Minimal topological cobordisms and signature type invariants of torus links

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1 Introduction

One of the early applications of the signature invariant was the determination of the 4-genus and the unknotting number of two-strand torus knots [14]. Indeed, all knots K with definite symmetrised Seifert form satisfy the equality

$$|\sigma(K)| = 2g_4(K) = 2g(K),$$

where $\sigma(K), g_4(K), g(K)$ denote the signature invariant, the topological 4-genus, and the Seifert genus of K, respectively. Precise computations of the signature invariant of torus links $T(m,n)$ in the early 80's revealed that the ratio between their signature invariant and their Seifert genus satisfies

$$\lim_{m,n\to\infty} \frac{\sigma(T(m,n))}{g(T(m,n))} = 1,$$

rather than 2 [10], making an easy determination of the smooth 4-genus of torus knots impossible. It took more elaborate techniques to achieve this: some kind of gauge theory or Khovanov homology [11, 16]. Interestingly, the topological 4-genus of torus links is still unknown. Recent results show that the ratio between the topological 4-genus and the Seifert genus satisfies

$$\lim_{m,n\to\infty} \frac{g_4(T(m,n))}{g(T(m,n))} \leq \frac{14}{27},$$

suggesting that the asymptotic ratio $\frac{\sigma}{g_4}$ might actually be 2, as for two-strand torus knots [2]. A positive answer to the following question would imply that.

Question 1. *Does* $\displaystyle\lim_{m,n\to\infty} \frac{g_4(T(m,n))}{mn} = \frac{1}{4}$ *hold?*

In order to get a better understanding of the topological 4-genus, we propose to study a relative version of it, the topological cobordism distance d_χ, defined as follows, compare [4, 5]. Let L_1, L_2 be two oriented links in S^3. We define $d_\chi(L_1, L_2)$ as the maximal absolute value of the Euler characteristic $\chi(\Sigma)$ among all topological submanifolds $\Sigma \subset S^3 \times [0,1]$ with $\partial\Sigma = L_1 \times \{0\} \cup \Sigma = L_2 \times \{1\}$, all whose connected components intersect both $\Sigma = L_1 \times \{0\}$ and $\Sigma = L_2 \times \{1\}$ non-trivially. With this definition, for all knots K, $2g_4(K) = d_\chi(K, O)$, where O denotes the trivial knot. The following statement is a slight strengthening of Theorem 1 in [5].

Theorem 1. *There exists a constant $c > 0$, so that for all $m, n, N \in \mathbb{N}$ with $N \geq \frac{3}{4}mn$:*

$$d_\chi(T(m,n), T(2,N)) = \sigma(T(2,N)) - \sigma(T(m,n)) + E(m,n),$$

with an error term $E(m,n)$ satisfying $|E(m,n)| \leq c(m+n+1)$.

Remark 1. The difference $\sigma(T(2,N)) - \sigma(T(m,n))$ is positive, since the values of the signature invariant involved are $\sigma(T(2,N)) = N - 1 \geq \frac{3}{4}mn - 1$ and $\sigma(T(m,n)) \approx \frac{mn}{2}$, see again [10]. For the same reason, the error term $E(m,n)$ is asymptotically negligable, making Theorem 1 much sharper than the corresponding bound on the cobordism distance derived in [1].

Remark 2. The condition $N \geq \frac{3}{4}mn$ in Theorem 1 may come a bit arbitrary; however, there is no way around a restriction on N, since for $N \approx \frac{mn}{2}$, the difference $\sigma(T(2,N)) - \sigma(T(m,n))$ vanishes up to linear order in m and n. Interestingly, a similar equality with the right hand side reversed, $\sigma(T(m,n)) - \sigma(T(2,N)) + E(m,n)$, could hold again for small values of N. In particular, the case $N = 1$ essentially boils down to a positive answer to Question 1.

In order to estimate the topological 4-genus, or the cobordism distance between links, we need tools that produce lower and upper bounds on these quantities. For the latter, we will use the method of nullhomologous twisting introduced by McCoy [13]. This is completely different from the construction of asymptotically minimal smooth cobordisms between torus links of type $T(d,d)$ and large 2-strand torus links in the recent work [3].

