Hyperbolic knots whose Upsilon invariants are convex

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1 Introduction

For a knot $K \subset S^3$, the *Upsilon invariant* $\Upsilon_K(t)\colon [0,2] \to \mathbb{R}$ is a concordance invariant introduced within the framework of knot Floer homology theory [18].

A knot is called an *L–space knot* if some positive Dehn surgery yields an L–space. For example, any positive torus knot is an L–space knot. In [2], Borodzik and Hedden showed that for an L–space knot, its Upsilon invariant is the Legendre transform of the function $2J(-x)$, where $J(x)$ is a certain function determined by its Alexander polynomial. As a corollary, we have the following.

Theorem 1.1 ([2])**.** *The Upsilon invariant of any L–space knot is a convex function.*

In view of this, Borodzik and Hedden gave the following question.

Question 1 (Question 1.4 of [2])**.** *For which knots is Υ_K a convex function?*

It is known that the following knots have convex Upsilon invariants:

- L–space knots ([2]).
- Alternating knots with a negative signature, more generally, *Floer thin knots* with a positive tau invariant ([1, 18]).
- Connected sum of knots whose Upsilon invariants are convex. (This follows immediately from the additivity for the connected sum operation, see subsection 2.2.)

On the other hand, there are knots whose Upsilon invariant is neither convex nor concave:

- Many cable operations for an L–space knot produce such Upsilon invariant. For example, the Upsilon invariant of the $(2,1)$–cable of the right handed trefoil is so. (see [21].)
- The closure of the 3–braid $(\sigma_1^2\sigma_2^2)^n\sigma_1\sigma_2$ for $n \geq 6$ (see [6]).

The following theorem is the main result.

Theorem 1.2. *There exist infinitely many hyperbolic knots satisfying the following*:

(1) *The Upsilon invariant is a convex function.*

(2) *Each knot is neither an L–space knot nor a Floer thin knot.*

(3) *The knots are mutually topologically non–concordant.*

The knots in Theorem 1.2 provide new answers to Borodzik and Hedden's question. Note that since the Upsilon invariant is a concordance invariant, if K is concordant to a knot whose Upsilon invariant is convex, Υ_K is also convex. Hence, it makes sense to construct mutually non-concordant knots.

The main theme of this paper is in accordance with [10].

2 Preliminaries

In this section, we review the *full knot Floer complex* CFK^∞ and the Upsilon invariant.

2.1 CFK^∞ and stably equivalence

For more details of CFK^∞, see, for example, [9, 14, 16].

For a knot $K \subset S^3$, the full knot Floer complex $\mathrm{CFK}^\infty(K)$ is a $\mathbb{Z} \oplus \mathbb{Z}$–filtered chain complex with $\mathbb{F}_2$ coefficient. $\mathrm{CFK}^\infty(K)$ can be drawn on the plane as follow: First, we assign each generator on the plane with filtration levels as the coordinates. Second, the differentials are drawn by arrows. Also, the generator with homological grading 0 shall be drawn with a white dot. see Figure 1.

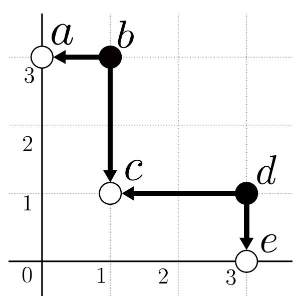

Figure 1: $\mathrm{CFK}^\infty(T(3,4))$. We can see that $\partial b = a + c$ and $\partial d = c + e$.
White vertices have homological grading 0, and hence each is a generator of $\mathrm{H}_0(\mathrm{CFK}^\infty(T(3,4))) \cong \mathbb{F}_2$.

Remark 2.1. More precisely, $\mathrm{CFK}^\infty(K)$ is a module over $\mathbb{F}_2[U, U^{-1}]$, where $\mathbb{F}_2[U, U^{-1}]$ is the Laurent polynomial ring with a formal variable U and $\mathbb{F}_2$ coefficient. The action of U commutes with the differential, lowers homological grading by 2 and lowers both filtration levels by 1. However, since the generator of the homology with 0 grading is important in our future discussion, we will not concern ourselves with this.

