Non-trivial cycles of the spaces of long embeddings detected by 2-loop graphs

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1 Introduction

Long embeddings are embeddings of $\mathbb{R}^j$ to $\mathbb{R}^n$ which coincide with the standard linear embedding outside a fixed ball in $\mathbb{R}^j$. We write $\mathcal{K}_{n,j}$ for the space $\mathrm{Emb}(\mathbb{R}^j, \mathbb{R}^n)$ of long embeddings. As long embeddings are *long immersions*, there is a map $\mathcal{K}_{n,j} \to \mathrm{Imm}(\mathbb{R}^j, \mathbb{R}^n)$ to the space of long immersions. Since the space $\mathrm{Imm}(\mathbb{R}^j, \mathbb{R}^n)$ is well-studied, we often consider the difference between the two spaces:

$$\overline{\mathcal{K}}_{n,j} = \overline{\mathrm{Emb}}(\mathbb{R}^j, \mathbb{R}^n) = \mathrm{hofib}_\iota(\mathcal{K}_{n,j} \to \mathrm{Imm}(\mathbb{R}^j, \mathbb{R}^n)).$$

Haefliger started to study higher dimensional embeddings. In 1966, Haefliger [Hae] computed

$$\pi_0 \mathcal{K}_{n,j} \quad (2n - 3j - 3 = 0,\ n - j \geq 3).$$

(See [Bud] for the difference between $\mathrm{Emb}(S^j, S^n)$ and $\mathcal{K}_{n,j}$.) The result depends only on parities of n and j. In 2004, Budney [Bud], by using Goodwillie's result, further showed

$$\pi_{2n-3j-3} \mathcal{K}_{n,j} \quad (n - j \geq 3,\ j \neq 1)$$

depends only on parities of n and j. This type of *bi-periodicity* motivates us to study not only isotopy classes of long embeddings but also higher homotopy of the space of long embeddings.

In 2017, Fresse, Turchin and Willwacher [FTW], following Arone and Turchin, [AT 1, AT 2] showed that if $n - j \geq 3$, $j \geq 1$, the rational homotopy $\pi_* \overline{\mathcal{K}}_{n,j} \otimes \mathbb{Q}$ depends only on the parities of n and j, up to degree shifts. This surprising result was shown by a homotopy theoretical approach, called Goodwillie–Weiss [GW] *embedding calculus*.

Behind this bi-periodicity, there is a combinatorial object, called the *hairy graph complex*. This complex $HGC_{n,j}$ is generated by some kind of graphs, and its differential is defined in terms of contractions of edges. The complex is defined so that it depends on the parities of n and j only.

Theorem 1.1. *[AT 1, AT 2, FTW] For $n - j \geq 3$, $j \geq 1$, there is an isomorphism*

$$\pi_* \overline{\mathcal{K}}_{n,j} \otimes \mathbb{Q} \cong H_*(HGC_{n,j}).$$

Dually, there exists a zigzag of quasi-isomorphisms (in $\mathbf{CDGA}_{\mathbb{Q}}$)

$$\bigwedge \widetilde{HGC}_{n,j} \overset{\simeq}{\leftarrow} \dots \overset{\simeq}{\rightarrow} A^*_{PL}(\overline{\mathcal{K}}_{n,j}).$$

Here $\bigwedge \widetilde{HGC}_{n,j}$ stands for the free polynomial algebra generated by the linear dual space of $HGC_{n,j}$. In other words, $\bigwedge \widetilde{HGC}_{n,j}$ is a *rational model* of $\overline{\mathcal{K}}_{n,j}$ $(n-j \geq 3)$.

Problem 1.2. *Give a geometric meaning to the map $I : H^*(\widetilde{HGC}_{n,j}) \to H^*(\overline{\mathcal{K}}_{n,j}, \mathbb{Q})$. Does $H^*(\widetilde{HGC}_{n,j})$ represent non-trivial cohomology of $\overline{\mathcal{K}}_{n,j}$ when $n-j=2$?*

One approach to this problem is *configuration space integrals*. Formally, configuration space integrals give a map (of coefficient $\mathbb{R}$)

$$I : \widetilde{GC}_{n,j} \longrightarrow A^*_{dR}(\overline{\mathcal{K}}_{n,j}) \quad (n-j \geq 2,\ j \geq 2)$$

from another graph complex $\widetilde{GC}_{n,j}$. In this approach, one must deal with potential obstructions for I to be a cochain map. Moreover, to show I is injective, one will be required to give dual cycles of $\overline{\mathcal{K}}_{n,j}$. Computing $H^*(\widetilde{GC}_{n,j})$ is also a difficult problem.

We have a decomposition with respect to the first Betti number g, and the *order* k of graphs

$$\widetilde{GC}_{n,j} = \bigoplus_{g \geq 0} \widetilde{GC}_{n,j}(g) = \bigoplus_{g \geq 0} \bigoplus_{k \geq 1} \widetilde{GC}_{n,j}(k,g).$$

Configuration space integrals in the case $g = 0, 1$, $* = $ top (see Definition 3.8) is developed by Bott[Bot] Cattaneo, Rossi[CR], Sakai and Watanabe [Sak, SW, Wat 1].

The aim of this article is the case $g = 2$, $* = $ top, which generalizes the simplest case ($g = 2$, $* = $ top,) $k = 3$, developed in the author's paper [Yos 1]. The results are very likely to be generalized to the case g is higher.

2 Main Result

As a graph complex for configuration space integrals, we use quite a different graph complex $\widetilde{DGC}_{n,j}$. In what follows, we often write DGC and HGC for the dual graph complexes and omit the symbol $\breve{}$ and the subscripts n, j. The coefficients of graph complexes are considered to be $\mathbb{R}$ when we define configuration space integrals.

