Hypersationary Subsets of $\mathcal{P}_\kappa\lambda$

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Abstract

Let κ be an uncountable regular cardinal, $\kappa \subseteq A$. We study the notion of n-stationarity on $\mathcal{P}_\kappa(A)$ introduced by H. Brickhill, S. Fuchino and H. Sakai and a minor modification of the same. We set a posible foundational framework for an exploration into the adaptability of results presented in Bagaria's article "Derived Topologies on Ordinals and Stationary Reflection" to the more general context of $\mathcal{P}_\kappa(A)$.

1 Introduction

The exploration of combinatorial properties of $\mathcal{P}_\kappa\lambda = \{x \subseteq \lambda : |x| < \kappa\}$ where κ denotes an uncountable regular cardinal and $\kappa \leq \lambda$, boasts a rich historical background [10, 11, 12, 6, 7, 8, 9, 14]. Appropriate formulations of the generalisation of properties from ordinals to the case of $\mathcal{P}_\kappa\lambda$ may mostly lead to compelling results with significantly higher levels of consistency strength.

In Bagaria's paper "Derived Topologies on Ordinals and Stationary Reflection" (See [1]), an iterated notion of stationary reflection for a given limit ordinal α was introduced. Specifically, $A \subseteq \alpha$ is 0-*stationary* in α if and only if it is unbounded in α. For $\xi > 0$, $A \subseteq \alpha$ is ξ-stationary in α if and only if for every $\zeta < \xi$ and every S ζ-stationary in α, there is $\beta < \alpha$ such that $S \cap \beta$ is ζ-stationary in β. Building upon this, Bagaria, Magidor, and Sakai demonstrated in [2] a profound connection between this stronger form of stationarity and the concept of indescribability. They proved that in L a regular cardinal is $n+1$-stationary if and only if it is Π^1_n indescribable.

Subsequently, in [3], Bagaria demonstrated that sets simultaneously reflecting pairs of ξ-stationarity subsets of ordinals (ξ-simultaneously-stationary sets) played a pivotal role in characterising the discreteness of derived topologies on ordinals. As a consequence, Bagaria established a correlation between this new notion of stationarity and the completeness of **GLP** logics [3, 4, 5], thus underscoring its significance beyond the realm of set theory. Bagaria also showed that the set $\mathcal{I}^\xi_\alpha$ -comprising all non-simultaneously-stationary subsets of α- is a proper ideal if and only if α is ξ-simultaneously-stationary in α. Lastly, he extended the findings from [2] to encompass arbitrary ordinals ξ by introducing a natural new notion of $\Pi\xi^1$ indescribability.

Inspired by insights from the exploration of higher stationarity on ordinals [1, 2, 3], a pioneering effort was initiated to define higher stationarity within $\mathcal{P}_\kappa\lambda$. In [16], H. Brickhill, S. Fuchino, and H. Sakai proposed a definition of n-stationarity in $\mathcal{P}_\kappa(A)$, where κ is a regular cardinal and $\kappa \subseteq A$. While the consistency strength of hyperstationarity on ordinals is rather low in the large-cardinal hierarchy (below a measurable cardinal), its generalisation to $\mathcal{P}_\kappa(A)$ is possibly much stronger. Thus, the formulation of the appropriate generalisation of hyperstationarity for $\mathcal{P}_\kappa(A)$ and the development of its theory, in analogy with the notion of hyperstationarity for cardinals should allow more interesting applications at a much higher level, in terms of consistency strength. Our objective, therefore, is to explore the consequences of this definition and its alignment with results obtained by Bagaria in [3], all within the framework of $\mathcal{P}_\kappa(A)$.

2 Notation and framework

Throughout the subsequent discussion, κ will represent an uncountable regular cardinal, and A any set such that $\kappa \subseteq A$. Recall that $\mathcal{P}_\kappa(A)$ signifies the set $x \subseteq A : |x| < \kappa$. In [10, 11, 12], Jech introduced the following definitions:

Definition 2.1. *(T. Jech) Let κ be an uncountable regular cardinal and let A be a set of ordinals such that $\kappa \subseteq A$.*

1. *$S \subseteq \mathcal{P}_\kappa(A)$ is unbounded in $\mathcal{P}_\kappa(A)$ iff for any $X \in \mathcal{P}_\kappa(A)$ there is some $Y \in S$ such that $X \subseteq Y$.*
2. *$S \subseteq \mathcal{P}_\kappa(A)$ is closed in $\mathcal{P}_\kappa(A)$ iff for any $\{X_\xi : \xi < \beta\} \subseteq S$ with $\beta < \kappa$ and $X_\xi \subseteq X_\zeta$ for $\xi \leq \zeta < \beta$, $\bigcup_{\xi<\beta} X_\xi \in S$.*
3. *$S \subseteq \mathcal{P}_\kappa(A)$ is club of $\mathcal{P}_\kappa(A)$ iff S is closed and unbounded in $\mathcal{P}_\kappa(A)$,.*
4. *$S \subseteq \mathcal{P}_\kappa(A)$ is stationary in $\mathcal{P}_\kappa(A)$ iff for any C club in $\mathcal{P}_\kappa(A)$, $S \cap C \neq \varnothing$.*

The following are some well-known facts that can be found easily in the literature [12, 13, 15]. We provide some proofs of the them.

Lemma 2.2. *If $S \subseteq \mathcal{P}_\kappa(A)$ is a club of $\mathcal{P}_\kappa(A)$, then it is stationary in $\mathcal{P}_\kappa(A)$. And if $S \subseteq \mathcal{P}_\kappa(A)$ is stationary in $\mathcal{P}_\kappa(A)$, then it is unbounded in $\mathcal{P}_\kappa(A)$.*

Proof : Let S be be a club of $\mathcal{P}_\kappa(A)$, and pick any club C of $\mathcal{P}_\kappa(A)$. We well prove that in fact $S \cap C$ is a club of $\mathcal{P}_\kappa(A)$. It is clear that $S \cap C$ is closed, so we will prove that it is unbounded in $\mathcal{P}_\kappa(A)$. Let $X_0 \in \mathcal{P}_\kappa(A)$, as S, C are unbounded in $\mathcal{P}_\kappa(A)$, we may construct the following ω-sequence

$$X_0 \subsetneq X_1 \subsetneq X_2 \subsetneq \cdots \subsetneq X_n \subsetneq X_{n+1} \subsetneq \cdots$$

Where $X_i \in S$ if $i > 0$ is even and $X_i \in C$ otherwise. Then, $\bigcup_{i<\omega} X_{2i} \in S$ and $\bigcup_{i<\omega} X_{2i+1} \in C$, but $\bigcup_{i<\omega} X_{2i} = \bigcup_{i<\omega} X_{2i+1}$, therefore $\bigcup_{i<\omega} X_i \in S \cap C$.

For the second statement take $X \in S$, consider the club subset $C = \{Y \in \mathcal{P}_\kappa(A) : X \subseteq Y\}$. Pick $Z \in S \cap C$, then $Z \in S$ and $X \subseteq Z$, this is S is stationary in $\mathcal{P}_\kappa(A)$. □

Lemma 2.3. *Let D be a directed system, then for each $X \subseteq D$, there is a directed system D' such that $X \subseteq D' \subseteq D$ and $|D'| \leq |X| + \aleph_0$.*

Proof : Consider the set $Y := \{\{x, y\} : x, y \in X\}$ of all pairs of elements of X. Notice that $|Y| \leq |X| + \aleph_0$, let us say $Y = \{z_\alpha : \alpha < |Y|\}$. Now, for each $\alpha < |Y|$ we have that $\cup z_\alpha \in X$. Then, the set $D' := Y \cup \{\cup z_\alpha : \alpha < |Y|\}$ is such that $X \subseteq D' \subseteq D$ and $|D'| \leq |X| + \aleph_0$. □

Proposition 2.4. *$C \subseteq \mathcal{P}_\kappa(A)$ is closed if and only if for every directed set $X \subseteq C$ of cardinality $< \kappa$, $\bigcup X \in C$.*

Proof : ($\Rightarrow$) We prove this direction by induction on $|X| = \gamma$. Suppose that $X = \{A_\alpha : \alpha < \gamma\}$. By induction on $\alpha < \gamma$ we will define a continuous sequence of inductive systems contained in X. Suppose that for each $\beta < \alpha$, D_β is an inductive system such that $A_\beta \in D_\beta$, $|D_\beta| \leq |\beta| + \aleph_0$ and $D_\delta \subseteq D_\beta$ for all $\delta < \beta$. Define $X_\alpha := \bigcup_{\beta<\alpha} D_\beta \cup \{A_\alpha\}$, then, $|X_\alpha| = |\bigcup_{\beta<\alpha} D_\beta| \leq |\alpha| < \gamma$. And by Lemma 2.3. choose D_α to be a direct system of X such that $X_\alpha \subseteq D_\alpha \subseteq X$ and $|D_\alpha| \leq |X_\alpha| + \aleph_0$. Then $A_\alpha \in D_\alpha$, $|D_\alpha| < \gamma$ and $D_\beta \subseteq D_\alpha$ for all $\beta < \alpha$. Since each D_α has cardinality less than γ, by induction hypothesis $\bigcup D_\alpha \in C$ for each $\alpha < \gamma$. Then as C is closed

$\bigcup X = \bigcup_{\alpha<\gamma} D_\alpha \in C$.

