Keisler's theorem and cardinal invariants at uncountable cardinals

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Contents

1 Introduction

The following is an important theorem in model theory proved by Keisler and Shelah. Keisler [Kei64] proved it by assuming GCH, but Shelah [She71] removed that assumption.

Theorem 1 (Keisler–Shelah)**.** For every (first-order) language $\mathcal{L}$ and two $\mathcal{L}$-structures $\mathcal{A}, \mathcal{B}$, the following are equivalent:

(1) $\mathcal{A} \equiv \mathcal{B}$ (that is, $\mathcal{A}$ and $\mathcal{B}$ are elementarily equivalent).

(2) There is a nonprincipal ultrafilter $\mathcal{U}$ over an infinite set such that the ultrapowers $\mathcal{A}^{\mathcal{U}}$ and $\mathcal{B}^{\mathcal{U}}$ are isomorphic.

The following theorem is also known in connection with the above theorem.

Theorem 2 (Keisler, Golshani and Shelah)**.** The following are equivalent:

(1) The continuum hypothesis.

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(2) For every countable language $\mathcal{L}$ and two $\mathcal{L}$-structures $\mathcal{A}, \mathcal{B}$ of size $\leq \mathfrak{c}$, if $\mathcal{A} \equiv \mathcal{B}$ then there is a nonprincipal ultrafilter $\mathcal{U}$ over ω such that the ultrapowers $\mathcal{A}^{\mathcal{U}}$ and $\mathcal{B}^{\mathcal{U}}$ are isomorphic.

For this theorem, Keisler [Kei64] showed (1) $\Rightarrow$ (2) and Golshani and Shelah [GS23] (2) $\Rightarrow$ (1).

In order to analyze these theorems in detail, we introduce the following principle.

Definition 3. Let κ, μ and λ be infinite cardinals. We define a criterion $\mathrm{KT}^{\mu}_{\kappa}(\lambda)$ by

$$\begin{aligned}\mathrm{KT}^{\mu}_{\kappa}(\lambda) \iff & \text{ for every language } \mathcal{L} \text{ of size } \leq \mu \text{ and} \\ & \text{every elementarily equivalent } \mathcal{L}\text{-structures } \mathcal{A}, \mathcal{B} \text{ of size } \leq \lambda, \\ & \text{there is a uniform ultrafilter } \mathcal{U} \text{ on } \kappa \text{ such that } \mathcal{A}^{\mathcal{U}} \simeq \mathcal{B}^{\mathcal{U}}.\end{aligned}$$

We also define a criterion $\mathrm{SAT}^{\mu}_{\kappa}(\lambda)$ by

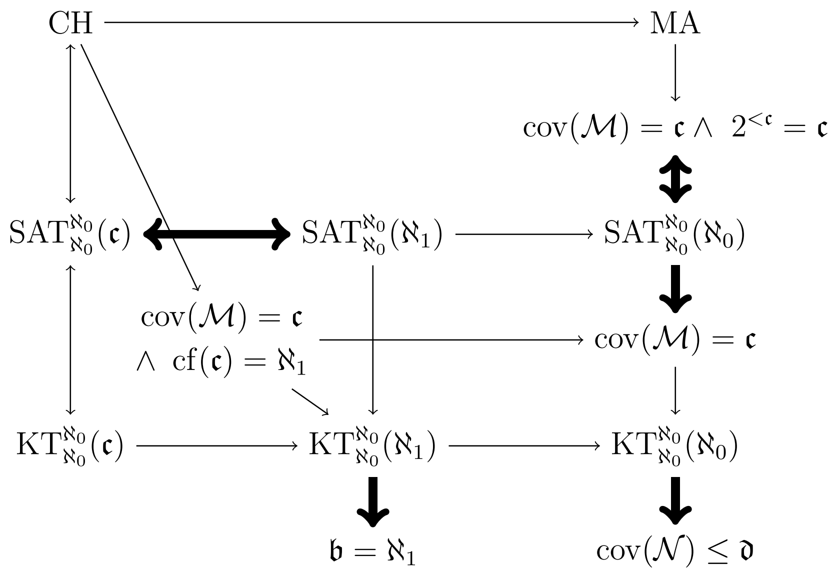
$$\begin{aligned}\mathrm{SAT}^{\mu}_{\kappa}(\lambda) \iff & \text{ there is a uniform ultrafilter } \mathcal{U} \text{ on } \kappa \text{ such that} \\ & \text{for every language } \mathcal{L} \text{ of size } \leq \mu \text{ and} \\ & \text{every sequence } \langle \mathcal{A}_i : i < \kappa \rangle \text{ of infinite } \mathcal{L}\text{-structures of size } \leq \lambda, \\ & \text{the ultraproduct } \left(\prod_{i \in \kappa} \mathcal{A}_i \right) / \mathcal{U} \text{ is saturated.}\end{aligned}$$

Keisler–Shelah's theorem means that $\mathrm{KT}^{2^\lambda}_{2^\lambda}(\lambda)$ holds for any infinite cardinal λ.