Theorem 2.1 (Y.)**.** *Assume $n - j \geq 2$ and $j \geq 2$. Then, there exists a graph complex DGC and a zigzag*

$$HGC \underset{p}{\leftarrow} DGC \underset{I}{\rightarrow} A^*_{dR}(\overline{\mathcal{K}}_{n,j})$$

of cochain maps such that

(1) $p^* : H^{top}(DGC) \to H^{top}(HGC)$ is surjective when $n-j$ is even.

(2) If $H \in H^{top}(DGC(g=2))$ and $I^*(H) = 0$, then $p^*(H) = 0$.

Example 2.2 (The simplest odd case [Yos 1]). Suppose $(n, j) = (\text{odd}, \text{odd})$, $n-j \geq 2$ and $j \geq 3$. There exists a non-trivial graph cocycle in $HGC^{top}(k=3, g=2)$ that includes the 2-loop hairy graph $\Theta(1,0,1)$. See Figure 4. Hence, by computing the degree of the graph $\Theta(1,0,1)$, we have

$$H^{3(n-j-2)+(j-1)}(\overline{\mathcal{K}}_{n,j}, \mathbb{Q}) \neq 0.$$

Example 2.3 (The simplest even case). Suppose $(n, j) = (\text{even}, \text{even})$, $n-j \geq 2$ and $j \geq 2$. There exists a non-trivial graph cocycle in $HGC^{top}(k=7, g=2)$ that includes $\Theta(3,2,1)$. Hence we have

$$H^{7(n-j-2)+(j-1)}(\overline{\mathcal{K}}_{n,j}, \mathbb{Q}) \neq 0.$$

As a cororally of our Main Result, we have an alternative proof of the result shown by Budney, Gabai [BG] and Watanabe [Wat 2].

Cororally 2.4. *The $(n-1)$-th homotopy group $\pi_{n-1}(\mathcal{K}_{n+2,n})_u$ of the unknot component of the space of n-dimensional long knots has an infinite-rank subgroup.*

3 Graph complexes and graph homologies

We introduce two graph complexes $PGC_{n,j}$ and $HGC_{n,j}$ with a natural projection $p : PGC_{n,j} \to HGC_{n,j}$. The graph complex $DGC_{n,j}$ in Theorem 2.1 is quasi-isomorphic to $PGC_{n,j}$. Both complexes have decompositions with respect to the first Betti number g of graphs and the *order* k of graphs, a notion that measures the complexity of graphs. See Definition 3.7.

Several parts of the cohomology of HGC are already computed. In particular, Conant, Costello, Turchin and Weed [CCTW] showed that $H^*(HGC^{top}(g=2))$ is infinite dimensional. On the other hand, PGC has more graphs and hence is suited as a source of morphisms.

The first graph complex PGC is a cochain complex generated by connected *plain graphs.*

Definition 3.1 (The plain graphs). *Plain graphs are graphs which have two types of vertices and two types of edges. White vertices have at least three dashed edges and no solid edges, while black vertices have an arbitrary number of solid and dashed edges. Each component has at least one black vertex. Double edges are allowed but no loop-edge is allowed. A plain graph is admissible if it satisfies both of the following.*

(I) Every black vertex without dashed edges must have at least three solid edges. Every component has a black vertex with a dashed edge.

(II) The restriction to solid edges consists of disjoint broken lines.

Note that (I) + (II) implies that every black vertex has at least one dashed edge and at most two solid edges.

Define the degree $|\Gamma| = deg(\Gamma)$ of a graph Γ by

$$(n-1)|E_{\cdots\cdot}| + (j-1)|E_{\_\_}| - n|V_\circ| - j|V_\bullet|,$$

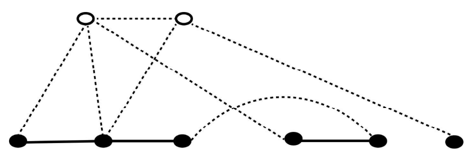


Figure 1: Example of an admissible plain graph

where $|E_{\cdots\cdots}|$, $|E_{\_\_}|$, $|V_\circ|$, $|V_\bullet|$ are the number of dashed edges, solid edges, white vertices and black vertices respectively.

A label of a plain graph is a choice of an ordering of the set of vertices and edges and a choice of orientations of edges. Each label gives an orientation of a graph. The orientation is defined so that it depends only on parities of n and j.

Definition 3.2 (The plain graph complex). *As a graded vector space,*

$$PGC_{n,j} = \frac{\mathbb{Q}\{\textit{Connected labeled admissible plain graphs}\}}{\textit{Orientation relations}}.$$

The differential d_{PGC} of PGC is defined by the sum of contractions of edges except for chords: dashed edges which connect two black vertices, and multiple edges: pairs of a dashed edge and a solid edge which connect two black vertices, and double edges. See Figure 2.

$$d_{PGC}(\Gamma) = \sum_{\substack{e \in E(\Gamma) \\ e \neq \bullet\text{-}\text{-}\text{-}\bullet}} \pm \Gamma/e.$$

The signs of $d_{PGC}(\Gamma)$ arises when labels of vertices and edges are permuted and when d "jumps" vertices.

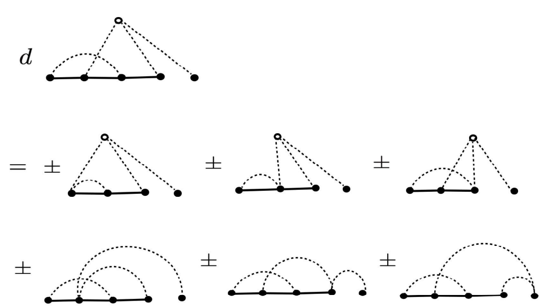


Figure 2: Example of the differential

Lemma 3.3. (PGC, d_{PGC}) *is a cochain complex.*

The second complex HGC is a cochain complex generated by *hairy graphs.*

Definition 3.4. *Hairy graphs are admissible plain graphs with no solid edge such that each black vertex has exactly one dashed edge. A segment* $\bullet$- - - *is called a hair.*

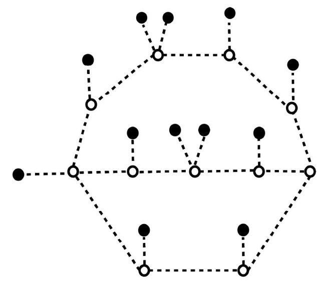


Figure 3: Example of a hairy graph

Definition 3.5 (The hairy graph complex)**.** *As a graded vector space,*

$$HGC_{n,j} = \frac{\mathbb{Q}\{\textit{Connected labeled hairy graphs}\}}{\textit{Orientation relations}}.$$

The differential d_{HGC} of HGC is defined by

$$d_{HGC} = \sum_{\substack{e \in E(\Gamma) \\ e = \circ - - - \circ}} \pm \Gamma/e.$$

Note that we do not perform contractions of hairs.

