Singh in [12].
Before that, we recall the definitions of racks and quandles.

Definition 5 ([11][16][8])**.** $(X, *)$ is called a *rack* if X is a set with a binary operation $*$ satisfying the following conditions for all $x, y, z \in X$:

$$* \, x : X \to X \text{ is a bijection,}$$
$$(x * y) * z = (x * z) * (y * z).$$

A rack which satisfies $x * x = x$ for all $x \in X$ is called a *quandle*.

We denote the inverse map of $* \, x$ by $\bar{*} \, x$.

Since the axioms of quandles correspond to the Reidemeister moves, quandle colorings of knot diagrams bring knot invariants, such as quandle coloring numbers and the fundamental quandle. We apply the same idea to Legendrian knots. Namely, we consider the axioms correspond to the Legendrian Reidemeister moves in order to obtain invariants of Legendrian knots.
We introduce a bi-Legendrian rack as a small modification of a Legendrian rack defined in [2].

Definition 6 ([13])**.** $(X, *, f, g)$ is called a *bi-Legendrian rack* if $(X, *)$ is a rack and f and g are maps on X satisfying the following conditions for all $x, y \in X$:

$$\begin{aligned} f \circ g &= g \circ f, \\ fg(x * x) &= x, \\ f(x * y) &= f(x) * y, \\ g(x * y) &= g(x) * y, \\ x * f(y) &= x * y, \\ x * g(y) &= x * y. \end{aligned}$$

Remark 3. If $(X, *, f)$ is a Legendrian rack in the sense of Definition 4 in [2], then $(X, *, f, f)$ is a bi-Legendrian rack. Hence a bi-Legendrian rack is a generalization of a Legendrian rack.

Remark 4. Let $(X, *, f, g)$ be a bi-Legendrian rack. Then $(X, *)$ is a quandle if and only if g is the inverse map of f.

Proposition 4.1 ([13])**.** Let $(X, *, f, g)$ be a bi-Legendrian rack. Then f and g are automorphisms of $(X, *)$.

Definition 7 ([12])**.** Let $(X_1, *_1, f_1, g_1)$ and $(X_2, *_2, f_2, g_2)$ be bi-Legendrian racks. A bi-Legendrian rack homomorphism $\varphi : (X_1, *_1, f_1, g_1) \to (X_2, *_2, f_2, g_2)$ is a map φ :

$X_1 \to X_2$ satisfying the following conditions:

$$\varphi(x *_1 y) = \varphi(x) *_2 \varphi(y),$$
$$\varphi \circ f_1 = f_2 \circ \varphi,$$
$$\varphi \circ g_1 = g_2 \circ \varphi.$$

Example 2. Let $(G, *)$ be a conjugation quandle, i.e. G is a group and $x * y = y^{-1}xy$ for $x, y \in G$. Take an elemmaent z contained in the center of G, and define a map f on G by $f(x) = zx$. Then $(G, *, f, f^{-1})$ is a bi-Legendrian quandle.

Example 3. Let $(X, *)$ be a constant rack, i.e., $x * y = \sigma(x)$ for some bijection σ on X. For maps f and g on X, we can easily check that $(X, *, f, g)$ is a bi-Legendrian rack if and only if

$$f \circ g = g \circ f = \sigma^{-1}$$

holds. Hence any constant bi-Legendrian rack is obtained as follows.
Let X be a set. Take bijections f and g on X such that they are commutative. Define a binary operation $*$ on X by

$$x * y := (f \circ g)^{-1}(x).$$

Then $(X, *, f, g)$ is a constant bi-Legendrian rack.

Example 4. $(\mathbb{Z}/8\mathbb{Z}, *, f, g)$ is a bi-Legendrian rack with $f \neq g$ if we define an operation $*$ and maps f and g on $\mathbb{Z}/8\mathbb{Z}$ by

$$x * y = 3x + 2y,$$
$$f(x) = x + 4,$$
$$g(x) = 5x + 4$$

for $x, y \in \mathbb{Z}/8\mathbb{Z}$.

In this article, an arc of a front diagram of a Legendrian knot is defined as a part of the diagram each of whose end is either an undercrossing or a cusp and which contains no undercrossings and no cusps in its interior.

Definition 8. Let D be the front diagram of a Legendrian knot and $(X, *, f, g)$ a bi-Legendrian rack. An $(X, *, f, g)$-*coloring* of D is a map from the set of the arcs of D to X such that at each crossing and each cusp the relations between colors of arcs shown in Figure 9 hold. The set of the $(X, *, f, g)$-colorings of D is denoted by $\mathrm{Col}(D, X)$.