($\Leftarrow$) Let C be a set of $\mathcal{P}_\kappa(A)$, and suppose $\{X_\xi : \xi < \beta\} \subseteq C$ whit $\beta < \kappa$ and $X_\xi \subseteq X_\zeta$ for $\xi \leq \zeta < \beta$. Let $X_{\xi_1}, X_{\xi_2} \in \{X_\xi : \xi < \beta\}$, we may assume $X_{\xi_1} \subseteq X_{\xi_2}$, then $X_{\xi_1} \cup X_{\xi_2} \subseteq X_{\xi_2}$. This is, $\{X_\xi : \xi < \beta\}$ is a directed subset of C of cardinality $\beta < \kappa$, then by hypothesis we have that $\bigcup_{\xi<\beta} X_\xi \in S$.

Our research builds upon the following definition proposed by H. Brickhill, S. Fuchino, and H. Sakai, as presented in [16], establishing a crucial starting point for our exploration.

Definition 2.5. *(H. Brickhill, S. Fuchino and H. Sakai [16]) Let $n < \omega$ and κ be a regular limit cardinal such that $\kappa \subseteq A$.*

1. *$S \subseteq \mathcal{P}_\kappa(A)$ is 0-stationary in $\mathcal{P}_\kappa(A)$ iff S is unbounded in $\mathcal{P}_\kappa(A)$.*
2. *$S \subseteq \mathcal{P}_\kappa(A)$ is n-stationary in $\mathcal{P}_\kappa(A)$ iff for all $m < n$ and all $T \subseteq \mathcal{P}_\kappa(A)$ m-stationary in $\mathcal{P}_\kappa(A)$, there is $B \in S$ such that*
   - *$\mu := B \cap \kappa$ is regular cardinal.*
   - *$T \cap \mathcal{P}_\mu(B)$ is m-stationary in $\mathcal{P}_\mu(B)$.*

We however introduced a subtle modification of the same, relaxing the condition over μ, this is, requiring only the existence of a μ regular contained in $B \cap \kappa$. And this is the definition of n-stationarity we are going to use from now, noticing when pertinent which results holds from the stronger Definition 2.5

Definition 2.6. *Let $n < \omega$ and κ be a regular limit cardinal such that $\kappa \subseteq A$.*

1. *$S \subseteq \mathcal{P}_\kappa(A)$ is 0-w-stationary in $\mathcal{P}_\kappa(A)$ iff S is unbounded in $\mathcal{P}_\kappa(A)$.*
2. *$S \subseteq \mathcal{P}_\kappa(A)$ is n-w-stationary in $\mathcal{P}_\kappa(A)$ iff for all $m < n$ and all $T \subseteq \mathcal{P}_\kappa(A)$ m-w-stationary in $\mathcal{P}_\kappa(A)$, there is $B \in S$ and μ regular cardinal such that*
   - *$\mu \subseteq B \cap \kappa$.*
   - *$T \cap \mathcal{P}_\mu(B)$ is m-w-stationary in $\mathcal{P}_\mu(B)$.*

Corollary 1. *For any $n < \omega$, if $S \subseteq \mathcal{P}_\kappa(A)$ is n-stationary in $\mathcal{P}_\kappa(A)$, then, $S \subseteq \mathcal{P}_\kappa(A)$ is n-w-stationary in $\mathcal{P}_\kappa(A)$.* $\square$

To enhance readability, we adopt the shorthand "S is n-w-stationary" instead of "S is n-w-stationary in $\mathcal{P}_\kappa(A)$" when the context is clear.

3 Results

Proposition 3.1. *If $S \subseteq \mathcal{P}_\kappa(A)$ is 1-w-stationary, then S is unbounded.*

Proof : Suppose that $S \subseteq \mathcal{P}_\kappa(A)$ 1-w-stationary and let $X \in \mathcal{P}_\kappa(A)$. The set $U_X := \{Y \in \mathcal{P}_\kappa(A) : X \subseteq Y\}$ is clearly unbounded in $\mathcal{P}_\kappa(A)$. Then there is $B \in S$ such that $\mu \subseteq B \cap \kappa$ is regular and $U_X \cap \mathcal{P}_\mu(B)$ is unbounded in $\mathcal{P}_\mu(B)$. Note that $\bigcup(U_X \cap \mathcal{P}_\mu(B)) = B$, because if $b \in B$, then $\{b\} \in \mathcal{P}_\mu(B)$ and so there is $Y \in U_X \cap \mathcal{P}_\mu(B)$ such that $\{b\} \subseteq Y$. Thus, $b \in Y \in U_X \cap \mathcal{P}_\mu(B)$ and $b \in \bigcup(U_X \cap \mathcal{P}_\mu(B)) = B$. Now we will see that $X \subseteq B$. Let $x \in X$. Then $x \in Y$ for all $Y \in U_X$, in particular $x \in Y$ for all $Y \in U_X \cap \mathcal{P}_\mu(B)$. Hence $x \in \bigcup(U_X \cap \mathcal{P}_\mu(B)) = B$. $\square$

Proposition 3.2. *$S \subseteq \mathcal{P}_\kappa(A)$ being n-w-stationary implies S is m-w-stationary for all $m < n$.*

Proof : We proceed by induction. The case $n = 0$ is precisely Proposition 3.1. Suppose we have the result for all $k < n$, and that $S \subseteq \mathcal{P}_\kappa(A)$ n-w-stationary. Let $m < n$ and take $T \subseteq \mathcal{P}_\kappa(A)$ to be l-w-stationary for some $l < m$. As S is n-w-stationary, there is some $B \in S$ and μ regular cardinal such that $\mu \subseteq B \cap \kappa$ and $T \cap \mathcal{P}_\mu(B)$ is l-w-stationary in $\mathcal{P}_\mu(B)$. Therefore, S is m-w-stationary. □

It is straightforward that if $S' \subseteq S \subseteq \mathcal{P}_\kappa(A)$ and S' is n-w-stationary, then S is n-w-stationary as well. The following proposition was stated by H. Brickhill, S. Fuchino and H. Sakai in [16] for Definition 2.5, we prove that this same result follows for Definition 2.6.

Proposition 3.3. *If $\mathcal{P}_\kappa(A)$ is 1-w-stationary in $\mathcal{P}_\kappa(A)$, then κ is weakly Mahlo.*

Proof : Suppose that $\mathcal{P}_\kappa(A)$ is 1-w-stationary in $\mathcal{P}_\kappa(A)$. We will prove that $R := \{\mu < \kappa : \mu \text{ is a regular limit cardinal}\}$ is stationary in κ. Let C be a club subset of κ and consider the following set $T_C = \{Y \in \mathcal{P}_\kappa(A) : \exists \alpha \in C \text{ such that } Y \cap \kappa \subsetneq \alpha \leq |Y|\}$.

- *T_C is unbounded in $\mathcal{P}_\kappa(A)$: Suppose $Y \in \mathcal{P}_\kappa(A)$ and let $\alpha \in C$ be such that $Y \cap \kappa \subsetneq \alpha$. Consider $\tilde{\alpha} := \{\delta \setminus \{0\} : \delta \in \alpha\}$, clearly $\tilde{\alpha} \cap \kappa = \{\varnothing\}$. Now $Z := Y \cup \{\tilde{\alpha}\}$ is such that $Z \cap \kappa = (Y \cup \{\tilde{\alpha}\}) \cap \kappa = (Y \cap \kappa) \cup (\{\tilde{\alpha}\} \cap \kappa) = Y \cap \kappa \subsetneq \alpha$. Moreover $\alpha \leq |\alpha| = |\tilde{\alpha}| \leq |Y \cup \tilde{\alpha}| = |Z|$, whence $Z \in T$. Hence, for every $Y \in \mathcal{P}_\kappa(A)$ there is $Z \in T$ such that $Y \subseteq Z$.*

Hence, by 1-w-stationary of $\mathcal{P}_\kappa(A)$, there is $B \in \mathcal{P}_\kappa(A)$ such that

- $\mu \subseteq B \cap \kappa$ is a regular cardinal ($\mu \in R$).
- $T_C \cap \mathcal{P}_\mu(B)$ is 0-w-stationary in $\mathcal{P}_\mu(B)$.

Note that $C \cap \mu$ is unbounded in μ : Let $\gamma < \mu$, then $\gamma \in \mu = B \cup \kappa \subseteq B$, also since μ is regular cardinal $|\gamma| < \mu$, thus $\gamma \in \mathcal{P}_\mu(B)$. Then, there is $Y \in T_C \cap \mathcal{P}_\mu(B)$ such that $\gamma \subseteq Y$ (and so $\gamma \subseteq Y \cap \kappa$). As $Y \in T$, there is some $\alpha \in C$ such that $Y \cap \kappa \subsetneq \alpha \leq |Y|$. But then $\gamma \subsetneq Y \cap \kappa \subsetneq \alpha \leq |Y| < \mu$. This is $\alpha \in C \cap \mu$ and $\gamma < \alpha$.