Remark 3.6. Strictly speaking, we must include loop edges for Theorem 1.1. However, as mentioned in [AT 2], the difference is only 1-dimensional, so we exclude loop edges for simplicity.

Definition 3.7. *Define the order of a plain graph Γ by $k(\Gamma) = |E_{\cdots\cdot}(\Gamma)| - |V_\circ(\Gamma)|$. Then we have a decomposition*

$$PGC = \bigoplus_{g \geq 0} \bigoplus_{k \geq 1} PGC(k, g).$$

with respect to the first Betti number g, and the order k. A similar decomposition exists for HGC.

Notation 3.8. *We say a non-admissible graph is non-degenerate if any white vertex has exactly one dashed edge and any black vertex has exactly one dashed edge. The subspace of PCG and HGC generated by non-degenerate graphs is written as PGC^{top} and HGC^{top}, respectively. The kernel of the differential is written as $H^{top}(PGC)$ and $H^{top}(HGC)$. Note that If $(n, j) = (3, 1)$, $H^{top}(HGC) = H^0(HGC)$ is isomorphic to the space of Vassiliev invariants.*

Theorem 3.9 ([Yos 2])**.** *The projection $PGC^{top} \to HGC^{top}$ induces an epimorphism between the top cohomologies.*

Proof. (Sketch) Dually, we show the map: $\chi_* : H_{top}(HGC) \to H_{top}(PGC)$ induced by the inclusion is injective. In fact, we can construct a left inverse

$$\sigma_* : H_{top}(PGC) \to H_{top}(HGC), \quad \sigma_* \chi_* = id$$

by induction on the number of black vertices. This construction of σ_* is motivated by Bar-Natan's construction [Bar] of $\chi^{-1} : \mathcal{A}(S^1) \to \mathcal{B}$. □

Remark 3.10. Consider the complex fPGC generated by plain graphs which satisfy the first condition (I) of Definition 3.1. fPGC has a good description in terms of graph operads used in [AT 2]. The author believes fPGC is quasi-isomorphic to HGC.

4 Construction of cycles: ribbon presentations

This section is the main part of this article and is based on the author's paper [Yos 1]. Let $k \geq 1$, $p, r \geq 1$, $q \geq 0$, $p+q+r+1 = k$. In this section, we construct $(k(n-j-2)+(j-1))$-cycles

$$d(\Theta(p,q,r)) : (S^{n-j-2})^k \times S^{j-1} \to \overline{\mathcal{K}}_{n,j}$$

by the operation which we call *perturbation of ribbon presentations* with one node. The cycle $d(\Theta(p,q,r))$ is detected by the hairy graph $\Theta(p,q,r)$ whose three edges of "Θ" have p, q, r hairs respectively. See Figure 4.

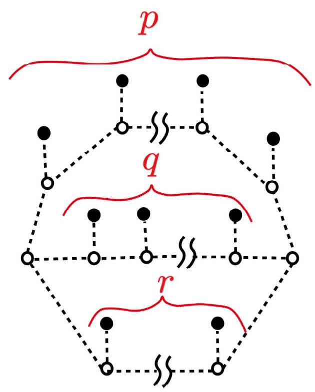


Figure 4: The hairy graph $\Theta(p,q,r)$

We construct the cycles as follows. First, we give a diagram $D(\Theta(p,q,r))$ from the hairy graph $\Theta(p,q,r)$. From this diagram, we give a ribbon presentation $P(\Theta(p,q,r))$. We see that such a ribbon presentation gives $S^{j-1} \times (S^{n-j-2})^{\times k}$ cycle of embedded submanifolds ($\approx \mathbb{R}^j$) in $\mathbb{R}^n$. Then the desired cycle $d(\Theta(p,q,r)) : S^{j-1} \times (S^{n-j-2})^{\times k} \to \overline{\mathcal{K}}_{n,j}$ is obtained by giving a path of immersions to the trivial immersion. The path also gives the parameterization of the embedded submanifolds.

Our construction is analogous to the construction of *wheel-like cycles* given by Sakai and Watanabe [Wat 1, SW],

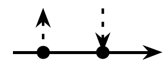
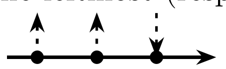
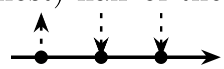
$$c_k : (S^{n-j-2})^{\times k} \to \overline{\mathcal{K}}_{n,j},$$

which are detected by 1-loop graphs. The additional parameter on S^{j-1} of $d(\Theta(p,q,r))$ arises from the *node* (see Notation 4.2), which only our ribbon presentations have.

Though we focus on 2-loop hairy graphs, this construction of cycles is very likely to be generalized to hairy graphs with an arbitrary number of loops.

4.1 The diagram $D(\Theta(p,q,r))$

The diagram $D(\Theta(p,q,r))$ is obtained from the hairy graph $\Theta(p,q,r)$ as follows.

- First, arbitrarily orient three edges of Θ.
- Replace each hair with the oriented line with two open chords . Exceptionally replace the leftmost (resp. rightmost) hair of the upper (resp. lower) edge of $\Theta(p,q,r)$ with  (resp. ).
- Finally connect ends of chords as expected from the graph $\Theta(p,q,r)$.