Proposition 4.2 ([13])**.** Let K be a Legendrian knot in $(\mathbb{R}^3, \xi_{std})$ and D the front diagram of K. Let $(X, *, f, g)$ be a bi-Legendrian rack. Then the number of the $(X, *, f, g)$-colorings of D is invariant under the Legendrian Reidemeister moves. Namely, $\#\mathrm{Col}(D, X)$ is an invariant of a Legendrian knot K, denoted by $\#\mathrm{Col}(K, X)$.

We present distinguishing results of Legendrian knots by using bi-Legendrian rack coloring numbers. We state that bi-Legendrian rack coloring numbers can distinguish all Legendrian unknots with the same Thurston-Bennequin number. We also consider

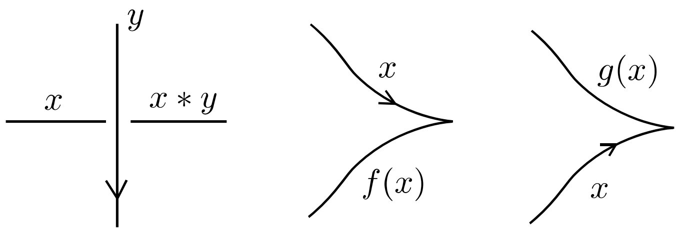

Figure 9: Relations at crossings and cusps (1).

pairs of Legendrian knots which cannot be distinguished by bi-Legendrian rack coloring numbers.

Recall that each Legendrian unknot is Legendrian isotopic to $K_{p,n}$ for some non-negative integers n and p with $p \leq n$. See Figure 8. Notice that $tb(K_{p,n}) = -1 - n$, $rot(K_{p,n}) = 2p - n$.

Theorem 4.3 ([13])**.** For any non-negative integer n, there exists a bi-Legendrian rack $(X_n, *, f, g)$ such that $n+1$ Legendrian unknots $K_{p,n}$ $(0 \leq p \leq n)$ are simultaneously distinguished by $\#\mathrm{Col}(K_{p,n}, X_n)$.

Remark 5. $K_{p,n}$ with a fixed n and different p cannot be distinguished by Legendrian rack coloring numbers used in [2].

Theorem 4.3 implies that bi-Legendrian rack coloring numbers can distinguish some Legendrian knots with the same Thurston-Bennequin number. A natural question to consider next is whether or not bi-Legendrian rack coloring numbers can distinguish pairs of Legendrian knots with the same Thurston-Bennequin number and the same rotation number. Although we do not have a complete answer, the following two theorems give a partial answer of this question.

Theorem 4.4 ([13])**.** If two Legendrian knots K_0 and K_1 are of the same knot type and satisfy $tb(K_0) = tb(K_1)$ and $rot(K_0) = rot(K_1)$, then $\#\mathrm{Col}(K_0, X) = \#\mathrm{Col}(K_1, X)$ for any bi-Legendrian quandle $(X, *, f, g)$.

Recall that K_2 and K_3 in Figure 6 are of the same knot type $m(5_2)$, $tb(K_2) = tb(K_3) = 1$ and $rot(K_2) = rot(K_3) = 0$.

Theorem 4.5 ([13])**.** Let K_2 and K_3 be the Chekanov knots shown in Figure 6. Then $\#\mathrm{Col}(K_2, X) = \#\mathrm{Col}(K_3, X)$ for any bi-Legendrian rack $(X, *, f, g)$.

We explain the fundamental bi-Legendrian rack of a Legendrian knot, which is introduced by Karmakar, Saraf and Singh in [12]. The fundamental bi-Legendrian rack of a Legendrian knot is a Legendrian analogue of the fundamental quandle of a topological knot.

Let S be a set. We define the set of words $W(S)$ generated by S satisfying the following conditions:

(i) $x \in W(S)$ for any $x \in S$,

(ii) $x * y$, $x \bar{*} y$, $f(x)$ and $g(x) \in W(S)$ for any $x, y \in W(S)$.

We define the *free bi-Legendrian rack* generated by S, denoted by $FbLR(S)$, as the set of equivalence classes of elements of $W(S)$ modulo the equivalence relation generated by the following relations:

$$\begin{aligned}
(x * y) \bar{*} y &\sim x \sim (x \bar{*} y) * y, \\
(x * y) * z &\sim (x * z) * (y * z), \\
f(g(x)) &\sim g(f(x)), \\
f(g(x * x)) &\sim x, \\
f(x * y) &\sim f(x) * y, \\
g(x * y) &\sim g(x) * y, \\
x * f(y) &\sim x * y, \\
x * g(y) &\sim x * y,
\end{aligned}$$

for any $x, y, z \in W(S)$.