As C is closed, $C \cap \mu$ is unbounded in μ implies $\mu \in C$. Therefore $\mu \in C \cap R$, and so $R = \{\mu < \kappa : \mu \text{ is a regular cardinal}\}$ is stationary in κ. □

Corollary 2. *(H. Brickhill, S. Fuchino and H. Sakai) If $\mathcal{P}_\kappa(A)$ is 1-stationary in $\mathcal{P}_\kappa(A)$, then κ is weakly Mahlo.* □

Previous Corollary follows straightforward from Corollary 1. The advantage of w-stationarity (Definition 2.6) is that, in fact, the converse of Poposition 3.3 is also true. Obtaining thereof κ weakly Mahlo as a necessary and sufficient condition for $\mathcal{P}_\kappa(A)$ to be 1-w-stationary.

Proposition 3.4. *If κ is weakly Mahlo, then $\mathcal{P}_\kappa(A)$ is 1-w-stationary in $\mathcal{P}_\kappa(A)$.*

Proof : Suppose that κ is weakly Mahlo. Then, the set $R = \{\mu < \kappa : \mu \text{ is a regular limit cardinal}\}$ is stationary in κ. Let $T \subseteq \mathcal{P}_\kappa(A)$ be 0-stationary in $\mathcal{P}_\kappa(A)$, and construct the following transfinite sequence

$X_0 \in T$.
$X_{\alpha+1} \in T$ is such that $X_{\alpha+1} \supsetneq X_\alpha \cup \alpha$.

$X_\gamma \in T$ is such that $X_\gamma \supsetneq \bigcup_{\alpha<\gamma}[X_\alpha \cup \alpha]$, for $\gamma < \kappa$ limit.

This sequence is well defined. Successor and limit step may be performed since T is unbounded and κ is regular; $|X_\alpha|, |\alpha| < \kappa$ and so $X_\alpha \cup \alpha \in \mathcal{P}_\kappa(A)$. Also from $\gamma < \kappa$ we get $\bigcup_{\alpha<\gamma}[X_\alpha \cup \alpha] \in \mathcal{P}_\kappa(A)$. So defined $\{X_\alpha : \alpha < \kappa\} \subseteq T$ is an strict ascending chain.

$\cdot\ U := \{\alpha < \kappa : \exists \beta < \kappa$ s.t. $|X_\beta| = \alpha\}$ is unbounded in κ : Let $\delta < \kappa$. As κ is a regular limit cardinal $|\delta|^+ < \kappa$. Then $X_{|\delta|^+ +1} \supseteq X_{|\delta|^+} \cup |\delta|^+$. Note that $\delta < |\delta|^+ \leq |X_{|\delta|^+ +1}| < \kappa$. Then, for $\alpha := |X_{|\delta|^+ +1}| < \kappa$, there exists $\beta := |\delta|^+ + 1 < \kappa$ such that $|X_\beta| = \alpha > \delta$. This is $\alpha \in U$ and $\delta < \alpha < \kappa$.

Since R is stationary in κ, there is $\mu \in R$ such that $U \cap \mu$ is unbounded in μ. We may now construct the following subsequence:

Pick $\delta < \mu$. Then, there is $\delta_0 \in U \cap \mu$ such that $\delta < \delta_0$, and so there is $\beta_0 < \kappa$ such that $|X_{\beta_0}| = \delta_0 < \mu$. Given X_{β_α} let $X_{\beta_{\alpha+1}}$ be such that $|X_{\beta_\alpha}| < |X_{\beta_{\alpha+1}}| < \mu$; and for $\alpha < \mu$ limit, let X_{β_α} be such that $|\bigcup_{\xi<\alpha} X_{\beta_\xi}| < |X_{\beta\alpha}| < \mu$. Notice that $\beta_\alpha \neq \beta_{\alpha'}$ for $\alpha \neq \alpha'$ and since $\{X_{\beta_\alpha} : \alpha < \mu\} \subseteq \{X_\alpha : \alpha < \kappa\}$, we have that $\{X_{\beta_\alpha} : \alpha < \mu\}$ is also a chain. Since $|X_{\beta_\alpha}| < \kappa$, for all $\alpha < \mu < \kappa$ and κ is regular, $\bigcup_{\alpha<\mu} X_{\beta_\alpha} \in \mathcal{P}_\kappa(A)$.

Let $B := \bigcup_{\alpha<\mu} X_{\beta_\alpha}$, and notice that since $\{X_{\beta_\alpha} : \alpha < \mu\}$ forms a strictly ascending chain, B is the union of at most μ many sets of cardinality less than μ, so that $|B| = \mu$. To conclude the proof we will show that B and μ are as we wanted, this is

(i) $\mu \subseteq B \cap \kappa$: First notice that, if $\alpha < \alpha'$ then $X_{\beta_\alpha} \subsetneq X_{\beta_{\alpha'}}$, and since $\{X_\alpha : \alpha < \kappa\}$ is strict ascending, this implies $\beta_\alpha < \beta_{\alpha'}$. Notice that, for all $\alpha < \mu$, we have $\beta_\alpha \subseteq X_{\beta_{\alpha+1}} \subseteq B$. Also, it is easily proved by induction that $\alpha \leq \beta_\alpha$ for all $\alpha < \mu$. Hence, $\sup_{\alpha<\mu} \beta_\alpha = \bigcup_{\delta<\alpha} \beta_\delta \subseteq B$ and $\mu = \sup_{\alpha<\mu} \alpha \leq \sup_{\alpha<\mu} \beta_\alpha$. Therefore $\mu \subseteq B$ and so $\mu \subseteq B \cap \kappa$.

(ii) $T \cap \mathcal{P}_\mu(B)$ is unbounded in $\mathcal{P}_\mu(B)$: Let $X \in \mathcal{P}_\mu(B)$. Then $X \subseteq \bigcup_{\alpha<\mu} X_{\beta_\alpha}$ and $|X| < \mu$. As $|B| = \mu$ is regular, we get that X is not unbounded in B. Then $X \subseteq X_{\beta_\alpha}$ for some $\alpha <\ \mu$. But $X_{\beta_\alpha} \subseteq \bigcup_{\alpha<\mu} X_{\beta_\alpha} = B$ and $|X_{\beta_\alpha}| < \mu$. Thus, there is $X_{\beta_\alpha} \in T \cap \mathcal{P}_\mu(B)$ such that $X \subseteq X_{\beta_\alpha}$. $\square$

Corollary 3. *$\mathcal{P}_\kappa(A)$ is 1-w-stationary in $\mathcal{P}_\kappa(A)$ if and only if κ is weakly Mahlo.* $\square$

Notice that, in the proof of Proposition 3.4 we can in fact start the sequence $\{X_\alpha : \alpha < \kappa\}$ with $X_0 \supseteq y$ for any given $y \in \mathcal{P}_\kappa(A)$. Thus at the end of the proof we will get $y \subseteq B$ and $B \cap \kappa$ contains a regular cardinal. Therefore, if κ is weakly Mahlo and $T \subseteq \mathcal{P}_\kappa(A)$ is unbounded in $\mathcal{P}_\kappa(A)$, the set $W := \{x \in \mathcal{P}_\kappa(A)$: exists μ is regular limit cardinal such that $\mu \subseteq x \cap \kappa$ and $T \cap \mathcal{P}_\mu(x)$ is unbounded in $\mathcal{P}_\mu(x)\}$ is unbounded in $\mathcal{P}_\kappa(A)$.

Proposition 3.5. *Let κ be the least weakly Mahlo cardinal, then $\mathcal{P}_\kappa(A)$ is not 2-w-stationary.*

Proof : Towards a contradiction, suppose that $\mathcal{P}_\kappa(A)$ is 2-stationary. As κ is weakly Mahlo, by Theorem 3.4 we have that $\mathcal{P}_\kappa(A)$ is 1-w-stationary. Then, there is $B \in \mathcal{P}_\kappa(A)$ and μ regular cardinal such that $\mu \subseteq B \cap \kappa$ such that $\mathcal{P}_\kappa(A) \cap \mathcal{P}_\mu(B)$ is 1-w-stationary in $\mathcal{P}_\mu(B)$. From $B \in \mathcal{P}_\kappa(A)$ and $\mu \subseteq B \cap \kappa$ we get that $\mu < \kappa$. But $\mathcal{P}_\kappa(A) \cap \mathcal{P}_\mu(B) = \mathcal{P}_\mu(B)$, and then $\mathcal{P}_\mu(B)$ is 1-w-stationary in $\mathcal{P}_\mu(B)$, but again by Proposition 3.3 this implies μ weakly Mahlo. $\square$

Proposition 3.6. *If κ is weakly Mahlo, then $C \subseteq \mathcal{P}_\kappa(A)$ club implies C is 1-w-stationary.*

Proof : Suppose that κ is weakly Mahlo, we may then perform a similar proof to the one we did for Proposition 3.4. For each unbounded T of $\mathcal{P}_\kappa(A)$, we will however, construct the main sequence as follows

$X_0 \in T$. And $Y_0 \in C$ such that $X_0 \subseteq Y_0$
$X_{\alpha+1} \in T$ is such that $X_{\alpha+1} \supsetneq X_\alpha \cup \alpha \cup Y_\alpha$. And $Y_{\alpha+1} \in C$ such that $X_{\alpha+1} \subseteq Y_{\alpha+1}$
$X_\gamma \in T$ is such that $X_\gamma \supsetneq \bigcup_{\alpha<\gamma}[X_\alpha \cup \alpha \cup Y_\alpha]$ for $\gamma < \kappa$ limit.