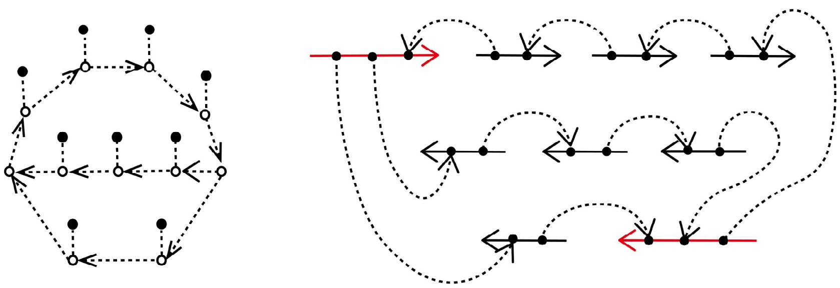


Figure 5: The diagram $D(\Theta(4,3,2)))$

4.2 Review of ribbon presentations

Habiro, Kanenobu and Shima [HS] [HKS] introduced the notion of ribbon presentations to define finite type invariants of ribbon n-knots.

Definition 4.1. *A ribbon presentation $P = \mathcal{D} \cup \mathcal{B}$ is an oriented immersed 2-disk in $\mathbb{R}^3$, where $\mathcal{D} = D_0 \cup D_1 \cdots \cup D_l$ is the (disjoint) union of 2-dimensional disks, $D_i \approx D^2$, and $\mathcal{B} = B_1 \cup \cdots \cup B_l$ is the (disjoint) union of bands, $B_i \approx I \times I$. There is a base point on the boundary of D_0. Each band connects two disks and can intersect with the interiors of disks except for D_0. These intersections are called crossings of this ribbon presentation. See Figure 6.*

Notation 4.2. *A disk without intersecting bands is called a node. A disk with an intersecting band is called a leaf if exactly one band is attached to the boundary of the disk.*

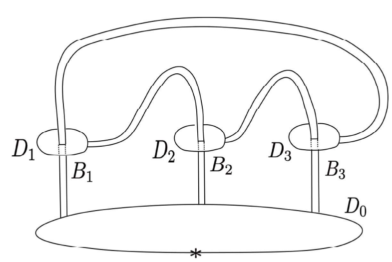


Figure 6: Example of a ribbon presentation

From a ribbon presentation P, we obtain a j-dimensional long embedding $\psi(P) \subset \mathbb{R}^n$ as follows. Let V_P be the thickened ribbon presentation

$$V_P = \mathcal{B} \times [-1/4, 1/4]^{j-1} \bigcup \mathcal{D} \times [-1/2, 1/2]^{j-1}.$$

Then, after smoothing of corners of V_P, define a long embedding $\psi(P)$ by the connected sum

$$\psi(P) = \partial V_P \# \iota(\mathbb{R}^j) \subset \mathbb{R}^n.$$

Corresponding to the i-th crossing of P, the long embedding $\psi(P)$ has a link of a punctured sphere $\hat{D}_i \approx S^j \setminus pt$ and a tube $\hat{B}_i \approx I \times S^{j-1}$. See Figure 7. We call these links crossings of $\psi(P)$.

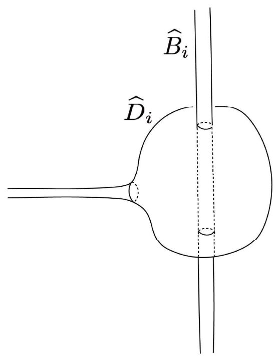


Figure 7: The i-th crossing

Habiro, Kanenobu and Shima [HS] [HKS] introduced several moves of ribbon presentation, which do not change isotopy classes of corresponding embeddings. Moves in Figure 8 are examples of the moves.

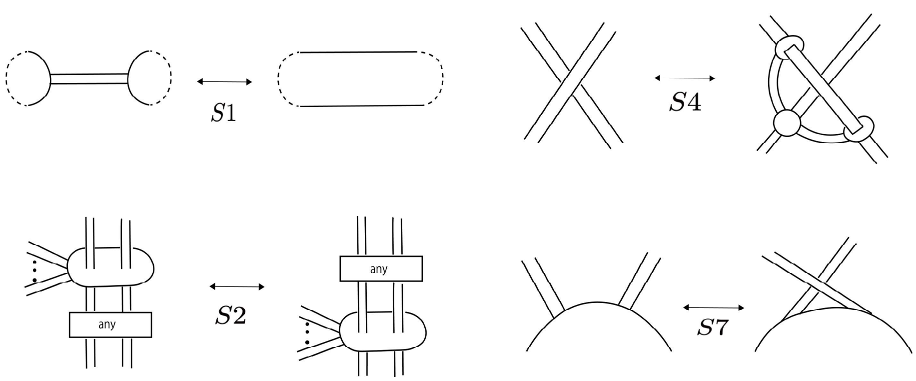
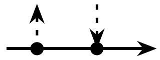
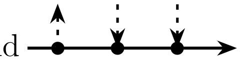
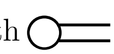
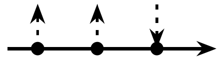
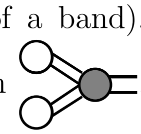
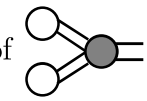


Figure 8: Example of moves of ribbon presentations

4.3 The ribbon presentation $P(\Theta(p, q, r))$

A ribbon presentation $P(\Theta(p, q, r))$ is obtained from the diagram $D(\Theta(p, q, r))$ as follows.

- Replace  and  with , where a vertex with an outgoing (resp. ingoing) open chord is replaced by a disk (resp. a segment of a band). Exceptionally, replace , which has two open chords, with . The disk with three bands (drawn in gray) is the node of our ribbon presentation.

- Intersect a disk with a band if they are connected by chords. Assign the label $\star$ to this crossing. This label is a sign of perturbation of a crossing which we later define. The orientation of the crossing is arbitrary, except that two end-disks of  must intersect with bands in opposite orientation.
- Connect the free ends of bands to the based disk.