Keisler's paper also gives an example showing the following.

Fact 4 (Keisler [Kei64]). Let κ be an infinite cardinals. Then $\neg \mathrm{KT}^{\kappa^+}_{\kappa}(\kappa^+)$ holds.

The author's paper [Got22] showed the implications of the bold lines in the following figure.



2 Results

In this section, we discuss the principles introduced in Section 1. The case where the cardinality μ of the language and the cardinality κ of the underlying set of the ultrafilter are both $\aleph_0$ was analyzed in detail in the author's paper [Got22]. Here, the more general case is investigated. However, most of the results are naive generalisations of the arguments in [Got22].

Before proceeding to the results, we recall the basic definitions of ultrafilters.

Definition 5. Let $\mathcal{U}$ be a ultrafilter on κ. We say $\mathcal{U}$ is *regular* if there is $\mathcal{E} \subseteq \mathcal{U}$ of size κ such that for every $i < \kappa$, the set $\{E \in \mathcal{E} : i \in E\}$ is finite.

Definition 6. For ultrafilters $\mathcal{U}, \mathcal{V}$ on I, J respectively, we define

$$\mathcal{U} * \mathcal{V} = \{A \subseteq I \times J : \{i \in I : \{j \in J : (i, j) \in A\} \in \mathcal{V}\} \in \mathcal{U}\}.$$

$\mathcal{U} * \mathcal{V}$ is called the *Fubini product* of $\mathcal{U}$ and $\mathcal{V}$.

Lemma 7. Let $\kappa \leq \kappa'$ be two infinite cardinals. Then $\mathrm{KT}^{\mu}_{\kappa}(\lambda)$ implies $\mathrm{KT}^{\mu}_{\kappa'}(\lambda)$.

Proof. Fix a language $\mathcal{L}$ of size $\leq \mu$ and two elementarily equivalent $\mathcal{L}$-structures $\mathcal{A}$ and $\mathcal{B}$ of size $\leq \lambda$. By $\mathrm{KT}^{\mu}_{\kappa}(\lambda)$, we can take a uniform ultrafilter $\mathcal{U}$ on κ. Fix a uniform ultrafilter $\mathcal{V}$ on κ'. Then the ultrapowers of $\mathcal{A}$ and $\mathcal{B}$ by the ultrafilter $\mathcal{U} * \mathcal{V}$ are isomorphic. □

Lemma 8. (1) $\mathrm{KT}^{\mu}_{\kappa}(\lambda)$ implies there exists a regular ultrafilter witnessing $\mathrm{KT}^{\mu}_{\kappa}(\lambda)$.

(2) If $\lambda \geq \kappa$, then every witness for $\mathrm{SAT}^{\mu}_{\kappa}(\lambda)$ is a regular ultrafilter.

Proof. First, we show (1). Take an ultrafilter $\mathcal{U}$ on κ witnessing $\mathrm{KT}^{\mu}_{\kappa}(\lambda)$. Take a regular ultrafilter $\mathcal{V}$ on κ. Then the product ultrafilter $\mathcal{U} * \mathcal{V}$ is regular and witnesses $\mathrm{KT}^{\mu}_{\kappa}(\lambda)$.

Next we show (2). Take a witness $\mathcal{U}$ for $\mathrm{SAT}^{\mu}_{\kappa}(\lambda)$. Let $M = ([\kappa]^{<\aleph_0}, \subseteq)$ and consider $M_* = M^{\kappa}/\mathcal{U}$. By an easy diagonal argument, we have $|M_*| \geq \kappa^+$. Define a set of formulas p with a free variable x by

$$p = \{\ulcorner \{\alpha\}_* \subseteq x \urcorner : \alpha < \kappa\},$$

where $\{\alpha\}_*$ is the equivalence class of the constant sequence of $\{\alpha\}$. It can be easily checked that p is finitely satisfiable and the number of parameters of p is κ, which is smaller than $|M_*|$. Therefore, by $\mathrm{SAT}^{\mu}_{\kappa}(\lambda)$, we can take $f \colon \kappa \to M$ such that $[f]$ satisfies p. This f clearly satisfies $\{i \in \kappa : \alpha \in f(i)\} \in \mathcal{U}$ for every $\alpha < \kappa$. Thus, $\mathcal{U}$ is a regular ultrafilter. □

Lemma 9. $\mathrm{SAT}^{\mu}_{\kappa}(\lambda)$ implies $\mathrm{KT}^{\mu}_{\kappa}(\lambda)$ for every $\lambda \leq 2^{\kappa}$.

Proof. By regularity (Lemma 8), the ultrapowers have same cardinality. Thus uniqueness of saturated models implies this lemma. □

Lemma 10. $\neg\mathrm{SAT}_\kappa^{\aleph_0}(\kappa^{++})$.