For the lower bound, we will use the signature invariant. Indeed, the known inequality $|\sigma(K)| \leq 2g_4(K)$ for all knots K, see for example [6], readily generalises to

$$|\sigma(L_1) - \sigma(L_2)| \leq d_\chi(L_1, L_2),$$

for all oriented links L_1, L_2. It turns out that Theorem 1 has an interesting consequence concerning all topological concordance invariants that share certain basic properties with the signature invariant. Inspired by [4], we define a maximal clover invariant to be an additive link invariant ρ with the following two properties:

(i) $\rho(T(2,N)) = \sigma(T(2,N))$, for all $N \in \mathbb{Z}$,

(ii) $|\rho(L_1) - \rho(L_2)| \leq d_\chi(L_1, L_2)$, for all oriented links L_1, L_2.

The first property asks that maximal clover invariants coincide with the signature invariant on closures of 2-braids. The second property implies that topologically concordant links, i.e. pairs of links with vanishing topological cobordism distance, have equal maximal clover invariants. As the term 'maximal' suggests, there is a more general notion of clover invariant, where the first property is asked to be true for the trefoil knot only: $\rho(3_1) = \sigma(3_1)$. As explained in [4], all Levine-Tristram signature invariants $\sigma_{e^{2\pi i\theta}}$ with $\theta \in (\frac{1}{6}, \frac{1}{2}]$ (see [12, 18] for a definition) are clover invariants, but only the classical signature invariant is a maximal clover invariant. Moreover, neither of the well-established smooth concordance invariants s and τ are clover invariants, since these coincide with the smooth 4-genus on torus knots, up to sign [16, 15]. In fact, we do not know any example of a maximal clover invariant, beside the signature invariant.

Question 2. *Does there exist a maximal clover invariant other than the classical signature invariant?*

A positive answer to the above question would yield a striking characterisation of the signature invariant. We do not expect this, even though the following consequence of Theorem 1 poses a strict restriction on the values of maximal clover invariants on torus links.

Corollary 1. *There exists a constant $C > 0$ with the following property. For all maximal clover invariants ρ, and for all $m, n \in \mathbb{N}$:*

$$\rho(T(m,n)) \geq \frac{1}{2}mn - C(m+n+1).$$

Here, as well as in Theorem 1, it is possible to extract an explicit constant C (resp. c) of order less than one hundred from the proof. Corollary 1 has an interesting consequence: the signature invariant is minimal among all maximal clover invariant on torus links, up to an affine term. Moreover, when combined with a positive answer to Question 1, Corollary 1 yields the following limit for all maximal clover invariants ρ:

$$\lim_{m,n\to\infty} \frac{\rho(T(m,n))}{mn} = \frac{1}{2}.$$

We present the proofs of Theorem 1 and Corollary 1 in the next two sections, respectively. As alluded to before, they are both variations on recent work about minimal cobordisms between torus links and clover invariants [5, 4].

2 Nullhomologous twisting

Freedman's work in the early 80's provided examples of knots that bound topological subsurfaces in the 4-ball, but not smooth subsurfaces of the same topological type. In particular, all knots with trivial Alexander polynomial are topologically slice, i.e. they bound topological submanifolds homeomorphic to a disc in the 4-ball [8]. This fact quickly led to the conclusion that the Thom conjecture about the 4-genus of algebraic links was not true in the topological setting [17]. As mentioned in the introduction, the topological 4-genus of torus links still waits to be determined. An important step in this direction was taken by McCoy, who derived an upper bound on the topological 4-genus from an operation called nullhomologous twisting [13].

Let $L \subset S^3$ be an oriented link and let $C \subset S^3 \setminus L$ be an unknotted circle whose total linking number with L is zero. The operation that introduces a positive or negative full twist along the circle C to the link L is called a positive or negative nullhomologous twist. An example of a negative nullhomologous twist is depicted in Figure 2. In that figure, all six strands are supposed to be oriented from left to the right; the boxes labelled ± 1 stand for a positive or negative full twist on three strands.