In general, it is difficult to compute $\mathrm{CFK}^\infty(K)$. However, the following methods are available:

- For a $(1,1)$–knot, there is a combinatorial method [8]. (A $(1,1)$–knot is a knot admitting 1-bridge decomposition by two solid tori.)
- For an L–space knot, CFK^∞ is determined by its Alexander polynomial [17].

- For a Floer thin knot, CFK^∞ is determined by its Alexander polynomial and its τ invariant [19].
- Computer program "SnapPy" [5].

Definition 2.2. Two full knot Floer complexes C_1 and C_2 are *stably equivalent* if there are acyclic $\mathbb{Z} \oplus \mathbb{Z}$–filtered chain complexes A_1 and A_2 such that $C_1 \oplus A_1$ is filtered chain homotopic to $C_2 \oplus A_2$.

The acyclic complex mainly used in this paper is a *box complex.* see Figure 2.

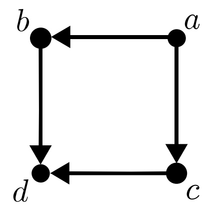

Figure 2: A box complex. The cycles are only d and $b + c$, but both are boundary cycles. Thus, it is an acyclic chain complex.

Example 2.3. Let C be a full knot Floer complex. Since a box complex is acyclic, the direct sum of C and a box complex is stably equivalent to the original complex C. For example, $\mathrm{CFK}^\infty(3_1)$, $\mathrm{CFK}^\infty(5_2)$ and $\mathrm{CFK}^\infty(7_2)$ are stably equivalent each other, see Figure 3.

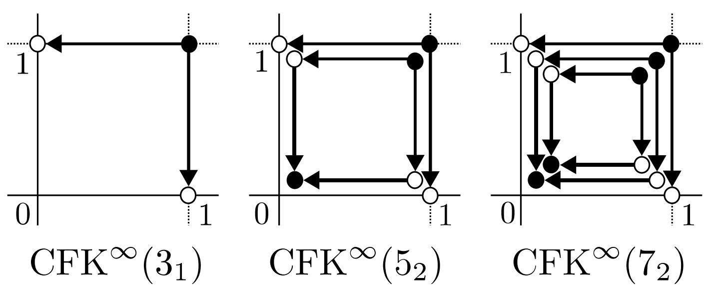

Figure 3: $\mathrm{CFK}^\infty(3_1)$, $\mathrm{CFK}^\infty(5_2)$ and $\mathrm{CFK}^\infty(7_2)$ are stably equivalent each other. The vertices of box complexes are actually on the grid, but are drawn slightly displaced.

Remark 2.4. If K_1 and K_2 are smoothly concordant, then $\mathrm{CFK}^\infty(K_1)$ and $\mathrm{CFK}^\infty(K_2)$ are stably equivalent [Hom]. Furthermore, to our knowledge, all concordance invariants derived from knot Floer theory (for example, τ, ε, Υ) are actually invariants on stably equivalent classes.

2.2 The Upsilon invariant

As mentioned above, for a knot $K \subset S^3$, the Upsilon invariant $\Upsilon_K(t)\colon [0,2] \to \mathbb{R}$ is a concordance invariant. It has the following features:

- $\Upsilon_K(t)$ is continuous and piecewise liner;
- $\Upsilon_K(t) = \Upsilon_K(2-t)$;

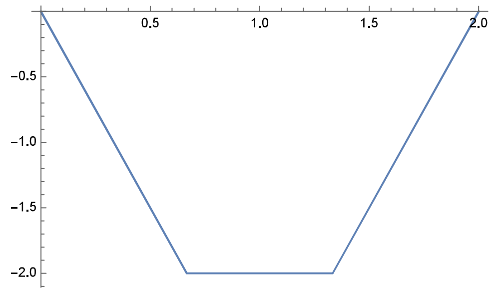

Figure 4: The Υ invariant of the torus knot $T(3,4)$.

- $\Upsilon_{-K}(t) = -\Upsilon_K(t)$ where $-K$ is the mirror image of K with reversed orientation;
- $\Upsilon_K(t)$ is additive under the connected sum, that is, $\Upsilon_{K\#L}(t) = \Upsilon_K(t) + \Upsilon_L(t)$;
- for small t, $\Upsilon_K(t) = -\tau(K)\cdot t$ where $\tau(K)$ is the *tau invariant* of K. (see [15] for the tau invariant.)