Proof. Take a witness $\mathcal{U}$ of $\mathrm{SAT}_\kappa^{\aleph_0}(\kappa^{++})$. Let $\mathcal{A} = (\kappa^{++}, <)$ and $\mathcal{A}_* = \mathcal{A}^{\mathcal{U}}$. We have $|\mathcal{A}_*| \geq |\mathcal{A}| = \kappa^{++}$. Consider the following set p of formulas with one free variable x:

$$p = \{\ulcorner \alpha_* < x < (\kappa^+)_* \urcorner : \alpha < \kappa^+\}.$$

This p is finitely satisfiable and the number of parameters occurring in p is κ^+. Thus, by $\mathrm{SAT}_\kappa^{\aleph_0}(\kappa^{++})$, we can take $f\colon \kappa \to \kappa^+$ such that $[f]$ realizes p. Put $\beta = \sup_{\alpha<\kappa} f(\alpha)$. By $\beta_* < [f]$, we have $\{\alpha < \kappa : \beta < f(\alpha)\} \in \mathcal{U}$. This contradicts the choice of β. □

Lemma 11. $\mathrm{SAT}_\kappa^{\aleph_0}(\kappa^+)$ implies $2^\kappa = \kappa^+$.

Proof. Take a witness $\mathcal{U}$ of $\mathrm{SAT}_\kappa^{\aleph_0}(\kappa^+)$ and assume $\kappa^+ < 2^\kappa$. Let $\mathcal{A}_* = (\kappa^+, <)^{\mathcal{U}}$. We have $|\mathcal{A}_*| = 2^\kappa$ since $\mathcal{U}$ is regular (Lemma 8). Consider the following set p of formulas with one free variable x:

$$p = \{\ulcorner \alpha_* < x \urcorner : \alpha < \kappa^+\}.$$

This p is finitely satisfiable and the number of parameters occurring in p is equal to κ^+, which is smaller than 2^κ. Thus, by $\mathrm{SAT}_\kappa^{\aleph_0}(\kappa^+)$, we can take $f\colon \kappa \to \kappa^+$ such that $[f]$ realizes p. Then, this f is unbounded, which contradicts that κ^+ is regular. □

Lemma 12. $\neg\,\mathrm{KT}_\kappa^{\aleph_0}(\kappa^{++})$.

Proof. This proof is based on [Tsu22]. Let $(\mathcal{M}, <)$ be a linearly ordered set with cofinality κ^{++}. We define an increasing continuous sequence $\langle A_i : i \leq \kappa^{++}\rangle$ of subsets of $\mathcal{M}$ such that:

(1) For every $i \leq \kappa^{++}$, A_i is an elementary substructure of $\mathcal{M}$.

(2) For every $i < \kappa^{++}$, there is $a_i \in A_{i+1}$ such that for every $b \in A_i$, we have $b < a_i$.

(3) For every $i \leq \kappa^{++}$, we have $|A_i| \leq |i| + \aleph_0$.

We show that the pair of A_{κ^+} and $A_{\kappa^{++}}$ is a counterexample of $\mathrm{KT}_\kappa(\kappa^{++})$. Let $\mathcal{U}$ be an ultrafilter on κ.

We claim that $(A_{\kappa^+})^{\mathcal{U}}$ has a cofinal increasing sequence of length κ^+. In fact, $\langle (a_i)_* : i < \kappa^+\rangle$ is a cofinal increasing sequence. In order to show it, take $[f] \in (A_{\kappa^+})^{\mathcal{U}}$. For each $\alpha < \kappa$, we can take $i_\alpha < \kappa^+$ such that $f(\alpha) \in A_{i_\alpha}$. Then $i = \sup_{\alpha<\kappa} i_\alpha$ satisfies $[f] < a_i$.

On the other hand, in $(A_{\kappa^{++}})^{\mathcal{U}}$, every κ^+-sequence is bounded. In order to check it, take $\langle b_i : i < \kappa^+\rangle$. We write b_i as $b_i = [f_i]$, where $f_i\colon \kappa \to A_{\kappa^{++}}$. Since the set $\{f_i(\alpha) : i < \kappa^+, \alpha < \kappa\}$ has size less than or equal to κ^+, we can take $\beta < \kappa^{++}$ such that all the elements of this set belong to A_β. Then a_β is a bound of all b_i.

So we have $(A_{\kappa^+})^{\mathcal{U}} \not\cong (A_{\kappa^{++}})^{\mathcal{U}}$. □

Theorem 13. Let κ and μ be infinite cardinals satisfying $\mu \leq \kappa$. Then the following are equivalent.

(1) $2^\kappa = \kappa^+$.

(2) $\mathrm{SAT}^\mu_\kappa(2^\kappa)$.