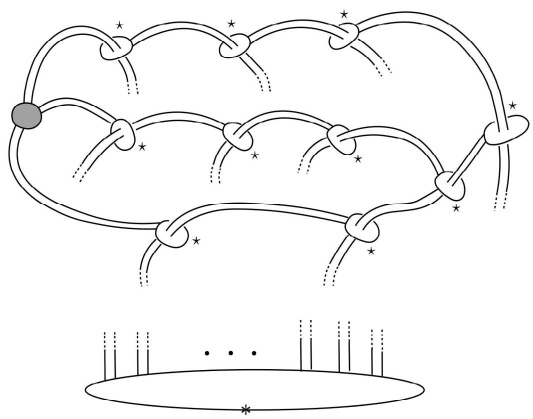


Figure 9: The ribbon presentation $P(\Theta(4,3,2)))$

4.4 The cycle $d(\Theta(p,q,r))$

Here, we construct cycles $d(\Theta(p,q,r)) : (S^{n-j-2})^k \times S^{j-1} \to \overline{\mathcal{K}}_{n,j}$. The parameter space $(S^{n-j-2})^k$ arises from *perturbation* of crossings, which Sakai and Watanabe introduced. We see $\mathbb{R}^n$ as $\mathbb{R}^3 \times \mathbb{R}^{n-j-2} \times \mathbb{R}^{j-1}$. The original ribbon presentations are constructed in $\mathbb{R}^3$, and they are thickened using parameters of $\mathbb{R}^{j-1}$. Represent the parameter space S^{n-j-2} as

$$\{(x_3, \dots, x_{n-j+1}) \in \mathbb{R}^{n-j-1} \mid (x_3-1)^2 + x_4^2 + \cdots + x_{n-j+1}^2 = 1\}.$$

Assume the x_3 coordinate is perpendicular to bands, near crossings.

Definition 4.3. *[Wat 1, SW] The perturbation of a crossing (with $\star$) is the operation to replace the band B with the band $B(v)$ ($v \in S^{n-j-2}$), which is perturbed to the direction v near the crossing. See Figure 10.*

Recall that our ribbon presentation $P = P(\Theta(p,q,r))$ has k crossings so that each band B_j has one or two crossings. For each parameter $\mathbf{v} = (v_1, \dots, v_k) \in (S^{n-j-2})^k$, set the new ribbon presentation $P_{\mathbf{v}}$ by

$$P_{\mathbf{v}} = \mathcal{D} \cup \mathcal{B}(\mathbf{v}) = \bigcup D_i \cup \bigcup B_j(v_1, v_2, \dots, v_k),$$

where $B_j(v_1, v_2, \dots, v_k)$ is the band obtained by perturbating all crossings c_k of B_j to the direction v_k.

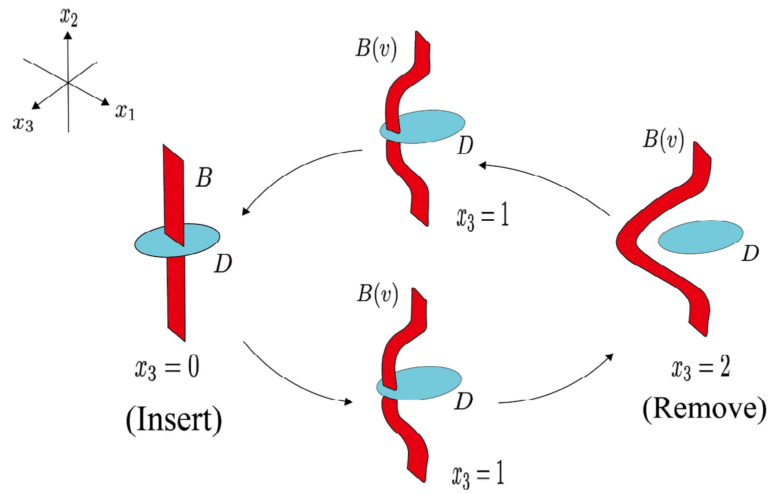
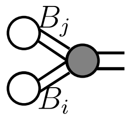


Figure 10: Perturbation of a crossing $(n - j = 3)$

Definition 4.4. *[Wat 1, SW] The cycle* $c(\Theta(p, q, r)) : (S^{n-j-2})^k \to \mathcal{K}_{n,j}$ *is defined by*

$$\mathbf{v} \longmapsto \psi(P_{\mathbf{v}}) = \partial V_{P_{\mathbf{v}}} \# \iota(\mathbb{R}^j).$$

Recall that our ribbon presentation has a node which has two bands, say B_i and B_j, connected to leaves 

. The additional S^{j-1} family is given by moving one tube $(\hat{B}_i)$ around the other tube $(\hat{B}_j)$. See Figure 11. Then we obtain a $(S^{n-j-2})^k \times S^{j-1}$ cycle of submanifolds in $\mathbb{R}^n$. Note that as (images of) immersions, there is a path to the trivial immersion. Using this path, we obtain the desired cycle

$$d(\Theta(p, q, r)) : (S^{n-j-2})^k \times S^{j-1} \to \overline{\mathcal{K}}_{n,j}.$$

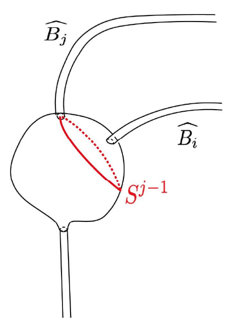
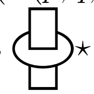
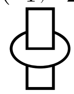
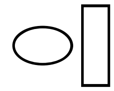


Figure 11: The additional S^{j-1} family

4.5 Properties of the ribbon presentation $P(\Theta(p, q, r))$

In Cororally 2.4, one must take the cycles from the unknot component. We introduce an important property to take them.