For example, Figure 4 shows the graph of the Upsilon invariant of the $(3,4)$–torus knot $T(3,4)$.

It was originally defined using the t–modified knot Floer complex, but Livingston gave an interpretation on the *full knot Floer complex* $\mathrm{CFK}^\infty(K)$ [13]. In fact, the Upsilon invariant Υ_K can be calculated from the filtration levels of generators of $\mathrm{H}_0(\mathrm{CFK}^\infty(K)) \cong \mathbb{F}_2$. Therefore, an acyclic chain complex does not affect the calculation of Υ_K. So, we have the following.

Proposition 2.5. *If* $\mathrm{CFK}^\infty(K_1)$ *and* $\mathrm{CFK}^\infty(K_2)$ *are stably equivalent, then* $\Upsilon_{K_1} \equiv \Upsilon_{K_2}$.

3 Proof of the main result

In this section, we give the details of Theorem 1.2.

Let n be a non-negative integer, and let $q \geq 4$ be an integer coprime to 3. Then $q = 3k+1$ or $q = 3k+2$ for some $k > 0$. A knot $K_n^{(3,q)}$ is defined as in Figures 5 and 6. Note that $K_0^{(3,q)}$ is the $(3,q)$–torus knot $T(3,q)$.

Two knots $K_n^{(3,q)}$ and $K_m^{(3,q')}$ are not equivalent when $n \neq m$ or $q \neq q'$ (this can be seen from their Alexander polynomials, see Lemma 3.2).

Theorem 1.2 follows from following lemmas.

Lemma 3.1. $\mathrm{CFK}^\infty(K_n^{(3,q)})$ *is stably equivalent to* $\mathrm{CFK}^\infty(T(3,q))$.

Proof. Since $K_n^{(3,q)}$ is a $(1,1)$–knot (in fact, we can give a specific $(1,1)$–decomposition of this knot), $\mathrm{CFK}^\infty(K_n^{(3,q)})$ can be combinatorially computed. (For the sake of space, we omit here the details of the calculation method, which is described in [10].) The results are shown in Figures 7 and 8. Recall that $K_0^{(3,q)} = T(3,q)$, hence $\mathrm{CFK}^\infty(K_n^{(3,q)})$ is stably equivalent to $\mathrm{CFK}^\infty(T(3,q))$.

□

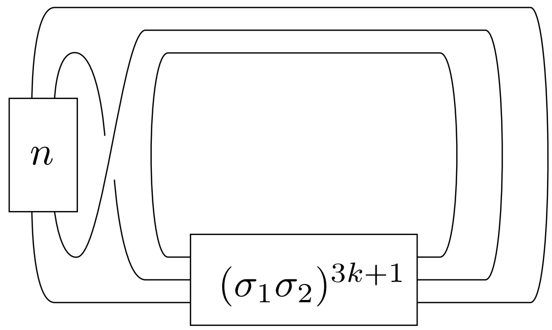

Figure 5: The case $q = 3k + 1$ ($k \geq 1$). σ_1 and σ_2 are the standard generators of the 3–braid group. The box with n contains n right handed vertical full-twists.

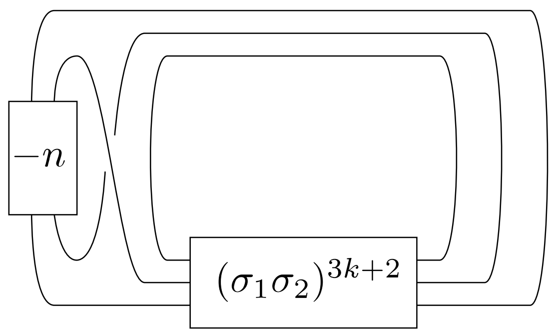

Figure 6: The case $q = 3k + 2$ ($k \geq 1$). The box with $-n$ contains n left handed vertical full-twists.