Free bi-Legendrian racks satisfy the universal property as follows.

Proposition 4.6 (Proposotion 4.2 [12])**.** Let S be a set. Then, for any bi-Legendrian rack $(X, *_1, f_1, g_1)$ and any map $\varphi : S \to X$, there exists a unique bi-Legendrian rack homomorphism $\tilde{\varphi} : (FbLR(S), *, f, g) \to (X, *_1, f_1, g_1)$ satisfying $\tilde{\varphi} \circ \iota = \varphi$, where ι is the inclusion from S to $FbLR(S)$.

We define the *fundamental bi-Legendrian rack* of a Legendrian knot K from the front diagram D of K as follows. Let S be the set of the arcs of D. Define the fundamental bi-Legendrian rack $bLR(D)$ associated to D as the quotient of the free bi-Legendrian rack $FbLR(S)$ by the equivalence relation generated by the relations at crossings and cusps shown in Figure 9.

Proposition 4.7 (Theorem 4.3 [12])**.** Let K be a Legendrian knot in $(\mathbb{R}^3, \xi_{std})$ and D the front diagram of K. Then the fundamental bi-Legendrian rack $bLR(D)$ associated to D is invariant under the Legendrian Reidemeister moves. Namely, the fundamental bi-Legendrian rack is an invariant of a Legendrian knot K, denoted by $bLR(K)$.

The fundamental bi-Legendrian rack of a Legendrian knot is the universal invariant for bi-Legendrian rack coloring numbers in the following sense.

Remark 6. Let D be the front diagram of a Legendrian knot K and $(X_1, *_1, f_1, g_1)$ a bi-Legendrian rack. An $(X_1, *_1, f_1, g_1)$-coloring of D defined in Definition 8 is a homomorphism from the fundamental bi-Legendrian rack $bLR(K)$ of K to $(X_1, *_1, f_1, g_1)$. Hence $\#\mathrm{Col}(K, X_1)$ is the cardinality of the set of the homomorphisms from $bLR(K)$ to $(X_1, *_1, f_1, g_1)$.

The fundamental quandle of a topological knot can be recovered from the fundamental bi-Legendrian rack of a Legendrian knot.

Remark 7. For a Legendrian knot K with the knot type $\mathcal{K}$, we obtain the fundamental quandle of $\mathcal{K}$ from the fundamental bi-Legendrian rack $bLR(K)$ of K by adding the relation $f(x) = g(x) = x$ for any generator x of $bLR(K)$.

At the end of this section, we define a 4-Legendrian rack, which is a generalization of a bi-Legendrian rack. We state that a 4-Legendrian rack coloring number and the fundamental 4-Legendrian rack are invariants of Legendrian knots.

Definition 9 ([14])**.** $(X, *, f_L, f_R, g_L, g_R)$ is called a 4-*Legendrian rack* if $(X, *)$ is a rack and f_L, f_R, g_L and g_R are maps on X satisfying the following conditions for all $x, y \in X$:

$$\begin{aligned} f_L \circ g_R = g_R \circ f_L = f_R \circ g_L = g_L \circ f_R, \\ f_L g_R(x * x) = x, \\ f_L(x * y) = f_L(x) * y, \\ f_R(x * y) = f_R(x) * y, \\ g_L(x * y) = g_L(x) * y, \\ g_R(x * y) = g_R(x) * y, \\ x * f_L(y) = x * f_R(y) = x * y, \\ x * g_L(y) = x * g_R(y) = x * y. \end{aligned}$$

Remark 8. If $(X, *, f, g)$ is a bi-Legendrian rack, then $(X, *, f, f, g, g)$ is a 4-Legendrian rack. Hence a 4-Legendrian rack is a generalization of a bi-Legendrian rack.

Remark 9. Let $(X, *, f_L, f_R, g_L, g_R)$ be a 4-Legendrian rack. Then $(X, *)$ is a quandle if and only if g_L is the inverse map of f_R and g_R is the inverse map of f_L.

Proposition 4.8 ([14])**.** Let $(X, *, f_L, f_R, g_L, g_R)$ be a 4-Legendrian rack. Then f_L, f_R, g_L and g_R are automorphisms of $(X, *)$.