And completely analogous to Proposition 3.4 we get $B := \bigcup_{\alpha<\mu} X_{\beta_\alpha} \in \mathcal{P}_\kappa(A)$ and μ regular cardinal, such that $\mu \subseteq B \cap \kappa$ and $T \cap \mathcal{P}_\mu(B)$ is unbounded in $\mathcal{P}_\mu(B)$.

So we are left to prove that $B \in C$. First, we will prove that $\bigcup_{\alpha<\mu} X_{\beta_\alpha} = \bigcup_{\alpha<\mu} Y_{\beta_\alpha}$. Let $z \in \bigcup_{\alpha<\mu} X_{\beta_\alpha}$, this is $z \in X_{\beta_\alpha}$ for some $\alpha < \mu$. By construction $X_{\beta_\alpha} \subseteq Y_{\beta_\alpha}$, then $z \in Y_{\beta_\alpha} \subseteq \bigcup_{\alpha<\mu} Y_{\beta_\alpha}$. Conversely, if $z \in \bigcup_{\alpha<\mu} Y_{\beta_\alpha}$ then $z \in Y_{\beta_\alpha}$ for some $\alpha < \mu$. Since for all $\alpha < \mu$, $X_{\beta_\alpha} \subsetneq X_{\beta_{\alpha+1}}$, we have $X_{\beta_\alpha+1} \subseteq X_{\beta_{\alpha+1}}$. Moreover, by construction (successor step) we have that $Y_{\beta_\alpha} \subseteq X_{\beta_\alpha+1} \subseteq X_{\beta_{\alpha+1}}$. Therefore $z \in X_{\beta_{\alpha+1}}$ and so $z \in \bigcup_{\alpha<\mu} X_{\beta_\alpha}$.

Now, $\{Y_{\beta_\alpha} : \alpha < \mu\}$ is clearly an ascending sequence of elements of C. Then, as C is closed, we get that $\bigcup_{\alpha<\mu} Y_{\beta_\alpha} \in C$. But $B = \bigcup_{\alpha<\mu} X_{\beta_\alpha} = \bigcup_{\alpha<\mu} Y_{\beta_\alpha}$, then $B \in C$. $\square$

Recall that in the ordinal case in [3]

$$S \subseteq \kappa \text{ club } \rightarrow S \text{ 1-w-stationary } \leftrightarrow S \text{ stationary } \rightarrow S \text{ unbounded}$$

By the previous propositions, in the case $\mathcal{P}_\kappa(A)$ when κ is weakly Mahlo, we have:

$$S \subseteq \mathcal{P}_\kappa(A) \text{ club } \rightarrow S \text{ 1-w-stationary } \rightarrow S \text{ stationary } \rightarrow S \text{ unbounded .}$$

Unfortunately, the correspondence between 1-stationarity and stationarity does not extend to $\mathcal{P}_\kappa(A)$.

Proposition 3.7. *The condition $S \subseteq \mathcal{P}_\kappa(A)$ is stationary in $\mathcal{P}_\kappa(A)$ does not imply that S is 1-w-stationary in $\mathcal{P}_\kappa(A)$.*

Proof : First let us prove the following facts:

- $C_0 = \{X \in \mathcal{P}_\kappa(A) : X \cap \kappa \text{ is a cardinal}\}$ is a club subset of $\mathcal{P}_\kappa(A)$: Let $Y \in \mathcal{P}_\kappa(A)$, and let α be the least cardinal less than κ such that $\alpha \geq \sup(Y \cap \kappa)$. (Such an ordinal exists because $|Y| < \kappa$ and κ weakly Mahlo). Define $X = Y \cup \alpha$, clearly $X \in \mathcal{P}_\kappa(A)$ and $X \cap \kappa = \alpha$. This is $Y \subseteq X \in C_0$. Consider now an increasing sequence $\langle X_\beta : \beta < \gamma\rangle$ of $\gamma < \kappa$ elements of C. Then $\langle X_\beta \cap \kappa : \beta < \gamma\rangle$ is an increasing sequence of cardinals less than κ, soits limit is also a cardinal less than κ. Hence $(\bigcup_{\beta<\gamma} X_\beta) \cap \kappa = \bigcup_{\beta<\gamma}(X_\beta \cap \kappa)$ is a cardinal, and so $\bigcup_{\beta<\gamma} X_\beta \in C_0$.
- $S = \{X \in \mathcal{P}_\kappa(A) : X \cap \kappa \text{ is a cardinal} \wedge cof(X \cap \kappa) < X \cap \kappa\}$ is a stationary subset of $\mathcal{P}_\kappa(A)$: Let C_1 be a club of $\mathcal{P}_\kappa(A)$ then $C := C_0 \cap C_1$ is also a club. Let $X_0 \in C$ be such that $X_0 \cap \kappa > \omega$ and $\langle X_n : n < \omega\rangle$ is an increasing sequence of elements of C, then $\langle X_n \cap \kappa : n < \omega\rangle$ is an increasing sequence of cardinals greater than ω, and so $cof(\bigcup_{n<\omega}(X_n \cap \kappa)) < \bigcup_{n<\omega}(X_n \cap \kappa)$. Hence $\bigcup_{n<\omega} X_n \in C \cap S$.

Now, towards a contradiction suppose S is 1-w-stationary. Then for C_0 it must exists $B \in S$ and $\mu < \kappa$ regular cardinal such that $\mu \subseteq B \cap \kappa$ and $T \cap \mathcal{P}_\mu(B)$ is unbounded in $\mathcal{P}_\mu(B)$. From $B \in S$ we get that $B \cap \kappa$ is a singular cardinal, then $\mu < B \cap \kappa$. Moreover, there is some cardinal α such that $\mu < \alpha < B \cap \kappa$ (take $\alpha = \mu^+$), whence $\alpha \in B$ and so $\{\alpha\} \in \mathcal{P}_\mu B$. Since $T \cap \mathcal{P}_\mu(B)$ is unbounded in $\mathcal{P}_\mu(B)$, there must be some $x \in T \cap \mathcal{P}_\mu(B)$ such that $\alpha \in x$. But then $\alpha \in x \cap \kappa$, and so $\mu < \alpha \leq x \cap \kappa \leq |x|$. Contradicting the fact that $x \in \mathcal{P}_\mu B$. $\square$

From previous proposition, and Corollary 1, we conclude that Proposition 3.7 also hols for Definition 2.5, this is:

Corollary 4. *The condition $S \subseteq \mathcal{P}_\kappa(A)$ is stationary in $\mathcal{P}_\kappa(A)$ does not imply that S is 1-stationary in $\mathcal{P}_\kappa(A)$.* $\square$

Theorem 3.8. *If $\mathcal{P}_\kappa(A)$ is 2-w-stationary in $\mathcal{P}_\kappa(A)$, then κ is 2-weakly Mahlo i.e. the set $\{\alpha < \kappa : \alpha$ is weakly mahlo $\}$ is stationary in κ.*

Proof : Suppose that $\mathcal{P}_\kappa(A)$ is 2-w-stationary in $\mathcal{P}_\kappa(A)$, we shall prove that the set $E := \{\mu < \kappa : \mu \text{ is weakly mahlo}\}$ is stationary in κ. By Proposition 3.2 the fact that $\mathcal{P}_\kappa(A)$ is 2-stationary implies $\mathcal{P}_\kappa(A)$ is 1-w-stationary and so κ is weakly Mahlo. Let C be a club subset of κ and consider the set $T := \{X \in \mathcal{P}_\kappa(A) : \exists \alpha \in C \text{ s.t. } X \cap \kappa \subseteq \alpha \leq |X|\}$.

- T is unbounded in $\mathcal{P}_\kappa(A)$: Suppose $Y \in \mathcal{P}_\kappa(A)$. Let $\alpha \in C$ be such that $Y \cap \kappa \subseteq \alpha$. Consider $\tilde{\alpha} := \{\delta \backslash \{0\} : \delta \in \alpha\}$, clearly $\tilde{\alpha} \cap \kappa = \{\varnothing\}$. Now $Z := Y \cup \{\tilde{\alpha}\}$ is such that $Z \cap \kappa = (Y \cup \{\tilde{\alpha}\}) \cap \kappa = (Y \cap \kappa) \cup (\{\tilde{\alpha}\} \cap \kappa) = Y \cap \kappa \subseteq \alpha$. Moreover $\alpha \leq |\alpha| = |\tilde{\alpha}| \leq |Y \cup \tilde{\alpha}| = |Z|$, whence $Z \in T$. Hence, for $Y \in \mathcal{P}_\kappa(A)$ there is $Z \in T$ such that $Y \subseteq Z$.