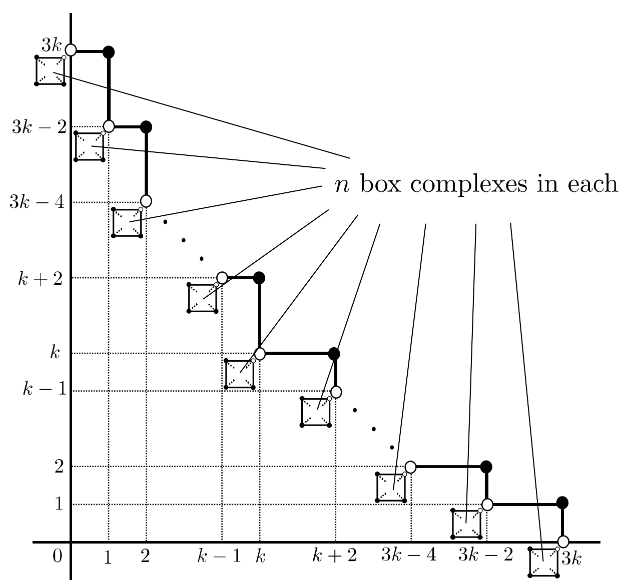

Figure 7: The complex $\mathrm{CFK}^\infty(K_n^{(3,3k+1)})$. This complex consists of the staircase complex, which is consistent with the complex of $\mathrm{CFK}^\infty(T(3, 3k + 1))$, and box complexes. The differential is represented by a line segment instead of an arrow, since the differential always lowers filtration levels.

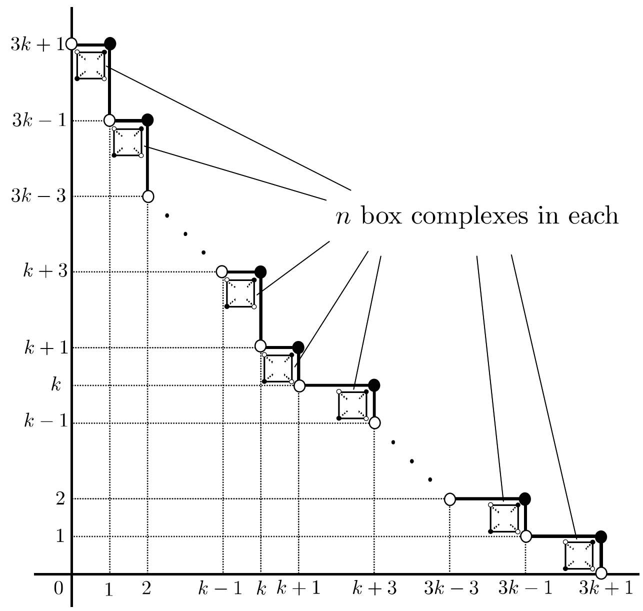

Figure 8: The complex $\mathrm{CFK}^\infty(K_n^{(3,3k+2)})$. The staircase complex is consistent with $\mathrm{CFK}^\infty(T(3,3k+2))$.

To prove the remain lemmas, the Alexander polynomial plays an important role.

Lemma 3.2. *The Alexander polynomial of $K_n^{(3,3k+1)}$ is given as*

$$\begin{aligned}\Delta_{K_n^{(3,3k+1)}}(t) = &\sum_{i=1}^{k}\{-nt^{3i+1}+(2n+1)t^{3i}-(n+1)t^{3i-1}\}\\ &-nt+(2n+1)-nt^{-1}\\ &+\sum_{i=1}^{k}\{-(n+1)t^{-3i+1}+(2n+1)t^{-3i}-nt^{-3i-1}\}.\end{aligned}$$

Also, the Alexander polynomial of $K_n^{(3,3k+2)}$ is given as

$$\begin{aligned}\Delta_{K_n^{(3,3k+2)}}(t) = &\sum_{i=1}^{k}\{(n+1)t^{3i+1}-(2n+1)t^{3i}+nt^{3i-1}\}\\ &(n+1)t-(2n+1)+(n+1)t^{-1}\\ &+\sum_{i=1}^{k}\{nt^{-3i+1}-(2n+1)t^{-3i}+(n+1)t^{-3i-1}\}.\end{aligned}$$

Proof. For a knot K, $\Delta_K(t)=\sum_{d,i}(-1)^d t^i\cdot \operatorname{rank}\widehat{\mathrm{HFK}}_d(K;i)$ [16]. $\widehat{\mathrm{HFK}}$ can be easily computed by $\mathrm{CFK}^\infty(K)$. So, we obtain the conclusion from the proof of Lemma 3.1. □

Lemma 3.3. *For $n \geq 1$, $K_n^{(3,q)}$ satisfies the following*:

(1) $K_n^{(3,q)}$ *is a hyperbolic knot.*

(2) $K_n^{(3,q)}$ *is neither an L–space knot nor a Floer thin knot.*

Proof. Let $n \geq 1$.