Notation 4.5. *Let $\varepsilon_i = \pm 1$. Write $P(\Theta(p,q,r))(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_k)$ for the ribbon presentation obtained by changing the jth crossing*  *to*  *when $\varepsilon_j = 1$, and to*  *when $\varepsilon_j = -1$.*

Proposition 4.6. *After several cross-change moves in Figure 12 are performed to $P(\Theta(p,q,r))$, the presentation $P(\Theta(p,q,r))(1,1,\ldots,1)$ is equivalent to the trivial presentation.*

Proof. We perform cross-change moves as in the Figure 13. Then the resulting presentation becomes trivial, after moves including $S4$. □

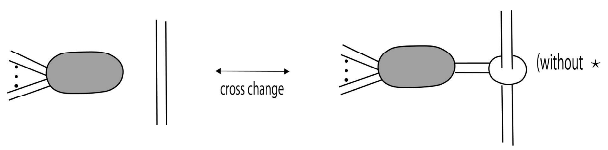


Figure 12: Cross-change move

Note that the cross-change move might change the cycles we later define. However, the move does not affect the pairing argument we later discuss.

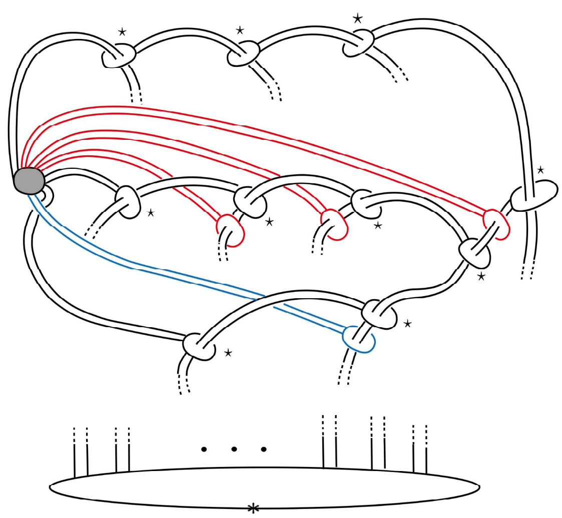


Figure 13: The ribbon presentation after cross-change moves

5 Construction of cocycles: configuration space integrals

From now on, the coefficients of graph complexes are considered to be $\mathbb{R}$. In this section, we give a geometric correspondence

$$I : PGC_{n,j} \to A_{dR}(\overline{\mathcal{K}}_{n,j}).$$

The map I is given by configuration space integrals in the same way as Bott [Bot], Cattaneo, Rossi [CR], Sakai and Watanabe [Sak, SW, Wat 1] give. It is unknown whether configuration space integrals give cochain maps, due to potential obstructions called *hidden faces*. However, we can show I is a cochain map "up to homotopy". See Theorem 5.4

Definition 5.1 (*Configuration spaces*)**.** *Define the k points configuration space of $\mathbb{R}^n$ by*

$$C_k(\mathbb{R}^n) = (\mathbb{R}^n)^{\times k} \setminus \Delta$$

where Δ is the fat diagonal $\bigcup_{1\le i\neq j\le k}\{y_i = y_j\}$.

Though $C_k(\mathbb{R}^n)$ is an open manifold, there exists a canonical compactification $\overline{C}_k(\mathbb{R}^n)$, called *Fulton–Macpherson compactification*. [Les] and [Sin] is a good reference for this type of compactification.

Recall $\overline{\mathcal{K}}_{n,j}$ consists of a family of immersions $\{\overline{\psi}_u\}_{u\in[0,1]}$, $\overline{\psi}_u \in \mathrm{Imm}(\mathbb{R}^j, \mathbb{R}^n)$ such that $\overline{\psi}_0$ is the trivial immersion and $\overline{\psi}_1 \in \mathcal{K}_{n,j}$.

We use the following bundle $E_{s,t}$ to define the configuration space integral associated to a plain graph with s black vertices and t white vertices.

Definition 5.2 (*Configuration space bundles*)**.** *$E_{s,t}$ is the bundle over $\overline{\mathcal{K}}_{n,j}$ defined by the pullback*

$$\begin{array}{ccc} E_{s,t} & \longrightarrow & C_{s+t}(\mathbb{R}^n) \\ \downarrow & & \downarrow \text{\textit{restriction}} \\ \overline{\mathcal{K}}_{n,j} \times C_s(\mathbb{R}^j) & \xrightarrow{\text{\textit{evaluation at }} u=1} & C_s(\mathbb{R}^n) \end{array}$$

The (typical) fiber of the projection $E_{s,t} \to \overline{\mathcal{K}}_{n,j}$ is written by $C_{s,t}$. $C_{s,t}$ is the space of configurations of $(s+t)$ points in $\mathbb{R}^n$ such that the first s vertices are images of s points on $\mathbb{R}^j$.

Let Γ be a labeled plain graph with s black vertices $\overset{(-j)}{\bullet}$ and t white vertices $\overset{(-n)}{\circ}$. Then each oriented dashed (resp. solid) edge e gives a map

$$P_e : E_{s,t} \to S^{n-1} \quad (\text{resp. } P_e : E_{s,t} \to S^{j-1}).$$

by assigning the direction from the initial point to the end point. See Figure 14.

Definition 5.3 (Configuration space integrals)**.** *Consider $A^*_{dR}(\overline{\mathcal{K}}_{n,j})$ as $\mathbf{Sset}(Sing^\infty_*(\overline{\mathcal{K}}_{n,j}), A^*_{dR}(\Delta))$. Define a form $I(\Gamma) \in A_{dR}(\overline{\mathcal{K}}_{n,j})$ as follows. For a simplex $f : \Delta_m \to \overline{\mathcal{K}}_{n,j}$, $I(\Gamma)(f)$ is given by the fiber integral*

$$I(\Gamma)(f) = \pi_*\Omega_f(\Gamma) = \int_{\overline{C}_{s,t}} \Omega_f(\Gamma) \in A_{dR}(\Delta^m),$$

where $\Omega_f(\Gamma)$ is the pullback of the volume forms by the direction map;

$$\Omega_f(\Gamma) = (P(\Gamma)\circ f)^*(\bigwedge \omega_{S^{j-1}} \wedge \bigwedge \omega_{S^{n-1}}).$$

.