(3) $\mathrm{SAT}^\mu_\kappa(\kappa^+)$

(4) $\mathrm{KT}^\mu_\kappa(2^\kappa)$.

Proof. Recall that there is a κ^+-good ultrafilter U on κ. That is, for every language $\mathcal{L}$ of size $\leq \kappa$, all U-ultraproducts of $\mathcal{L}$-structures are κ^+-saturated. The implication $2^\kappa = \kappa^+ \implies \mathrm{SAT}^\mu_\kappa(2^\kappa)$ follows from this fact.

The implication $\mathrm{SAT}^\mu_\kappa(\kappa^+) \implies 2^\kappa = \kappa^+$ is just Lemma 11.

The implication $\mathrm{KT}^\mu_\kappa(2^\kappa) \implies 2^\kappa = \kappa^+$ follows from Lemma 12. $\square$

Theorem 14. Let κ be a regular cardinal. Then $\mathrm{KT}^{\aleph_0}_\kappa(\kappa^+)$ implies $\mathfrak{b}_\kappa = \kappa^+$.

Proof. Take the same structure $\mathcal{M}$ as in Lemma 12. Consider two elementary substructures A_κ and A_{κ^+}.

Take a regular ultrafilter $\mathcal{U}$ on κ that witnesses $\mathrm{KT}^{\aleph_0}_\kappa(\kappa^+)$. As we saw in Lemma 12, we have $\mathrm{cf}((A_{\kappa^+})^\mathcal{U}) = \kappa^+$.

On the other hand, we have $\mathrm{cf}(A_\kappa) = \kappa$. So it holds that $\mathrm{cf}((A_\kappa)^\mathcal{U}) = \mathrm{cf}(\kappa^\kappa/\mathcal{U})$.

Since the ultrafilter $\mathcal{U}$ is uniform, we have $\mathfrak{b}_\kappa \leq \mathrm{cf}(\kappa^\kappa/\mathcal{U})$.

By $\mathrm{KT}^{\aleph_0}_\kappa(\kappa^+)$, the two models $(A_\kappa)^\mathcal{U}$ and $(A_{\kappa^+})^\mathcal{U}$ are isomorphic. So we have $\mathfrak{b}_\kappa \leq \mathrm{cf}(\kappa^\kappa/\mathcal{U}) = \kappa^+$. The other inequality is obvious. $\square$

Theorem 15. $\mathrm{SAT}^{\aleph_0}_\kappa(\kappa)$ implies $2^{<2^\kappa} = 2^\kappa$.

Proof. Fix a witness $\mathcal{U}$ for $\mathrm{SAT}^{\aleph_0}_\kappa(\kappa)$. Let $\lambda < 2^\kappa$. Define a language $\mathcal{L}$ and $\mathcal{L}$-structure $\mathcal{A}$ by $\mathcal{L} = \{\subseteq\}$ and $\mathcal{A} = ([\kappa]^{<\omega}, \subseteq)$. We have $|\mathcal{A}| = \kappa$. Put $\mathcal{A}_* = \mathcal{A}^\kappa/\mathcal{U}$. Since $\mathcal{U}$ is regular (Lemma 8), we have $|\mathcal{A}_*| = \kappa^\kappa = 2^\kappa$. Let $\iota\colon \kappa^\kappa/\mathcal{U} \to \mathcal{A}_*$ be the function defined by:

$$\iota([x]) = [\langle \{x(\alpha)\} : \alpha < \kappa \rangle].$$

Fix $F \subseteq \kappa^\kappa/\mathcal{U}$ of size λ. For $X \subseteq F$, we define a set $p_X(z)$ of formulas with a free variable z by:

$$p_X(z) = \{\ulcorner \iota(y) \subseteq z \urcorner : y \in X\} \cup \{\ulcorner \iota(y) \not\subseteq z \urcorner : y \in F \setminus X\}.$$

Each $p_X(z)$ is finitely satisfiable and the number of parameters occurring in $p_X(z)$ is λ. Therefore, by $\mathrm{SAT}^{\aleph_0}_\kappa(\kappa)$, for each $X \subseteq F$, we can take $[z_X] \in \mathcal{A}_*$ satisfying $p_X(z)$. For distinct $X, Y \subseteq F$, we have $[z_X] \neq [z_Y]$. Thus we have $2^\lambda = |\{[z_X] : X \subseteq F\}| \leq \mathcal{A}_* = 2^\kappa$. Since $\lambda < 2^\kappa$ was arbitrary chosen, we have $2^{<2^\kappa} = 2^\kappa$. $\square$

Theorem 16. Let κ be a regular cardinal. Let μ be a cardinal less than 2^κ. Then $\mathrm{cov}(\mathcal{M}_\kappa) = 2^\kappa$ implies $\mathrm{KT}^\mu_\kappa(\kappa)$.