- T is closed in $\mathcal{P}_\kappa(A)$: Let $\{X_\beta : \beta < \mu\}$ be an ascending sequence of elements of T. Notice that, for each X_β there is some α_β such that $X_\beta \cap \kappa \subseteq \alpha_\beta \leq |X|$. Consider $\alpha := \sup\{\alpha_\beta : \beta < \mu\}$. As C is closed, $\alpha \in C$. Moreover, from $X_\beta \cap \kappa \subseteq \alpha$ for each $\beta < \mu$, we get that $(\bigcup_{\beta<\mu} X_\beta) \cap \kappa \subseteq \sup\{\alpha_\beta : \beta < \mu\} = \alpha$. Also from $\alpha_\beta \leq |X_\beta|$ for each $\beta < \mu$, we get that $\alpha \leq \sup\{|X_\beta| : \beta < \mu\} = |\sup\{X_\beta : \beta < \mu\}| = |\bigcup_{\beta<\mu} X_\beta|$. This is, $(\bigcup_{\beta<\mu} X_\beta) \cap \kappa \subseteq \alpha \leq |\bigcup_{\beta<\mu} X_\beta|$, so that $\bigcup_{\beta<\mu} X_\beta \in T$.

Hence T is a club subset of $\mathcal{P}_\kappa(A)$, and so it is 1-w-stationary (Proposition 3.6). Now, since $\mathcal{P}_\kappa(A)$ is 2-stationary, there are $B \in \mathcal{P}_\kappa(A)$ and μ regular cardinal such that

- $\mu \subseteq B \cap \kappa$.
- $T \cap \mathcal{P}_\mu(B)$ is 1-w-stationary in $\mathcal{P}_\mu(B)$.

Since $T \cap \mathcal{P}_\mu(B)$ is 1-w-stationary in $\mathcal{P}_\mu(B)$, then $\mathcal{P}_\mu(B)$ is 1-w-stationary in $\mathcal{P}_\mu(B)$. Then, by Proposition 3.3, μ is weakly Mahlo. Moreover, we claim that $\mu \in C$. To see that, we will prove that $C \cap \mu$ is unbounded in $\mu < \kappa$. As C is closed, that will imply $\mu \in C$.

- $C \cap \mu$ is unbounded in μ : Let $\gamma < \mu$, then $\gamma \in \mathcal{P}_\mu(B)$. So, there is $X \in T \cap \mathcal{P}_\mu(B)$ such that $\gamma \subseteq X$ (and so $\gamma \subseteq X \cap \kappa$). As $X \in T$, there is some $\alpha \in C$ such that $X \cap \kappa \subseteq \alpha \leq |X|$. But then $\gamma \subseteq X \cap \kappa \subseteq \alpha \leq |X| < \mu$. This is, $\alpha \in C \cap \mu$ and $\gamma \leq \alpha$.

Therefore $\mu \in C \cap E$, whence E is stationary in κ. This shows κ is 2-weakly Mahlo. $\square$

So we have that κ being 2-weakly Mahlo is a necessary condition for the 2-w-stationarity of $\mathcal{P}_\kappa(A)$.

Is this also a sufficient condition? In other words, do we have an analogous of Proposition 3.4? Recall that in the ordinal case the existence of 1-w-stationary and 2-w-stationary sets respectively, jumped from the condition $cof(\kappa) \geq \omega_1$ to the condition of being weakly inaccessible or the successor of a singular cardinal. This suggests that the condition of κ being 2-weakly-Mahlo is too weak as a sufficient condition for 2-w-stationarity in $\mathcal{P}_\kappa(A)$.

Definition 3.9. *We say that a subset $X \subseteq \mathcal{P}_\kappa(A)$ n-**reflects** at $B \in \mathcal{P}_\kappa(A)$ iff there is μ regular cardinal such that $\mu \subseteq B \cap \kappa$ and $X \cap \mathcal{P}_\mu(B)$ is n-w-stationary in $\mathcal{P}_\mu(B)$.*

Notice that if κ is weakly Mahlo, then every unbounded subset T of $\mathcal{P}_\kappa(A)$ 0-reflects to some element of $\mathcal{P}_\kappa(A)$. More in general, if $\mathcal{P}_\kappa(A)$ is n-w-stationary, then every m-w-stationary subset S of $\mathcal{P}_\kappa(A)$ for $m < n$, m-reflects to some $B \in \mathcal{P}_\kappa(A)$.

Definition 3.10. *Let $S \subseteq \mathcal{P}_\kappa(A)$ and $n < \omega$, we define $d_n(S) := \{X \in \mathcal{P}_\kappa(A) : S$ n-reflects at $X\}$.*

Proposition 3.11. *Let κ be weakly Mahlo, and let $T, T_1, \ldots, T_l$ be unbounded in $\mathcal{P}_\kappa(A)$ for some $l < \omega$. Then $d_0(T_1) \cap \cdots \cap d_0(T_l)$ is 1-w-stationary in $\mathcal{P}_\kappa(A)$.*

Proof : To prove that $d_0(T_1) \cap \cdots \cap d_0(T_l)$ is 1-w-stationary in $\mathcal{P}_\kappa(A)$ we will prove that for any T_0 unbounded in $\mathcal{P}_\kappa(A)$ we have $d_0(T_0) \cap d_0(T_1) \cap \cdots \cap d_0(T_l) \neq \varnothing$. As κ is weakly Mahlo, we can perform an analogous proof of the one we did for Proposition 3.4, with $T = T_0$ and splitting the successor step in such a way that for $\alpha + m$ with $m \leq l$, $X_\alpha \in T_m$. Therefore $B = \bigcup_{\alpha<\mu} X_{\beta_\alpha} = \bigcup_{\alpha<\mu} X_{\beta_\alpha+m}$ for all $m \leq l$ and so $B \in d_0(T_1) \cap \cdots \cap d_0(T_l)$. □

Definition 3.12. *Let $NS^n_{\kappa,A}$ be the set of non n-w-stationary subsets of $\mathcal{P}_\kappa(A)$, this is $NS^n_{\kappa,A} := \{S \subseteq \mathcal{P}_\kappa(A) : S$ is not n-stationary in $\mathcal{P}_\kappa(A)\}$. Moreover let $F^n_{\kappa,A} := \{\mathcal{P}_\kappa(A) \setminus X : X \in NS^n_{\kappa,A}\}$, this is, $F^n_{\kappa,A} := (NS^n_{\kappa,A})^*$.*

Proposition 3.13. *Let $\mathcal{P}_\kappa(A)$ be n-w-stationary and let $X \in \mathcal{P}_\kappa(A)$. Then $X \in F^n_{\kappa,A}$ if and only if there is $T_X \subseteq \mathcal{P}_\kappa(\lambda)$ m-w-stationary for some $m < n$ such that $d_m(T_X) \subseteq X$.*

Proof : ($\Rightarrow$) Let $X \in F^n_{\kappa,A}$. Then $X = \mathcal{P}_\kappa(A) \setminus Y$ for some $Y \in NS^n_{\kappa,A}$. Since Y is not n-w-stationary, there is $T_X \subseteq \mathcal{P}_\kappa(\kappa)$ m-w-stationary with $m < n$ such that, for all $B \in Y$ and all $\mu \subseteq B \cap \kappa$ regular, $T_X \cap \mathcal{P}_\mu(B)$ is not m-w-stationary in $\mathcal{P}_\mu(B)$ (*).

We claim that $d_m(T_X) \subseteq X$. To see this it is enough to prove that $d_m(T_X) \cap Y = \varnothing$. Towards a contradiction, suppose that $W \in d_m(T_X) \cap Y$. Then, $W \in Y$ and T_X m-reflects at W. This is, $W \in Y$ and there is $\mu < \kappa$ regular such that $\mu \subseteq W \cap \kappa$ and $T_X \cap \mathcal{P}_\mu(W)$ is m-w-stationary in $\mathcal{P}_\mu(W)$, but this is a contradiction to (*).

($\Leftarrow$) Suppose that $X \in \mathcal{P}_\kappa(A)$ is such that there is $T_X \subseteq \mathcal{P}_\kappa(\lambda)$ m-w-stationary for some $m < n$ such that $d_m(T_X) \subseteq X$. Let us consider $Y := \mathcal{P}_\kappa(A) \setminus X$. We shall prove that $Y \in NS^n_{\kappa,\lambda}$. By contradiction, suppose Y is n-w-stationary. Then, for the m-w-stationary set $T_X \subseteq \mathcal{P}_\kappa(A)$, there is $B \in Y$ and $\mu \subseteq B \cap \kappa$ such that $T_X \cap \mathcal{P}_\mu(B)$is m-w-stationary in $\mathcal{P}_\mu(B)$. From the latter, we conclude that $B \in d_m(T_X) \subseteq X$. But B is also an element of Y, this is $B \in \mathcal{P}_\kappa\lambda \setminus X$, contradicting the fact that $B \in X$. □

Then, in analogy with the ordinal case, whenever $\mathcal{P}_\kappa(A)$ is n-w-stationary;

$$F^n_{\kappa,A} = \{X \subseteq \mathcal{P}_\kappa(A) : \exists T_X \subseteq \mathcal{P}_\kappa(A) \;\; m\text{-w-stationary for some } m < n, \text{ such that } d_m(T_X) \subseteq X\}.$$

Lemma 3.14. *If T_1, T_2 are both not unbounded subsets of $\mathcal{P}_\kappa(A)$, then $T_1 \cup T_2$ is not unbounded either.*