The main result in [13], Theorem 1, implies the following: let $K \subset S^3$ be a knot that can be transformed into the trivial knot by a sequence of k positive nullhomologous twists and k negative nullhomologous twists, then $g_4(K) \leq k$. Similar to Feller's upper bound on

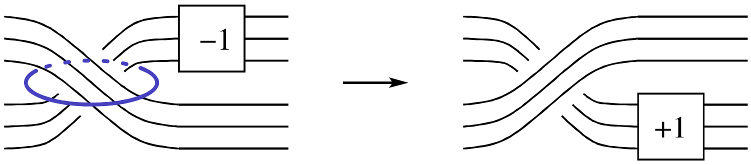

Figure 1: negative null-homologous twist

the topological 4-genus via the Alexander polynomial of knots [7], $2g_4(K) \leq \deg(\Delta_K(t))$, McCoy's bound takes Freedman's disc theorem as main input. However, the applications of these two upper bounds are quite different. It turns out that nullhomologous twisting is better suited for torus knots. Indeed, McCoy showed that the ratio between the topological and smooth 4-genus of torus links was essentially bounded above by $\frac{2}{3}$, a bound that was quickly improved to $\frac{14}{27}$ in [2]. In these applications, only negative nullhomologous twists are used. The full strength of McCoy's result comes to play in [5], where the topological cobordism distance between 'thick and thin' torus knots is computed, in particular between torus knots of type $T(m, m+1)$ with large $m \in \mathbb{N}$, and $T(2, N)$ with $N \geq \frac{3}{4}m^2$. In the rest of this section, we present the method of proof by deriving a slightly more general result, Theorem 1.

Proof of Theorem 1. Fix $m, n, N \in \mathbb{N}$ with $N \geq \frac{3}{4}mn$. The two expressions $\sigma(T(m, n))$ and $d_\chi(T(m, n), T(2, N))$ involved in the statement of Theorem 1 enjoy a kind of Lipschitz-continuity in both parameters m, n, as follows: the torus link $T(m, n)$ can be transformed into $T(m, n-1)$ by smoothing n crossings in its standard diagram, the closure of the braid $(\sigma_1 \cdots \sigma_{m-1})^n$. Smoothing a crossing of a link is a special case of an operation called saddle move, which changes the signature invariant of that link, and its cobordism distance to any other link, by at most one. As a consequence, we may assume that both parameters m, n are even, without changing the validity of the statement of Theorem 1, since we allow for an affine error term $E(m, n)$ with $|E(m, n)| \leq c(m + n + 1)$. Moreover, for simplicity, we will assume $m = 2p$, $n = 2q$ with $\gcd(p, q) = 1$, which ensures that $T(p, q)$ is a knot. The case $\gcd(p, q) > 1$ can be dealt with by replacing the link $T(p, q)$ by a knot connected to the latter by at most $p - 1$ saddle moves, compare the proof of Theorem 1 in [4]. Now comes a key observation: the link $T(2p, 2q)$ is a 2-cable of the knot $T(p, q)$. It is a well-known fact that the knot $T(p, q)$ can be transformed into the trivial knot by a sequence of $\frac{1}{2}(p-1)(q-1)$ negative crossing changes. This is the 'easy' part of the Milnor conjecture. When looking at $T(2p, 2q)$ as a 2-cable of $T(p, q)$, we obtain the following consequence, which was already observed by McCoy [13]: the link $T(2p, 2q)$ can be transformed into the link $T(2, 2pq)$ by a sequence of $t = \frac{1}{2}(p-1)(q-1) \approx \frac{1}{2}pq$ negative nullhomologous twists. The special case of a single nullhomologous negative twist transforming the link $T(4, 6)$ into $T(2, 12)$ is sketched in Figure 2. The interested reader is invited to check that the two braids depicted there indeed close up to the links $T(4, 6)$ and $T(2, 12)$.

At this point, we invoke the method of 'twisting up' used in [4] and [5]. When passing from $T(2p, 2q)$ to $T(2, 2pq)$ by t negative nullhomologous twists, we may further apply t positive nullhomologous twists that increase the framing of the resulting two-strand