Proof. Note that the assumption $\mathrm{cov}(\mathcal{M}_\kappa) = 2^\kappa$ is equivalent to $\mathrm{MA}_{<2^\kappa}(\mathrm{Fn}_\kappa(\kappa, 2))$.

Fix a enumeration of 2^κ.

Let $\mathcal{L}$ be a language of size $\leq \mu$ and $\mathcal{A}^0$ and $\mathcal{A}^1$ are $\mathcal{L}$-structures of size $\leq \kappa$ which are elementarily equivalent.

Enumerate $(\mathcal{A}^i)^\kappa$ for $i = 0, 1$ as

$$(\mathcal{A}^i)^\kappa = \{f^i_\alpha : \alpha < 2^\kappa\}.$$

By a back-and-forth method, we construct a sequence of triples $\langle(\mathcal{U}_\alpha, g^0_\alpha, g^1_\alpha) : \alpha < 2^\kappa\rangle$ satisfying:

(1) $g^0_\alpha \in (\mathcal{A}^0)^\kappa$,

(2) $g^1_\alpha \in (\mathcal{A}^1)^\kappa$,

(3) $\mathcal{U}_\alpha$ is a filter on κ generated by $\kappa + |\alpha|$ sets,

(4) $\langle \mathcal{U}_\alpha : \alpha < 2^\kappa\rangle$ is an increasing continuous sequence,

(5) If $\varphi(x_0, \dots, x_{n-1})$ is an $\mathcal{L}$-formula and $\beta_0, \dots, \beta_n \leq \alpha$, then the set

$$\{\xi \in \kappa : \mathcal{A}^0 \models \varphi(g^0_{\beta_0}(\xi), \dots, g^0_{\beta_{n-1}}(\xi)) \iff \mathcal{A}^1 \models \varphi(g^1_{\beta_0}(\xi), \dots, g^1_{\beta_{n-1}}(\xi))\}$$

belongs to $\mathcal{U}_{\alpha+1}$.

In the construction, when α is even, we put $g^0_\alpha = f^0_\gamma$ where γ is the least ordinal $f^0_\gamma \notin \{g^0_\beta : \beta < \alpha\}$. And $\mathbb{P}$ is the poset of partial functions of size $<\kappa$ from κ to $\mathcal{A}^1$. This poset is forcing equivalent to $\mathrm{Fn}_\kappa(\kappa, 2)$.

Take a generating set $\mathcal{F}$ of $\mathcal{U}_\alpha$ of size $\aleph_0 + |\alpha|$. Then by using $\mathrm{MA}_{<2^\kappa}(\mathrm{Fn}_\kappa(\kappa, 2))$, take a $\mathbb{P}$-generic filter G with respect to the following family of dense sets of $\mathbb{P}$:

$$D_\xi = \{p \in \mathbb{P} : \xi \in \mathrm{dom}\, p\} \text{ (for } \xi \in \kappa)$$

and

$$\begin{aligned} E_{X, \langle \varphi_\iota : \iota \in I\rangle, \langle \gamma^\iota_1, \dots, \gamma^\iota_{n_\iota} : \iota \in I\rangle} =& \{p \in \mathbb{P} : (\exists \xi \in \mathrm{dom}(p) \cap X)(\forall \iota \in I) \\ & (\mathcal{A}^0 \models \varphi_\iota(g^0_{\gamma^\iota_1}(\xi), \dots g^0_{\gamma^\iota_{n_\iota}}(\xi), g^0_\alpha(\xi)) \Leftrightarrow \\ & \mathcal{A}^1 \models \varphi_\iota(g^1_{\gamma^\iota_1}(\xi), \dots g^1_{\gamma^\iota_{n_\iota}}(\xi), p(\xi))\}), \end{aligned}$$

where $X \in \mathcal{F}$, $\langle \varphi_\iota : \iota \in I\rangle$ is a finite sequence of $\mathcal{L}$-formulas and $\gamma^\iota_1, \dots, \gamma^\iota_{n_\iota}$ for $\iota \in I$ are ordinals less than α.

We now prove that $E := E_{X, \langle \varphi_\iota : \iota \in I\rangle, \langle \gamma^\iota_1, \dots, \gamma^\iota_{n_\iota} : \iota \in I\rangle}$ is dense. Let $p \in \mathbb{P}$. For each $\xi \in \kappa$ and $\iota \in I$, put

$$v(\xi, \iota) = \begin{cases} 1 & \text{if } \mathcal{A}^0 \models \varphi_\iota(g^0_{\gamma^\iota_1}(\xi), \dots, g^0_{\gamma^\iota_{n_\iota}}(\xi), g^0_\alpha(\xi)) \\ 0 & \text{otherwise.} \end{cases}$$

And for each $\xi \in \kappa$ put

$$v(\xi) = \langle v(k, \iota) : \iota \in I\rangle.$$

Then by finiteness of ${}^I 2$, for some $v_0 \in {}^I 2$, we have $\kappa \setminus v^{-1}(v_0) \notin \mathcal{U}_\alpha$.