(1) By Lemma 3.2, $K_n^{(3,q)}$ is not a torus knot. Hence, assume that $K_n^{(3,q)}$ is a satellite knot. First, since $K_n^{(3,q)}$ is a $(1,1)$–knot, it is a prime knot. Also, $K_n^{(3,q)}$ admits a three-bridge decomposition. By [20], a prime satellite knot has a bridge number at least 4. This is a contradiction. Therefore, $K_n^{(3,q)}$ is a hyperbolic knot.

(2) Since non-zero coefficients of the Alexander polynomial of an L–space knot are ± 1 [17], the knot $K_n^{(3,q)}$ is not an L–space knot by Lemma 3.2. (In fact, the knot Floer full complex of an L–space knot is a staircase type [12, 17], but that of $K_n^{(3,q)}$ $(n \geq 1)$ is not such type.)

By [19], the full knot Floer complex of a Floer thin knot consists of staircase complexes and box complexes such that all arrows have length one. So, $K_n^{(3,q)}$ is not a Floer thin knot since $\mathrm{CFK}^\infty(K_n^{(3,q)})$ has an arrow of length two.

□

Lemma 3.4. *For a fixed integer q, the family $\{K_n^{(3,q)}\}_{n=0}^\infty$ contains infinitely many mutually topologically non-concordant knots.*

Proof. The Fox–Milnor condition [7] implies that if K and L are topologically concordant, then the product of the determinant of K and one of L is a square number. By Lemma 3.2, we have

$$\det(K_n^{(3,3k+1)}) = \begin{cases} 4n+3 & k\text{: odd} \\ 4n+1 & k\text{: even,} \end{cases}$$

$$\det(K_n^{(3,3k+2)}) = \begin{cases} 4n+1 & k\text{: odd} \\ 4n+3 & k\text{: even.} \end{cases}$$

For the case $q = 3k+1$ with odd k, if $4n+3$ and $4m+3$ are distinct prime numbers, then $\det(K_n^{(3,3k+1)})\det(K_m^{(3,3k+1)}) = (4n+3)(4m+3)$ is not a square number, and hence $K_n^{(3,3k+1)}$ and $K_m^{(3,3k+1)}$ are not topologically concordant. The same is true in other cases.

There are infinitely many prime integers of form $4n+1$ or $4n+3$. So, there are infinitely many n, m such that $K_n^{(3,q)}$ and $K_m^{(3,q)}$ are not topologically concordant. □

Remark 3.5. We expect that $K_n^{(3,q)}$ and $K_m^{(3,q)}$ are not concordant for $n \neq m$, but we could not prove it. For several n, m, we can verify that they are not concordant by the irreducibility of the Alexander polynomial (see Example 3.6) or the Levine–Tristram signature.

Example 3.6. Consider two knots $K_1^{(3,4)}$ and $K_{15}^{(3,4)}$. Since $\det(K_1^{(3,4)}) \cdot \det(K_{15}^{(3,4)}) = 21^2$, we cannot determine whether they are concordant by using the manner in the proof of Lemma 3.4. However, according to the program Mathematica, their Alexander polynomials are irreducible. This shows that $K_1^{(3,4)}$ and $K_{15}^{(3,4)}$ are not topologically concordant from the Fox–Milnor condition.