Proof : Suppose $T_i \subseteq \mathcal{P}_\kappa(A)$ is not unbounded for $i \in \{1,2\}$, then, there is $X_i \in \mathcal{P}_\kappa(A)$ such that for all $Y \in T_i$, $X_i \not\subseteq Y$. Towards a contradiction, suppose that $T_1 \cup T_2$ is unbounded in $\mathcal{P}_\kappa(A)$. Then, there is $Y_1 \in T_1 \cup T_2$ such that $X_1 \subseteq Y_1$. Notice that $Y_1 \notin T_1$. Also, there is $Y_2 \in T_1 \cup T_2$ such that $Y_1 \cup X_2 \subseteq Y_2$. Then $X_1 \subseteq Y_1 \cup X_2 \subseteq Y_2$. So, if $Y_2 \in T_1$ then $X_1 \subseteq Y_2$ contradicts that for all $Y \in T_1$, $X_1 \not\subseteq Y$. Similarly if $Y_2 \in T_2$ then $X_2 \subseteq Y_2$ contradicts that for all $Y \in T_2$, $X_2 \not\subseteq Y$. Hence $Y_2 \notin T_1 \cup T_2$, which is a contradiction. □

Proposition 3.15. *If $\mathcal{P}_\kappa(A)$ has the property that for all T_1, T_2 m^*-stationary, there is some T m-w-stationary such that $d_m(T) \subseteq d_{m^*}(T_1) \cap d_{m^*}(T_2)$, where $m \leq m^*$. Then, the set $NS^n_{\kappa,A}$ is an ideal over $\mathcal{P}_\kappa(A)$. Moreover $\mathcal{P}_\kappa(A)$ is n-w-stationary if and only if $NS^n_{\kappa,A}$ is a proper ideal.*

Proof : Clearly $\varnothing \in NS^n_{\kappa,A}$. Moreover, if $X \in NS^n_{\kappa,A}$ and $Y \in \mathcal{P}_\kappa(A)$ is such that $Y \subseteq X$, then $Y \in NS^n_{\kappa,A}$. Now, suppose that we have the result for all $m < n$, and let $X_1, X_2 \in NS^n_{\kappa,A}$. Then $\mathcal{P}_\kappa(A) \setminus X_1, \mathcal{P}_\kappa(A) \setminus X_2 \in F^n_{\kappa,A}$, by Proposition 3.13, there are T_{X_1} m_1-w-stationary and T_{X_2} m_2-w-stationary with $m_1, m_2 < n$, such that $d_{m_1}(T_{X_1}) \subseteq \mathcal{P}_\kappa(A) \setminus X_1$ and $d_{m_2}(T_{X_2}) \subseteq \mathcal{P}_\kappa(A) \setminus X_2$. But $d_{m_1}(T_{X_1}) \cap d_{m_2}(T_{X_2}) \subseteq (\mathcal{P}_\kappa(A) \setminus X_1) \cap (\mathcal{P}_\kappa(A) \setminus X_2) = \mathcal{P}_\kappa(A) \setminus (X_1 \cup X_2)$. Then, $d_{m^*}(T_{X_1}) \cap d_{m^*}(T_{X_2}) \subseteq \mathcal{P}_\kappa(A) \setminus (X_1 \cup X_2)$. Now, applying our hypothesis we get that there is $m \leq m^* < n$ and T m-w-stationary such that $d_m(T) \subseteq d_{m^*}(T_{X_1}) \cap d_{m^*}(T_{X_2})$. But this implies that $d_m(T) \subseteq \mathcal{P}_\kappa(A) \setminus (X_1 \cup X_2)$. By 3.13, we conclude that $\mathcal{P}_\kappa(A) \setminus (X_1 \cup X_2) \in F^n_{\kappa,A}$ and so $X_1 \cup X_2 \in NS^n_{\kappa,A}$.

Finally, suppose that $\mathcal{P}_\kappa(A)$ is n-w-stationary , then $\mathcal{P}_\kappa(A) \notin NS^n_{\kappa,A}$ and so $NS^n_{\kappa,A}$ is non-trivial. □

Corollary 5. *The set of non-1-w-stationary subsets of $\mathcal{P}_\kappa(A)$ when is an ideal, is contained in the ideal of non-stationary subsets of $\mathcal{P}_\kappa(A)$. This is, $NS_{\kappa,A} \subseteq NS^1_{\kappa,A}$.*

Our interest extends to the conditions needed on κ to ensure that $\mathcal{P}_\kappa(A)$ is n-w-stationary or $\mathcal{P}_\kappa(A)$ is n-stationary. Specifically, we aim to determine the minimal set of conditions required, as we did in the case $n = 1$ in by means of Proposition 3.4. To systematically address this inquiry, we will focus now on Definition 2.5 and we shall obtain the same results for Definition 2.6 as a consequence of Corollary 1. We begin by examining the dynamics within $\mathcal{P}_\kappa(\kappa)$. For the more general case of $\mathcal{P}_\kappa\lambda$ we proceed by addressing a proposition stated by Sakai in [16] and thereof providing a proof of the same. Notice that when $|A| = |B|$, then $\langle \mathcal{P}_\kappa(A), \subseteq \rangle$ is isomorphic to $\langle \mathcal{P}_\kappa(B), \subseteq \rangle$. Then, the study of $\mathcal{P}_\kappa(A)$ is analogous to that of $\mathcal{P}_\kappa\lambda$, where $|A| = \lambda \geq \kappa$. In this section we will expose two sufficient conditions for n-stationarity, in $\mathcal{P}_\kappa\lambda$.

Lemma 3.16. *Let κ be a regular cardinal. Then, the formula $\varphi_n(S)$: "$S \subseteq \mathcal{P}_\kappa(\kappa)$ is n-stationary in $\mathcal{P}_\kappa(\kappa)$" is Π^1_n over $\langle V_\kappa, \in, S \rangle$. Moreover, if $x \in \mathcal{P}_\kappa(\kappa)$, then $\varphi'_n(T)$: "$T \subseteq \mathcal{P}_{B\cap\kappa}(B)$ is n-stationary in $\mathcal{P}_{B\cap\kappa}(B)$" is a Π^1_0 sentence over $\langle V_\kappa, \in \rangle$, in the parameters T, B.*

Proof : First we will show that $\mathcal{P}_\kappa(\kappa) \in V_{\kappa+1} \setminus V_\kappa$ and $\mathcal{P}_{B\cap\kappa}(B) \in V_\kappa$. If $y \in \mathcal{P}_\kappa(\kappa)$, then $y \subseteq \alpha$ for some $\alpha < \kappa$. So we have $\text{rank}(y) \leq \text{rank}(\alpha) < \text{rank}(\kappa) = \kappa$, this is $y \in \{z : \text{rank}(z) < \kappa\} = V_\kappa$, whence $\mathcal{P}_\kappa(\kappa) \subseteq V_\kappa$ and so $\mathcal{P}_\kappa(\kappa) \in V_{\kappa+1}$. Since $\kappa \subseteq \mathcal{P}_\kappa(\kappa)$, $\kappa = \text{rank}(\kappa) \leq \text{rank}(\mathcal{P}_\kappa(\kappa))$, and this implies $\mathcal{P}_\kappa(\kappa) \notin V_\kappa$. Moreover, if $B \in S \subseteq \mathcal{P}_\kappa(\kappa) \subseteq V_\kappa$, $B \in V_\alpha$ for some $\alpha < \kappa$. So that $\mathcal{P}(B) \in V_{\alpha+1} \subseteq V_\kappa$, and so $\mathcal{P}_{B\cap\kappa}(B) \in V_\kappa$.

Notice that $Y \in \mathcal{P}_\kappa(\kappa)$ if and only if $\langle V_\kappa, \in \rangle \models \psi(Y)$ where $\psi(Y) : \exists\alpha(OR(\alpha) \wedge Y \subseteq \alpha)$. So defined $\psi(Y)$ is a Π^1_0 formula. In fact, $\psi(Y)$ is a Σ_1 formula with Y as a free variable.

We will now prove the lemma by simultaneous induction. Let $n = 0$. $S \subseteq \mathcal{P}_\kappa(\kappa)$ is 0-stationary in $\mathcal{P}_\kappa(\kappa)$ if and only if $\langle V_\kappa, \in \rangle \models \varphi_0(S)$ where

$$\varphi_0(S) : \ \forall Y \ (\psi(Y) \to \exists Y \in S \ (Y \subseteq Y) \ )$$

Y is a first-order variable, because it ranges over elements of $\mathcal{P}_\kappa(\kappa) \subseteq V_\kappa$. Thus $\varphi_0(S)$ is first order, i.e., Π^1_0.

Given $B \in \mathcal{P}_\kappa(\kappa)$, such that $B \cap \kappa$ is a regular cardinal, we have that $T \subseteq \mathcal{P}_{B\cap\kappa}(B)$ is 0-startionary in $\mathcal{P}_{B\cap\kappa}(B)$ if and only if $\langle V_\kappa, \in\rangle \models \varphi'_0(T, B)$ where

$$\varphi'_0(T, B) :\ \forall Y\ (Y \in \mathcal{P}_{B\cap\kappa}(B) \rightarrow \exists W \in T\ (Y \subseteq W)\ )$$

Since $T \subseteq \mathcal{P}_{B\cap\kappa}(B) \in V_\kappa$ and $Y \in \mathcal{P}_{B\cap\kappa}(B) \in V_\kappa$, $\varphi'_0(T; B)$ is a Π_1 formula, and so it is Π^1_0 in the parameters T and B.