For each $\iota \in I$, put

$$\varphi_\iota^+(x_1^\iota, \dots, x_{n_\iota}^\iota, y) \equiv \begin{cases} \varphi_\iota(x_1^\iota, \dots, x_{n_\iota}^\iota, y) & \text{if } v_0(\iota) = 1 \\ \neg\varphi_\iota(x_1^\iota, \dots, x_{n_\iota}^\iota, y) & \text{otherwise.} \end{cases}$$

Put

$$\psi \equiv \exists y \bigwedge_{\iota \in I} \varphi_\iota^+(x_1^\iota, \dots, x_{n_\iota}^\iota, y).$$

Then by the induction hypothesis (5), $Y_{\psi, \langle \gamma_1^\iota, \dots \gamma_{n_\iota}^\iota : \iota \in I\rangle} \in \mathcal{U}_\alpha$. So take $\xi \in X \cap v^{-1}(v_0) \cap Y_{\psi, \langle \gamma_1^\iota, \dots \gamma_{n_\iota}^\iota : \iota \in I\rangle} \setminus \mathrm{dom}(p)$.

Since $M^0 \models \psi(\langle g^0_{\gamma_1^\iota}(\xi), \dots g^0_{\gamma_{n_\iota}^\iota}(\xi) : \iota \in I\rangle)$, we have $M^1 \models \psi(\langle g^1_{\gamma_1^\iota}(\xi), \dots g^1_{\gamma_{n_\iota}^\iota}(\xi) : \iota \in I\rangle)$.

By the definition of ψ, we can take $y \in M^1$ such that $M^1 \models \varphi_\iota^+(g^1_{\gamma_1^\iota}(\xi), \dots, g^1_{\gamma_{n_\iota}^\iota}(\xi), y)$ for every $\iota \in I$. We now put $q = p \cup \{(\xi, y)\}$. This witnesses denseness of E.

Then we put $g_\alpha^1 = \bigcup G$ and letting $\mathcal{U}_{\alpha+1}$ contain $\mathcal{U}_\alpha$ and the sets in (5) and have either the α-th element of the enumeration of 2^κ or its complement.

When α is odd, do the same construction above except for swapping 0 and 1.

Then the construction guarantees that $\mathcal{U} = \bigcup_{\alpha < 2^\kappa} \mathcal{U}_\alpha$ is an ultrafilter and that the function

$$\langle ([g_\alpha^0]_\mathcal{U}, [g_\alpha^1]_\mathcal{U}) : \alpha < 2^\kappa\rangle$$

is an isomorphic from $(\mathcal{A}^0)^\mathcal{U}$ to $(\mathcal{A}^1)^\mathcal{U}$. □

Fact 17 ([Vlu23, Theorem 4.3])**.** Let κ be an inaccessible cardinal. Then $\mathrm{cov}(\mathcal{M}_\kappa) \geq \lambda$ holds iff for every $X \subseteq \kappa^\kappa$ of size $<\lambda$ there is $S \in \prod_{i<\kappa} [\kappa]^{\leq |i|+1}$ such that for all $x \in X$ we have $\{i < \kappa : x(i) \in S(i)\}$ is cofinal in κ.

Fact 17 does not seem to generalize to anything other than inacessible cardinals. In fact, it is known that when κ is a successor cardinal, the cardinal invariant determined by slaloms as claimed above is equal to $\mathfrak{d}_\kappa$.

Theorem 18. Let κ be an inaccessible cardinal. Then $\mathrm{SAT}_\kappa^{\aleph_0}(\kappa)$ implies $\mathrm{cov}(\mathcal{M}_\kappa) = 2^\kappa$.

Proof. Let $\mathcal{U}$ be a regular ultrafilter on κ witnessing $\mathrm{SAT}_\kappa^{\aleph_0}(\kappa)$. Let $X \subseteq \kappa^\kappa$ of size $<2^\kappa$. Define a language $\mathcal{L}$ by $\mathcal{L} = \{\subseteq\}$. For $i < \kappa$, define a $\mathcal{L}$-structure $\mathcal{A}_i$ by $\mathcal{A}_i = ([\kappa]^{<|i|}, \subseteq)$. Since κ is inaccessible, we have $|\mathcal{A}_i| = \kappa$. For $x \in \kappa^\kappa$, we define $S_x = \langle \{x(i)\} : i < \kappa\rangle$. Put $\mathcal{A}_* = \prod_{i<\kappa} \mathcal{A}_i / \mathcal{U}$. Consider a set of formulas $p(S)$ defined by

$$p(S) = \{\ulcorner [S_x] \subseteq S \urcorner : x \in X\}.$$

Then $p(S)$ is finitely satisfiable and the number of parameters occurring in $p(S)$ is $<2^\kappa$. Thus, by $\mathrm{SAT}_\kappa^{\aleph_0}(\kappa)$, we can take $[S] \in \mathcal{A}_*$ realizing $p(S)$. Then we have

$$(\forall x \in X)(\{i < \kappa : x(i) \in S(i)\} \in \mathcal{U}).$$

But since our ultrafilter $\mathcal{U}$ is uniform, we have

$$(\forall x \in X)(\{i < \kappa : x(i) \in S(i)\} \text{ is cofinal}).$$

So by Fact 17, we showed $\operatorname{cov}(\mathcal{M}_\kappa) = 2^\kappa$. $\square$