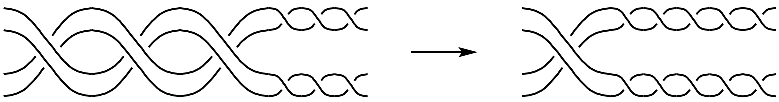

Figure 2: negative twist relating $T(4,6)$ and $T(2,12)$

torus link to $T(2, 2pq + (p-1)(q-1)) \approx T(2, 3pq)$. This is true, since we can transform $T(2,N)$ into $T(2,N+2)$ by a single positive crossing change. Altogether, this means that we can transform the link $T(2p,2q)$ into $T(2,3pq)$ by a sequence of about t negative nullhomologous twists and t positive nullhomologous twists. At last, a 'relative version' of McCoy's upper bound $g_4 \leq t$, explained in the proof of Theorem 1 in [5], implies

$$d_\chi(T(2p,2q), T(2,3pq)) \leq 2t \approx pq,$$

up to an affine error term in p, q (in a nutshell, this relative version derives from McCoy's bound via the equation $d_\chi(K_1,K_2) = 2g_4(K_1 \# \overline{K}_2)$, for all knots $K_1, K_2 \subset S^3$, where $K_1 \# \overline{K}_2$ denotes the connected sum of K_1 with the mirror image of K_2). Recalling $m = 2p$, $n = 2q$, $\sigma(T(m,n)) \approx \frac{1}{2}mn$, $\sigma(T(2,N)) = N-1$, we obtain

$$d_\chi(T(m,n), T(2,N)) \leq \sigma(T(2,N)) - \sigma(T(m,n)) + E(m,n),$$

with an error term $E(m,n)$ satisfying $|E(m,n)| \leq c(m+n+1)$, for $N = 3pq = \frac{3}{4}mn$. We conclude the proof of Theorem 1 with the following two remarks. First, the last inequality can be replaced by an equality, since the difference $\sigma(T(2,N)) - \sigma(T(m,n))$ is also a lower bound for $d_\chi(T(m,n), T(2,N))$. Second, the case $N \geq \frac{3}{4}mn$ is an immediate consequence of the limit case $N = \frac{3}{4}mn$, since increasing N by one increases the right hand side by one, while it changes the left hand side by at most one.

□

3 Maximal clover invariants

In this short section, we derive a lower bound on the values of maximal clover invariants on torus links. Recall that a maximal clover invariant is an additive link invariant ρ which coincides with the signature invariant on closures of 2-braids, and satisfies the following Lipschitz property for all links L_1, L_2:

$$|\rho(L_1) - \rho(L_2)| \leq d_\chi(L_1, L_2).$$

Proof of Proposition 1. Let ρ be a maximal clover invariants and let $m, n \in \mathbb{N}$. Choose $N \geq \frac{3}{4}mn$. Theorem 1 together with the Lipschitz property of ρ implies:

$$\begin{aligned} |\rho(T(m,n)) - \rho(T(2,N)| &\leq d_\chi(T(m,n), T(2,N)) \\ &\leq \sigma(T(2,N)) - \sigma(T(m,n)) + c(m+n+1) \\ &\leq \sigma(T(2,N)) - \frac{1}{2}mn + C(m+n+1), \end{aligned}$$

for a suitable constant $C > 0$, easy to extract from Theorem 5.2. in [10]. Finally, the normalisation $\rho(T(2,N)) = \sigma(T(2,N))$ implies

$$\rho(T(m,n)) \geq \frac{1}{2}mn - C(m+n+1).$$

□

A similar bound for all clover invariants was derived in [4], with the quadratic term $\frac{1}{2}mn$ replaced by $\frac{5}{18}mn$. As a consequence of that result, the Levine-Tristram signature invariant $\sigma_{e^{2\pi i \frac{1}{6}}}$ is minimal among all clover invariant on torus links, up to an affine term. There is a hierarchy of clover invariants ρ_N, obtained by requiring $\rho_N(T(2,k)) = \sigma(T(2,k))$, for all $k \in \mathbb{Z}$ with $|k| \leq N$, for any given $N \geq 3$. In the case of odd N, we may expect similar (sharp) lower bounds on $\rho_N(T(m,n))$, with the minimum possibly attained by the Levine-Tristram signature invariant $\sigma_{e^{2\pi i (\frac{1}{2} - \frac{1}{N})}}$. Here the number $e^{2\pi i \left(\frac{1}{2} - \frac{1}{N}\right)} \in S^1$ is the location of the 'last jump' of the signature function of the knot $T(2,N)$. This could yield a purely topological interpretation of the results by Gambaudo and Ghys on the asymptotic profile of the Levine-Tristram signature function of torus links [9].

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