Definition 10. Let $(X_1, *_1, f_{1,L}, f_{1,R}, g_{1,L}, g_{1,R})$ and $(X_2, *_2, f_{2,L}, f_{2,R}, g_{2,L}, g_{2,R})$ be 4-Legendrian racks. A 4-Legendrian rack homomorphism

$$\varphi : (X_1, *_1, f_{1,L}, f_{1,R}, g_{1,L}, g_{1,R}) \to (X_2, *_2, f_{2,L}, f_{2,R}, g_{2,L}, g_{2,R})$$

is a map $\varphi : X_1 \to X_2$ satisfying the following conditions:

$$\begin{aligned} \varphi(x *_1 y) = \varphi(x) *_2 \varphi(y), \\ \varphi \circ f_{1,L} = f_{2,L} \circ \varphi, \\ \varphi \circ f_{1,R} = f_{2,R} \circ \varphi, \\ \varphi \circ g_{1,L} = g_{2,L} \circ \varphi, \\ \varphi \circ g_{1,R} = g_{2,R} \circ \varphi. \end{aligned}$$

Definition 11. Let D be the front diagram of a Legendrian knot and $(X, *, f_L, f_R, g_L, g_R)$ be a 4-Legendrian rack. An $(X, *, f_L, f_R, g_L, g_R)$-*coloring* of D is a map from the set of the arcs of D to X such that at each crossing and each cusp the relations between colors of arcs shown in Figure 10 hold. The set of the $(X, *, f_L, f_R, g_L, g_R)$-colorings of D is denoted by $\mathrm{Col}(D, X)$.

The difference between bi-Legendrian rack colorings and 4-Legendrian rack colorings is to distinguish left cusps and right cusps.

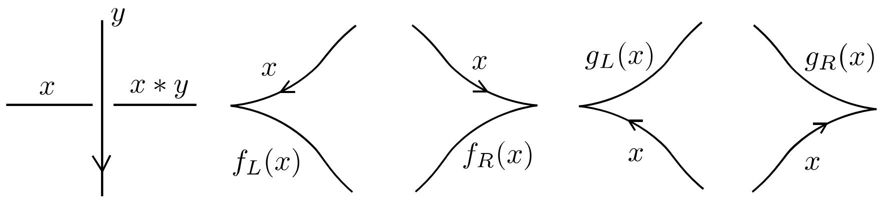

Figure 10: Relations at crossings and cusps (2).

Proposition 4.9 ([14])**.** Let K be a Legendrian knot in ($\mathbb{R}^3$, ξ_{std}) and D the front diagram of K. Let $(X, *, f_L, f_R, g_L, g_R)$ be a 4-Legendrian rack. Then the number of the $(X, *, f_L, f_R, g_L, g_R)$-colorings of D is invariant under the Legendrian Reidemeister moves. Namely, $\#\mathrm{Col}(D, X)$ is an invariant of a Legendrian knot K, denoted by $\#\mathrm{Col}(K, X)$.

Recall that we state in Theorem 4.4 that two Legendrian knots with the same knot type and the same classical invariants cannot be distinguished by bi-Legendrian quandle coloring numbers. Even if we make use of 4-Legendrian quandle coloring numbers, these Legendrian knots still cannot be distinguished as follows.

Theorem 4.10 ([14])**.** Let K_0 and K_1 be Legendrian knots in ($\mathbb{R}^3$, ξ_{std}). If K_0 and K_1 are of the same knot type, $tb(K_0) = tb(K_1)$ and $rot(K_0) = rot(K_1)$, then $\#\mathrm{Col}(K_0, X) = \#\mathrm{Col}(K_1, X)$ for any 4-Legendrian quandle $(X, *, f_L, f_R, g_L, g_R)$.

We also define free 4-Legendrian racks and the fundamental 4-Legendrian rack of a Legendrian knot in the similar way as the bi-Legendrian rack case.

We define the *fundamental* 4*-Legendrian rack* of a Legendrian knot K from the front diagram D of K as follows. Let S be the set of the arcs of D. Define the fundamental 4-Legendrian rack $4LR(D)$ associated to D as the quotient of the free 4-Legendrian rack $F4LR(S)$ by the equivalence relation generated by the relations at crossings and cusps shown in 10.

Proposition 4.11 ([14])**.** Let K be a Legendrian knot in ($\mathbb{R}^3$, ξ_{std}) and D the front diagram of K. Then the fundamental 4-Legendrian rack $4LR(D)$ associated to D is invariant under the Legendrian Reidemeister moves. Namely, the fundamental 4-Legendrian rack is an invariant of a Legendrian knot K, denoted by $4LR(K)$.

The fundamental 4-Legendrian rack of a Legendrian knot is the universal invariant for 4-Legendrian rack coloring numbers in the following sense.