Theorem 19. Let κ be a regular cardinal. Then $\operatorname{cov}(\mathcal{M}_\kappa) = 2^{<2^\kappa} = 2^\kappa$ implies $\mathrm{SAT}^\kappa_\kappa(\kappa)$.

Proof. Let $\langle b_\alpha : \alpha < 2^\kappa \rangle$ be an enumeration of κ^κ.

Let $\mathcal{L}^+ = \mathcal{L} \cup \{c_\alpha : \alpha < \mathfrak{c}\}$ where the c_α's are new constant symbols and let $\mathrm{Fml}(\mathcal{L}^+)$ be the set of all $\mathcal{L}^+$ formulas with one free variable.

Let $\langle (\mathcal{L}_\xi, T_\xi, \mathcal{B}_\xi, \Delta_\xi) : \xi < 2^\kappa \rangle$ be an enumeration of tuples $(\mathcal{L}, T, \mathcal{B}, \Delta)$ such that $\mathcal{L}$ is a language of size $\leq \kappa$, $T \colon \kappa \to \kappa + 1$, $\mathcal{B} = \langle \mathcal{A}_i : i < \kappa \rangle$ is a κ-sequence of $\mathcal{L}$-structures with i-th universe $T(i)$ and Δ is a subset of $\mathrm{Fml}(\mathcal{L}^+)$ with $|\Delta| < 2^\kappa$. Here we used $(2^\kappa)^{<2^\kappa} = 2^\kappa$. Ensure each $(\mathcal{L}, T, \mathcal{B}, \Delta)$ occurs cofinally in this sequence.

For $\mathcal{B}_\xi = \langle \mathcal{A}^\xi_i : i < \kappa \rangle$, we put

$$\mathcal{B}_\xi(i) = \langle \mathcal{A}^\xi_i, b_0(i) \upharpoonright T_\xi(i), b_1(i) \upharpoonright T_\xi(i), \dots \rangle,$$

which is a $\mathcal{L}^+$-structure. Here $\alpha \upharpoonright \beta = \begin{cases} \alpha & \text{if } \alpha < \beta \\ 0 & \text{otherwise} \end{cases}$ for α and β are ordinals.

Let $\langle X_\xi : \xi < 2^\kappa \rangle$ be an enumeration of $\mathcal{P}(\kappa)$. We construct a sequence of filters $\langle F_\xi : \xi < 2^\kappa \rangle$ satisfying following conditions:

(1) F_0 is the filter generated by a regularizing set for κ.

(2) $F_\xi \subseteq F_{\xi+1}$ and $F_\xi = \bigcup_{\alpha<\xi} F_\alpha$ for a limit ξ.

(3) $X_\xi \in F_{\xi+1}$ or $\kappa \setminus X_\xi \in F_{\xi+1}$.

(4) F_ξ is generated by $< 2^\kappa$ members.

(5) If

$$(\forall \Gamma \in [\Delta_\xi]^{<\aleph_0})(\{i < \kappa : \Gamma \text{ is satisfiable in } \mathcal{B}_\xi(i)\} \in F_\xi) \tag{$*$}$$

Then there is $f \in \prod_{i<\kappa} T_\xi(i)$ such that for every $\varphi \in \Delta_\xi$ we have $\{i < \kappa : f(i) \text{ satisfies } \varphi \text{ in } \mathcal{B}_\xi(i)\} \in F_{\xi+1}$.

Suppose that F_ξ is constructed and $(*)$ holds. Let

$$\mathbb{P} = \{p : p \text{ is a partial function of size} < \kappa \text{ from } \kappa \text{ to } \kappa\}$$

This forcing notion $\mathbb{P}$ is forcing equivalent to the forcing adding a κ-Cohen function.