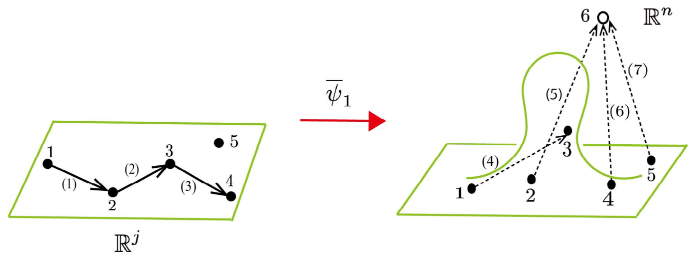


Figure 14: Example of the direction map

$$\begin{array}{ccccc} f^*E_{s,t} & \xrightarrow{\quad f \quad} & E_{s,t} & \xrightarrow{P(\Gamma)=\prod_e P_e} & \prod S^{j-1}\times\prod S^{n-1} \\ \downarrow \pi & & \downarrow \pi & & \\ \Delta^m & \xrightarrow{f:\ a\ \text{“smooth” simplex}} & \overline{\mathcal{K}}_{n,j} & & \end{array}$$

Unfortunately, the map of graded vector spaces

$$I : PGC \to A_{dR}(\overline{\mathcal{K}}_{n,j})$$

is not necessarily a cochain map. In fact, by Stokes' theorem, we have

$$(-1)^{|\Gamma|} dI(\Gamma) = \int_{\partial \overline{C}_{s,t}} \Omega(\Gamma) = \sum_{\substack{S \subseteq V(\Gamma)\cup\infty \\ |S|\geq 2}} \int_{\widetilde{C}_S} \Omega(\Gamma),$$

where $\widetilde{C}_S$ is the configuration such that the vertices of S are infinitely close. The obstructions

$$dI(\Gamma) - I(d\Gamma) = (-1)^{|\Gamma|} \Big(\sum_{\substack{S \subseteq V(\Gamma) \\ |S|\geq 3}} + \sum_{S = V(\bullet\!-\ -\!\bullet)} \Big) \int_{\widetilde{C}_S} \Omega(\Gamma)$$

are called *hidden face contributions*. Some hidden faces vanish by symmetries and rescaling of the faces. Other faces are canceled by introducing correction terms. We interpret adding correction terms as replacing graph complexes. The following non-obvious fact says all hidden faces are canceled by introducing correction terms.

Theorem 5.4 (Y. advised by Turchin)**.** *When $n - j \geq 2$, there exists a graph complex DGC and a zigzag*

$$PGC_{n,j} \underset{p}{\overset{\simeq}{\leftarrow}} DGC_{n,j} \underset{I}{\rightarrow} A^*_{dR}(\overline{\mathcal{K}}_{n,j}),$$

of cohain maps.

Proof. We only give a very rough sketch. Let $pPGC$ be the graph complex generated by possibly non-admissible plain graphs. Then DGC is a kind of tensor product of $pPGC$

and the bar construction $B(A, A.*) = A \otimes BA \simeq \mathbb{R}$. Here, A is a dg algebra describing all hidden faces and there is a map of dgas

$$A \to \mathrm{Inj}(\mathbb{R}^j, \mathbb{R}^n)$$

to the space of linear injective maps. The differential d_{DGC} is decomposed into two commutative differentials:

$$d_{DGC} = d_V + d_H.$$

The differential d_V is induced by the differential of $d_{A\otimes BA}$. The differential d_H is defined as the sum of contractions, including hidden faces. By using a spectral sequence filtered by the number of vertices, we can show that DGC and PGC are quasi-isomorphic.

Let $P(\mathrm{Imm}(\mathbb{R}^j, \mathbb{R}^n)) = P(\mathrm{Imm}(\mathbb{R}^j, \mathbb{R}^n), \bullet, \iota)$ be the path space of $\mathrm{Imm}(\mathbb{R}^j, \mathbb{R}^n)$ whose one end is fixed at the trivial immersion. The integral from decorated graphs whose plain part is one black vertex:

$$I : \bullet \otimes A \otimes BA \to A_{dR}(P(\mathrm{Imm}(\mathbb{R}^j, \mathbb{R}^n))) \to A_{dR}(\overline{\mathcal{K}}_{n,j})$$

is defined as follows. First, we have a map

$$E_{1,0} \to P(\mathrm{Inj}(\mathbb{R}^j, \mathbb{R}^n))$$

defined by the differential. (Recall $E_{1,0}$ is isomorphic to $C_1(\mathbb{R}^j) \times P(\mathrm{Imm}(\mathbb{R}^j, \mathbb{R}^n))$). The form on the path space $P(\mathrm{Inj}(\mathbb{R}^j, \mathbb{R}^n))$ is given by Chen's iterated integrals

$$A \otimes BA \to A_{dR}(P(\mathrm{Inj}(\mathbb{R}^j, \mathbb{R}^n))).$$

Then, the desired form on $P(\mathrm{Imm}(\mathbb{R}^j, \mathbb{R}^n))$ is obtained by the fiber integral along $C_1(\mathbb{R}^j)$. □

6 The cocycle-cycle pairing

Finally, we perform the pairing between the cycles in Section 4 and the cocycles in Section 5. The pairing is reduced to pairing between graphs and diagrams. Recall we have a zigzag of cochain maps

$$HGC_{n,j} \underset{p}{\leftarrow} PGC_{n,j} \overset{\simeq}{\leftarrow} DGC_{n,j} \underset{I}{\to} A^*_{dR}(\overline{\mathcal{K}}_{n,j}).$$

In Theorem 3.9, we showed $p^* : H^{top}(PGC) \to H^{top}(HGC)$ is surjective. As mentioned in Section 3, the 2-loop part $H^{top}(HGC(g = 2))$ is infinite-dimensional. In the rest of this article, we show the following injection property of the configuration space integrals.