Remark 10. Let D be the front diagram of a Legendrian knot K and $(X_1, *_1, f_{1,L}, f_{1,R}, g_{1,L}, g_{1,R})$ a 4-Legendrian rack. An $(X_1, *_1, f_{1,L}, f_{1,R}, g_{1,L}, g_{1,R})$-coloring of D is a homomorphism from the fundamental 4-Legendrian rack $4LR(K)$ of K to $(X_1, *_1, f_{1,L}, f_{1,R}, g_{1,L}, g_{1,R})$. Hence $\#\mathrm{Col}(K, X_1)$ is the cardinality of the set of the homomorphisms from $4LR(K)$ to $(X_1, *_1, f_{1,L}, f_{1,R}, g_{1,L}, g_{1,R})$.

The fundamental bi-Legendrian rack can be recovered from the fundamental 4-Legendrian rack.

Remark 11. We obtain the fundamental bi-Legendrian rack $bLR(K)$ of a Legendrian knot K from the fundamental 4-Legendrian rack $4LR(K)$ of K by adding the relation $f_L(x) = f_R(x)(= f(x))$ and $g_L(x) = g_R(x)(= g(x))$ for any generator x of $4LR(K)$.

Recall we state in Theorem 4.5 that the Chekanov knots cannot be distinguished by bi-Legendrian rack coloring numbers. Even if we make use of the fundamental 4-Legendrian rack, the Chekanov knots still cannot be distinguished as follows.

Theorem 4.12 ([14])**.** Let K_2 and K_3 be the Chekanov knots shown in Figure 6. Then $4LR(K_2)$ and $4LR(K_3)$ are isomorphic.

Though invariants of Legendrian knots derived from 4-Legendrian racks are theoretically stronger than those from bi-Legendrian racks, it has not been known yet whether they are actually stronger. We have not found a pair of Legendrian knots which cannot be distinguished by invariants from bi-Legendrian racks but can be distinguished by those from 4-Legendrian racks.

References

[1] D. Bennequin, Entrelacements et équations de Pfaff, Third Schnepfenried geometry conference, Vol. 1, Astérisque, **107–108** (1983), 87–161.

[2] J. Ceniceros, M. Elhamdadi and S. Nelson, Legendrian rack invariants of Legendrian knots, *Commun. Korean Math. Soc.* **36** (2021), no. 3, 623–639.

[3] Y. Chekanov, Differential algebra of Legendrian links, *Invent. Math.* **150** (2002), no. 3, 441–483.

[4] W. Chongchitmate and L. Ng, An atlas of Legendrian knots, *Exp. Math.* **22** (2013), no. 1, 26–37.

[5] Y. Eliashberg and M. Fraser, Topologically trivial Legendrian knots, *J. Symplectic Geom.* **7** (2009), no. 2, 77–127.

[6] J. Etnyre, Legendrian and transversal knots, *Handbook of knot theory*, 105–185, Elsevier B. V., Amsterdam, 2005.

[7] J. Etnyre and K. Honda, Knots and contact geometry. I. Torus knots and the figure eight knot, *J. Symplectic Geom.* **1** (2001), no. 1, 63–120.

[8] R. Fenn and C. Rourke, Racks and links in codimension two, *J. Knot Theory Ramifications* **1** (1992), no. 4, 343–406.

[9] D. Fuchs and S. Tabachnikov, Invariants of Legendrian and transverse knots in the standard contact space, *Topology* **36** (1997), no. 5, 1025–1053.

[10] H. Geiges, *An introduction to contact topology*, Cambridge Studies in Advanced Mathematics, 109. Cambridge University Press, Cambridge, 2008.

[11] D. Joyce, A classifying invariant of knots, the knot quandle, *J. Pure Appl. Algebra* **23** (1982), no. 1, 37–65.

[12] B. Karmakar, D. Saraf and M. Singh, Cohomology of generalized Legendrian racks and state-sum invariants of Legendrian links, (2023), arXiv:2301.06854.

[13] N. Kimura, Bi-Legendrian rack colorings of Legendrian knots, *J. Knot Theory Ramifications* **32** (2023), no. 4, Paper No. 2350029, 16 pp.

[14] N. Kimura, Rack coloring invariants of Legendrian knots, Ph. D. thesis, Waseda University, 2024.

[15] D. Kulkarni and V. Prathamesh, On rack invariants of Legendrian knots, (2017), arXiv:1706.07626.

[16] S. V. Matveev, Distributive groupoids in knot theory, *Mat. Sb. (N.S.)* **119(161)** (1982), no. 1, 78–88, 160.

[17] J. Swiatkowski, On the isotopy of Legendrian knots, *Ann. Global Anal. Geom.* **10** (1992), no.3, 195–207.

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