Let $Reg(z)$ be the formula "z is a regular cardinal ". For $n = 1$, $S \subseteq \mathcal{P}_\kappa(\kappa)$ is 1-w-stationary in $\mathcal{P}_\kappa(\kappa)$ if and only if $\langle V_\kappa, \in\rangle \models \varphi_1(S)$ where

$$\varphi_1(S) :\ \forall Y\ \phi_1(S, Y)$$

$$\phi_1(S, Y) : (\forall Z(Z \in Y \rightarrow \psi(Z)) \wedge \varphi_0(S))\ \rightarrow\ \sigma_1(S, Y)$$

$$\sigma_1(S, Y) : \exists B(B \in S \wedge Reg(B \cap \kappa) \wedge \varphi'_0(Y \cap \mathcal{P}_{B\cap\kappa}(B)))$$

Y is a second order variable because its possible values are subsets of $\mathcal{P}_\kappa(\kappa)$. Note that Z ranges over elements of V_κ ($Y \in V_{\kappa+1}$ and $Z \in Y$ implies $Z \in V_\kappa$). Then, as $\varphi'_0(Y \cap \mathcal{P}_{B\cap\kappa}(B))$ is Π^1_0, so is $\sigma_1(S, Y)$. Together with the fact that $\psi(Z)$ and $\varphi_0(S)$ are also Π^1_0, we get that $\varphi_1(S)$ is Π^1_1.

Given $B \in \mathcal{P}_\kappa(\kappa)$ such that $B \cap \kappa$ is a regular cardinal, we have that $T \subseteq \mathcal{P}_{B\cap\kappa}(B)$ is 1-w-stationary in $\mathcal{P}_{B\cap\kappa}(B)$ if and only if $\langle V_\kappa, \in\rangle \models \varphi'_1(T; B)$ where

$$\varphi'_1(T; B) :\ \forall Y\ \phi'_1(Y, T; B)$$

$$\phi'_1(T; B) : (Y \subseteq \mathcal{P}_{B\cap\kappa}(B) \wedge \varphi'_0(Y; B))\ \rightarrow\ \sigma'_1(T, Y)$$

$$\sigma'_1(T, Y) : \exists B'(B' \in T \wedge Reg(B' \cap \kappa) \wedge \varphi'_0(Y \cap \mathcal{P}_{B'\cap\kappa}(B'); B'))$$

Here Y is a first-order variable because its possible values are subsets of $\mathcal{P}_{B\cap\kappa}(B) \in V_\kappa$, and $\varphi'_0(Y; B), \varphi'_0(Y \cap \mathcal{P}_{B\cap\kappa'}(B'); B')$ are Π_1 formulas. Then, $\sigma'_1(T, Y)$ is a Σ_2 formula, whence $\varphi'_1(T; B)$ is a Π_3 formula and so a Π^1_0 formula .

Suppose now, that $S \subseteq \mathcal{P}_\kappa(\kappa)$ is m-stationary in $\mathcal{P}_\kappa(\kappa)$ if and only if $\langle V_\kappa, \in\rangle \models \varphi_m(S)$, where $\varphi_m(S)$ is a Π^1_m formula for all $m < n$. And let us prove the result for n.

Then, $\varphi_m(S)$ is of the form $\forall\ \mathbf{Y}^m_1\ \exists\ \mathbf{Y}^m_2\ \ldots\ Q\ \mathbf{Y}^m_m\ \phi_m(S, \mathbf{Y}^m_1, \ldots, \mathbf{Y}^m_m)$ where $Q = \forall$ if m is odd, $Q = \exists$ if m is even, $\mathbf{Y}^m_j = Y_1, \ldots, Y_{k_j}$ for $j \in \{1, \ldots, n\}$ and $\phi_m(S, \mathbf{Y}^m_1, \ldots, \mathbf{Y}^m_m)$ is a Π^1_0 formula. We have, $S \subseteq \mathcal{P}_\kappa(\kappa)$ is n-stationary in $\mathcal{P}_\kappa(\kappa)$ if and only if $\langle V_\kappa, \in\rangle \models \varphi_n(S)$, where

$$\varphi_n(S) : \varphi_{n-1}(S)\ \wedge\ \forall Y((\forall Z(Z \in Y \rightarrow \psi(Z)) \wedge \varphi_{n-1}(S)) \rightarrow \sigma_n(S, Y)\ )$$

From the inductive hypothesis, we know that $\varphi_{n-1}(S)$ is of the form $\forall\ \mathbf{Y}^{n-1}_1\ \exists\ \mathbf{Y}^{n-1}_2\ \ldots\ Q\ \mathbf{Y}^{n-1}_{n-1}$ $\phi_{n-1}(S, \mathbf{Y}^{n-1}_1, \ldots, \mathbf{Y}^{n-1}_{n-1})$, and so, we have that

$$\forall Y((\forall Z(Z \in Y \rightarrow \psi(Z)) \wedge \varphi_{n-1}(S)) \rightarrow \sigma_n(S, Y)\ ) \equiv \forall Y\ \ \exists\ \mathbf{Y}^{n-1}_1\ \forall\ \mathbf{Y}^{n-1}_2\ \cdots$$

$$\bar{Q}\ \mathbf{Y}^{n-1}_{n-1}((\forall Z(Z \in Y \rightarrow \psi(Z)) \wedge \phi_{n-1}(S, \mathbf{Y}^{n-1}_1, \ldots, \mathbf{Y}^{n-1}_{n-1})) \rightarrow \sigma_n(S, Y)\ )$$

where $\bar{Q} = \forall$ if $Q = \exists$ and $\bar{Q} = \exists$ if $Q = \forall$, and σ_n is the first order formula

$$\sigma_n(S,Y) : \exists B(B \in S \wedge Reg(B \cap \kappa) \wedge B \cap \kappa \subseteq B \wedge \varphi'_{n-1}(Y \cap \mathcal{P}_{B\cap\kappa}(B)))$$

Therefore, if $(\mathbf{Y}_1 := Y, \mathbf{Y}_1^1, \ldots, \mathbf{Y}_1^{n-1}), \ldots, (\mathbf{Y}_i := \mathbf{Y}_i^i, \ldots, \mathbf{Y}_i^{n-1}, \mathbf{Y}_{i-1}^{n-1}), \ldots, (\mathbf{Y}_n := \mathbf{Y}_{n-1}^{n-1})$, we may write $\varphi_n(S)$ in the following form

$$\varphi_n(S) \equiv \forall \mathbf{Y}_1 \;\; \exists \mathbf{Y}_2 \; \forall \; \mathbf{Y}_3 \; \ldots \; \bar{Q} \; \mathbf{Y}_n(\phi_1(S, \mathbf{Y}_1) \wedge \phi_2(S, \mathbf{Y}_1, \mathbf{Y}_2)$$

$$\wedge \cdots \wedge \phi_{n-1}(S, \mathbf{Y}_1, \ldots, \mathbf{Y}_{n-1}) \wedge$$

$$\wedge \; ((\forall Z(Z \in Y \to \psi(Z)) \wedge \phi_{n-1}(S, \mathbf{Y}_1, \ldots, \mathbf{Y}_{n-1})) \to \sigma_n(S,Y)) \;)$$

Since $\phi_j(S, \mathbf{Y}_1, \ldots, \mathbf{Y}_i)$ and $\sigma_n(S,Y)$ are Π^1_0 formulas for $j \in \{1, \ldots, n-1\}$, we get that $\varphi_n(S)$ is a Π^1_n formula.

Suppose now, that for $B \in \mathcal{P}_\kappa(\kappa)$, $T \subseteq \mathcal{P}_{B\cap\kappa}(B)$ is m-stationary in $\mathcal{P}_{B\cap\kappa}(B)$ if and only if $\langle V_\kappa, \in \rangle \models \varphi'_m(T,B)$, where $\varphi'_m(T,B)$ is a Π^1_0 formula for all $m < n$.