Theorem 6.1 (Y.). *If $H \in H^{top}(DGC(g = 2))$ and $I^*(H) = 0$, we have $p^*(H) = 0$*

Suppose $k \geq 1$ is an integer and suppose p, q, r are integers which satisfy $p+q+r+1 = k$, $p, r \geq 1$, $q \geq 0$. Let H be a 2-loop, top graph cocycle of order $\leq k$, expressed as

$$H = \sum_i \frac{w(\Gamma_i)}{|\mathrm{Aut}(\Gamma_i)|}\Gamma_i.$$

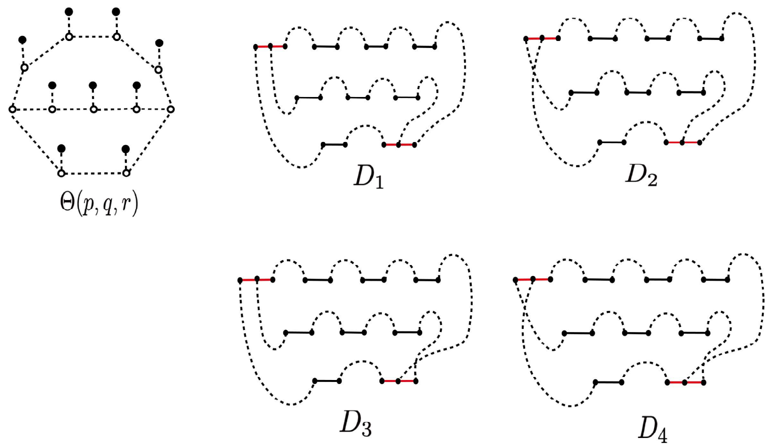


Figure 15: Graphs D_1, D_2, D_3 and D_4

Here, $w(\Gamma_i)$ is the coefficient of a graph Γ_i, divided by the number of automorphisms $|\mathrm{Aut}(\Gamma_i)|$. We always assume Γ_i has no orientation reversing automorphisms. Let $p_k^* : DGC^{top}(g=2) \to HGC^{top}(g=2,k)$ be the projection to the hairy graphs of order k. The following is the key proposition to show Theorem 6.1.

Proposition 6.2 (Counting formula)**.**

$$< I(H), d(\Theta(p,q,r)) >= \pm w(\Theta(p,q,r)).$$

where $\pm$ depends only on the oriented graph $\Theta(p,q,r)$.

Proof of Thm. Assume Proposition 6.2. Then if $I(H)$ is exact, $p_k^*(H) = 0$. □

We proceed to show Proposition 6.2. Consider the four graphs D_1, D_2, D_3, D_4 in Figure 15, which are obtained by performing STU relations to the hairy graph $\Theta(p,q,r)$. We can take orientations of these graphs so that the relation

$$w(\Theta(p,q,r)) = w(D_1) + w(D_2) + w(D_3) + w(D_4).$$

is satisfied in the graph cocycle $H = \sum_i \frac{w(\Gamma_i)}{|\mathrm{Aut}(\Gamma_i)|}\Gamma_i$. We show the integrals which do not vanish on the cycle $d(\Theta(p,q,r))$, are only the integrals associated with the graphs D_1, D_2, D_3, D_4.

Notation 6.3. *Suppose Γ_i has no orientation reversing automorphism. Define the paring of Γ_i and D_j by*

$$< \Gamma_i, D_j >= \begin{cases} 0 & (\textit{if } \Gamma_i \textit{ is not isomorphic to } D_j) \\ \pm 1 & (\textit{if } \Gamma_i \textit{ is isomorphic to } D_j) \end{cases}$$

The sign is positive (resp. negative) if the isomorphism preserves (resp. reserves) the orientation.

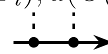
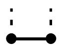
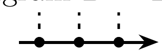
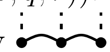
Lemma 6.4. *If the order of a graph Γ_i is less than or equal to k, we have*

$$< I(\Gamma_i), d(\Theta(p,q,r)) >= \pm \sum_{j=1,2,3,4} |Aut(D_j)| < \Gamma_i, D_j >,$$

Proof of Key prop. Assuming Lemma 6.4, we have

$$
\begin{aligned}
< I(H), d(\Theta(p,q,r)) > &= \pm \sum_i \frac{w(\Gamma_i)}{|\mathrm{Aut}(\Gamma_i)|} \sum_{j=1,2,3,4} |\mathrm{Aut}(D_j)| < \Gamma_i, D_j > \\
&= \pm(w(D_1) + w(D_2) + w(D_3) + w(D_4)) \\
&= \pm w(\Theta(p,q,r)).
\end{aligned}
$$

□

Proof of Lemma 6.4. After some observations, we can show that the pairing $< I(\Gamma_i), d(\Theta(p,q,r)) >$ is equal to counting graphs on the diagram $D = D(\Theta(p,q,r))$. On the segment , only  is counted. On the segment , only  or  is allowed, and  is not counted. On the other hand, we can show that decorated graphs are not counted. Then, there are four plain graphs counted, which are $D_1, \ldots, D_4$. Figure 16 shows how the graph D_2 is counted.

□

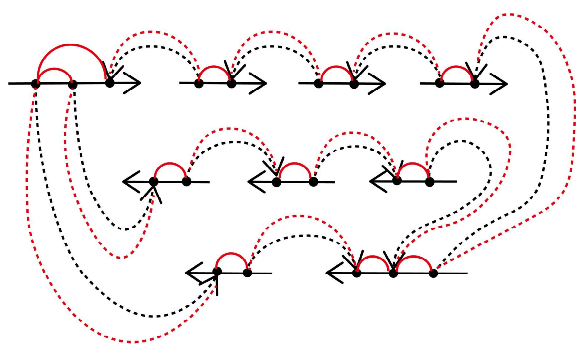


Figure 16: Graph D_2 is counted on the diagram $D(\Theta(p,q,r))$

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