Proof of Theorem 1.2. By Proposition 2.5 and Lemma 3.1, $\Upsilon_{K_n^{(3,q)}} \equiv \Upsilon_{T(3,q)}$. Since any positive torus knot is an L–space knot, $\Upsilon_{K_n^{(3,q)}}$ is a convex function. The conclusion follows from Lemmas 3.3 and 3.4. □

Comments. In general, it is quite difficult to construct a knot which has a given full knot Floer complex. Hence, it is also difficult to construct a knot with a given concordance invariant. The author computed the full knot Floer complexes for knots that are neither L–space knots nor quasi–alternating knots in the knot table, starting from the lowest crossing number. Then, it turns out that the knot 10_{128} has the full knot Floer complex that is stably equivalent to that of the $(3,4)$–torus knot. The knots $K_n^{(3,q)}$ are constructed based on this knot. However, because of the discussion made in the proof of Lemma 3.4, the family does not contain the knot 10_{128}.

A Appendix: The integral of the Upsilon invariant

The Upsilon invariant is continuous, so it is integrable. For $K \subset S^3$, let $\int \Upsilon_K$ denote $\int_0^2 \Upsilon_K(t)dt$. There are two previous studies using $\int \Upsilon$.

- In [21], Tange gives the explicit formula of $\int \Upsilon_K$ for the torus knot and a large class of iterated cable L–space knots. His motivation of this value seems to be the S^1–integral value of the Levine–Tristram signature.
- In a recent work [4], Borodzik and Teragaito associated $\tau, \int \Upsilon_K$ with an invariant of a singularity for a corresponding algebraic knot.

Remark that the purposes of the above two studies are elsewhere.

A simple question comes to mind: is there a pair of knots (K, L) that is $\int \Upsilon_K = \int \Upsilon_L$ but $\Upsilon_K \not\equiv \Upsilon_L$? Of course, there are many such examples, as information is lost when integrating. Then, based on [4], consider a pair of knots (K, L) satisfying

$$\int \Upsilon_K = \int \Upsilon_L \text{ and } \tau(K) = \tau(L), \text{ but } \Upsilon_K \not\equiv \Upsilon_L.$$

As a result, we obtain the following.

Theorem A.1. *For each of the following tuples of knots, they have same τ invariants and $\int \Upsilon_K$, but have distinct Υ invariants:*

(1) $(T(5, 25k+2),\ T(3, 9k+1)_{2,64k+5},\ \mathrm{BK}_{10k,-10k})$ $(k \geq 1)$,

(2) $(T(2l, 2l^2+1),\ T(2l+1, 2l^2-2l+1))$ $(l \geq 2)$,

(3) $(T(3, 9n+1; 2, s),\ \mathrm{BK}_{n,-(s+5n-2)})$ $(n \geq 1,\ s \geq 2)$,

where

	torus	satellite	hyperbolic
torus	$T(2l, 2l^2+1)$, $T(2l+1, 2l^2-l+1)$	$T(5, 25k+2)$, $T(3, 9k+1)_{2,64k+5}$	$T(5, 25k+2)$, $\mathrm{BK}_{10k,-10k}$
satellite		The same cables of another*	$T(3, 9k+1))_{2,64k+5}$ $\mathrm{BK}_{10k,-10k}$
hyperbolic			$T(3, 9n+1; 2, s)$, $\mathrm{BK}_{n,-(s+5n-2)}$

Table 1: The table of the knots in Theorem A.1, distributed into torus knots, satellite knots and hyperbolic knots entries. Each parameter satisfies $l \geq 2$, $k \geq 1$, $n \geq 1$ and $s \geq 2$, respectively. About the mark *, by [21], the (p, q)–cable operation with $2gp \leq q$ for an L–space knot with genus g yields expected knots.

- *$T(p, q)_{r,s}$ is the (r, s)–cable of the torus knot $T(p, q)$,*
- *$T(p, q; r, s)$ is the twisted torus knot, which are defined to be the (p, q) torus knot with s full twists on r adjacent strands where $0 < r < p$, and*
- $\mathrm{BK}_{n,m}$ *is the knot defined in Section 3 of* [3].

Table 1 is the knots in Therem A.1, distributed into torus knots, satellite knots and hyperbolic knots entries. Remark that $T(3, 9k+1)_{2,64k+5}$ $(k \geq 1)$ is an algebraic knot, since $64k + 5 > 2 \cdot 3 \cdot (9k + 1)$ holds. Also, all knots in Theorem 1.2 are L–space knots. The proof of Theorem A.1 is in preparation [11].

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