$T \subseteq \mathcal{P}_{B\cap\kappa}(B)$ is n-stationary in $\mathcal{P}_{B\cap\kappa}(B)$ if and only if $\langle V_\kappa, \in \rangle \models \varphi'_n(T,B)$, where

$$\varphi'_n(T,B) : \varphi'_{n-1}(T,B) \;\; \wedge \;\; \forall Y((Y \subseteq \mathcal{P}_{B\cap\kappa}(B) \wedge \varphi'_{n-1}(Y,B)) \;\; \to \;\; \sigma'_n(T,Y) \;)$$

and where

$$\sigma'_n(T,Y) : \exists B'(B' \in T \wedge Reg(B' \cap \kappa) \wedge \varphi'_{n-1}(Y \cap \mathcal{P}_{B\cap\kappa'}(B'); B')).$$

Here, Y is a first-order variable because its possible values are subsets of $\mathcal{P}_{B\cap\kappa}(B) \in V_\kappa$, and $\varphi'_{n-1}(Y \cap \mathcal{P}_{B\cap\kappa}(B), B')$ and $\sigma'_n(T,Y)$ are first-order formulas. Then $\varphi'_n(T,B)$ is a first-order formula and so it is Π^1_0. □

Theorem 3.17. *Let $n < \omega$. If κ is Π^1_n indescribable, then $\mathcal{P}_\kappa(\kappa)$ is $n+1$ stationary.*

Proof : Suppose κ is Π^1_n indescribable. Let $S \subseteq \mathcal{P}_\kappa(\kappa)$ be m-stationary, some $m < n+1$. Consider the Π^1_m sentence $\varphi_m(S)$ in $\langle V_\kappa, \in, S \rangle$. Then, we have

$$\langle V_\kappa, \in, S \rangle \models \varphi_m(S).$$

As κ is Π^1_n indescribable and $m \leq n$, there is some $\mu < \kappa$ regular such that

$$\langle V_\mu, \in, S \cap V_\mu \rangle \models \varphi_m(S \cap V_\mu).$$

Now, note that $\mathcal{P}_\kappa(\kappa) \cap V_\mu = \mathcal{P}_\mu(\mu)$. For if $X \in \mathcal{P}_\kappa(\kappa) \cap V_\mu$ then $X \subseteq \kappa \cap V_\mu = \mu$. Also $|X| < \mu$, otherwise rank$(X) = \mu$ and so $X \notin V_\mu$. Hence $X \in \mathcal{P}_\mu(\mu)$.

Thus, since $S = S \cap \mathcal{P}_\kappa(\kappa)$, we have that $S \cap V_\mu = S \cap \mathcal{P}_\kappa(\kappa) \cap V_\mu = S \cap \mathcal{P}_\mu(\mu)$. Therefore, we have $\langle V_\mu, \in, S \cap \mathcal{P}_\mu(\mu) \rangle \models \varphi_m(S \cap \mathcal{P}_\mu(\mu))$, and so $S \cap \mathcal{P}_\mu(\mu)$ is m-stationary in $\mathcal{P}_\mu(\mu)$. □

Corollary 6. *If κ is totally indescribable, then $\mathcal{P}_\kappa(\kappa)$ is n-stationary for any $n < \omega$ (and so $\mathcal{P}_\kappa(\kappa)$ is n-w-stationary for any $n < \omega$).*□

Now, we will provide a proof for the the assertion made by Sakai in [16], showing threof a sufficient condition to have n-stationarity in $\mathcal{P}_\kappa\lambda$. We will use the fact that, if f is an isomorphism between $\mathcal{P}_\kappa\lambda$ and $\mathcal{P}_\kappa(\delta)$, then, $S \subseteq \mathcal{P}_\kappa\lambda$ is m-stationary in $\mathcal{P}_\kappa\lambda$ if and only if $f[S]$ is m-stationary in $\mathcal{P}_\kappa(\delta)$. The proof of this fact is follows immediately form definition of n-stationarity in $\mathcal{P}_\kappa\lambda$.

Proposition 3.18. *([16]) If κ is λ-supercompact and $\lambda^{<\kappa} = \lambda$ then $\mathcal{P}_\kappa(\lambda)$ is n-stationary for any $n \in \mathbb{N}$.*

Proof : Let $n < \omega$ and take $S \subseteq \mathcal{P}_\kappa(\lambda)$ be m-stationary for a given $m < n$. Suppose that κ is λ-supercompact, this is, there is an elementary embedding $j : V \preceq M$ such that $\text{crit}(j) = \kappa$, $\lambda < j(\kappa)$ and ${}^\lambda M \subseteq M$, where M is transitive.

Recall that $j``x = \{j(y) : y \in x\}$, we claim that $j``\alpha \in M$, for all $\alpha \leq \lambda$. We prove this by induction on OR, $j``0 = 0 \in M$ because $j|_\kappa = Id|_\kappa$. If $j``\alpha \in M$ for $\alpha < \lambda$, then $j``(\alpha + 1) = j``\alpha \cup \{j(\alpha)\} \in M$. And if $\alpha \leq \lambda$ limit and $j``\beta \in M$ for all $\beta < \alpha$ then $j``\alpha = \{j``\beta : \beta < \alpha\}$ which is a sequence of $\alpha \leq \lambda$ elements of $M$, whence $j``\alpha \in^\lambda M \subseteq M$.

Since $j \restriction_\kappa = Id \restriction_\kappa$, we have that, $j``\kappa = \{j(\alpha) : \alpha < \kappa\} = \{\alpha : \alpha < \kappa\} = \kappa \in M$. Then, it follows that $\mathcal{P}_{j``\kappa}(j``\lambda) = \mathcal{P}_\kappa(j``\lambda) \subseteq M$. Moreover, as $|j``\lambda| = |\lambda|$, then $|\mathcal{P}_\kappa(j``\lambda)| = |j``\lambda|^{<\kappa} = \lambda^{<\kappa} = \lambda$, and so $\mathcal{P}_\kappa(j``\lambda) \in M$. Now, notice that there is an isomorphism f between $\mathcal{P}_\kappa\lambda$ and $\mathcal{P}_\kappa(j''\lambda)$ given by $X \mapsto j''X$.

By hypothesis, we have that $S \subseteq \mathcal{P}_\kappa(\lambda)$ is m-stationary in $\mathcal{P}_\kappa\lambda$, and so $f[S] = j``S \subseteq \mathcal{P}_\kappa(j``\lambda)$ is m-stationary in $\mathcal{P}_\kappa(j``\lambda)$. Therefore, as $j''S \subseteq j(S)$ we have that

$$V \models \text{`` } j(S) \cap \mathcal{P}_\kappa(j``\lambda) \text{ is } m\text{-stationary in } \mathcal{P}_\kappa(j``\lambda) \text{ ''}$$

Since $\mathcal{P}_\kappa(j``\lambda) \in M$, we have that $\mathcal{P}(\mathcal{P}_\kappa(j``\lambda)) \subseteq M$. So, since being m-stationary depends only on the subsets of $\mathcal{P}_\kappa(j``\lambda)$.

$$M \models \text{`` } j(S) \cap \mathcal{P}_\kappa(j``\lambda) \text{ is } m\text{-stationary in } \mathcal{P}_\kappa(j``\lambda) \text{ ''}.$$

In M we have that κ is regular and such that $\kappa < j(\kappa)$. If we define $B := j``\lambda$, then $\kappa = j``\kappa \subseteq j``\lambda = B$, and so $\kappa \subseteq B \cap j(\kappa)$. In fact $\kappa = B \cap j(\kappa)$; if $\alpha \in (B \cap j(\kappa)) \setminus \kappa$, then $\alpha = j(\beta)$ for some $\kappa < \beta < \lambda$ and $\alpha < j(\kappa)$, but $\kappa < \beta$ implies $j(\kappa) < j(\beta) = \alpha$, and this is a contradiction. Besides, as $|j``\lambda| = \lambda < j(\kappa)$, we have that $B \in \mathcal{P}_{j(\kappa)}(j(\lambda))$. Hence the following holds, witnessed by $\mu = \kappa$ and $B = j``\lambda$

$$M \models \exists B(\ Reg(B \cap j(\kappa)) \wedge B \in \mathcal{P}_{j(\kappa)}(j(\lambda)) \wedge$$
$$\text{``}j(S) \cap \mathcal{P}_{B\cap j(\kappa)}(B) \text{ is } m\text{-stationary in } \mathcal{P}_{B\cap j(\kappa)}(B)\text{'' }).$$

As j is an elementary embedding we get that

$$V \models \exists B(\ Reg(B \cap j^{-1}(j(\kappa))) \wedge B \in \mathcal{P}_{j^{-1}(j(\kappa))}(j^{-1}(j(\lambda))) \wedge$$
$$\text{``}j^{-1}(j(S)) \cap \mathcal{P}_{B\cap j^{-1}(j(\kappa))}(B) \text{ is } m\text{-stationary in } \mathcal{P}_{B\cap j^{-1}(j(\kappa))}(B)\text{'' }).$$

and since $j^{-1}(j(\kappa)) = \kappa$, $j^{-1}(j(\lambda)) = \lambda$ and $j^{-1}(j(S)) = S$,

$$V \models \exists B(\ Reg(B \cap \kappa) \wedge B \in \mathcal{P}_\kappa\lambda \wedge \text{``}S \cap \mathcal{P}_{B\cap\kappa}(B) \text{ is } m\text{-stationary in } \mathcal{P}_{B\cap\kappa}(B)\text{'' }).$$

This is, for each $m < n$ if $S \subseteq \mathcal{P}_\kappa(\lambda)$ is m-stationary, there is $B \in \mathcal{P}_\kappa\lambda$ such that $\mu \subseteq B \cap \kappa$ is regular and $S \cap \mathcal{P}_\mu(B)$ is m-stationary in $\mathcal{P}_\mu(B)$. And this is precisely to say that $\mathcal{P}_\kappa\lambda$ is n-stationary. □

Corollary 7. *If κ is λ-supercompact and $\lambda^{<\kappa} = \lambda$, $\mathcal{P}_\kappa(\lambda)$ is n-w-stationary for any $n \in \mathbb{N}$.* $\square$

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