Fix a generating set F'_ξ of F_ξ of size $< 2^\kappa$. For each $A \in F'_\xi$ and $\varphi_1, \dots, \varphi_n \in \Delta_\xi$, we put

$$E_{A,\varphi_1,\dots,\varphi_n} = \{p \in \mathbb{P} : (\exists i \in \mathrm{dom}(p) \cap A)(p(i) \text{ is element of } T_\xi(i) \text{ and satisfies } \varphi_1, \dots, \varphi_n \text{ in } \mathcal{B}_\xi(i))\}$$

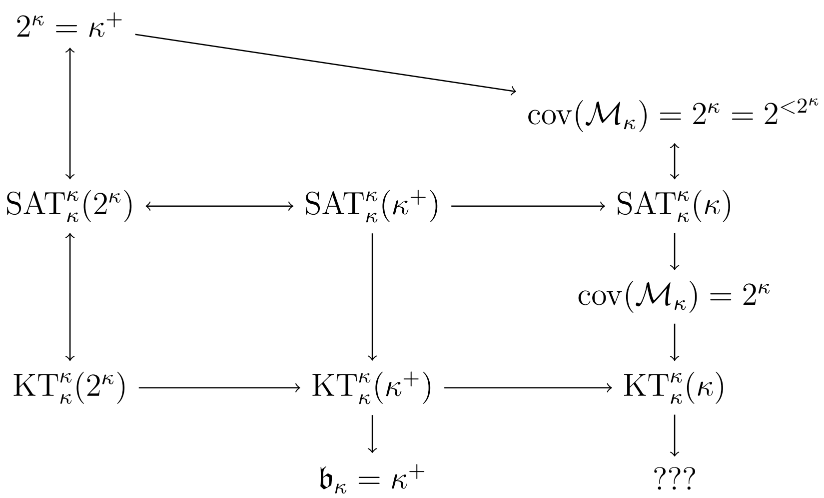
By assumption $(*)$, these $E_{A,\varphi_1,\dots,\varphi_n}$'s are dense subsets in $\mathbb{P}$.

So using $\mathrm{MA}_{<2^\kappa}(\mathbb{P})$, we have a filter G of $\mathbb{P}$ that intersects all $E_{A,\varphi_1,\dots,\varphi_n}$'s. Put $f(i) = (\bigcup G)(i) \restriction T_\xi(i)$. Then we can extend our filter F_ξ to $F_{\xi+1}$ such that for every $\phi \in \Delta_\xi$ $\{i < \kappa : f(i)$ satisfies φ in $\mathcal{B}_\xi(i)\} \in F_{\xi+1}$. Moreover we can extend this filter satisfying $X_\xi \in F_{\xi+1}$ or $\kappa \setminus X_\xi \in F_{\xi+1}$. This finishes the construction.

In order to check that the resulting ultrafilter $F = \bigcup_{\xi<2^\kappa} F_\xi$ witnesses $\mathrm{SAT}^\kappa_\kappa(\kappa)$, let $\mathcal{L}$ be a language of size $\leq \kappa$ and $\mathcal{B} = \langle \mathcal{A}_i : i \in \kappa \rangle$ be a sequence of $\mathcal{L}$-structures. We may assume that, for each $i < \kappa$, the universe of $\mathcal{A}_i$ is an ordinal. Let $T(i) =$ the universe of $\mathcal{A}_i$. Let Δ be a subset of $\mathrm{Fml}(\mathcal{L}^+)$ with $|\Delta| < 2^\kappa$. Assume that for all $\Gamma \subseteq \Delta$ finite, $X_\Gamma := \{i \in \kappa : \Gamma$ is satisfiable in $\mathcal{B}(i)\} \in F$. By the regularity of 2^κ which follows from the cardinal arithmetical assumption of the theorem, we have $\alpha < 2^\kappa$ such that for all $\Gamma \subseteq \Delta$ finite, $X_\Gamma \in F_\alpha$. Let $\xi \geq \alpha$ be satisfying $(\mathcal{L}_\xi, T_\xi, \mathcal{B}_\xi, \Delta_\xi) = (\mathcal{L}, T, \mathcal{B}, \Delta)$. Then by (5), there is a $f \in \prod_{i<\kappa} T(i)$ such that for all $\varphi \in \Delta$, $\{i \in \kappa : f(i)$ satisfies φ in $\mathcal{B}(i)\} \in F$. Thus $\prod_{i\in\kappa} \mathcal{A}_i/F$ is saturated. $\square$

3 Discussion

From the results of Section 2, the following figure can be drawn for an inaccessible cardinal κ.



In light of this, the following two questions naturally arise.

Question 20. (1) Can we eliminate the inaccessibility assumption from the result which states $\mathrm{SAT}^{\aleph_0}_\kappa(\kappa)$ implies $\mathrm{cov}(\mathcal{M}_\kappa) = 2^\kappa$?

(2) Can we prove the consistency of $\neg\,\mathrm{KT}^\kappa_\kappa(\kappa)$?

For the second question the answer Yes is obtained when $\kappa = \aleph_0$ ([She92]). [Got22] improves on that result, showing $\mathrm{KT}^{\aleph_0}_{\aleph_0}(\aleph_0) \Rightarrow \mathrm{cov}(\mathcal{N}) \leq \mathfrak{d}$. By generalizing the proof to an inaccessible cardinal, we obtain the following.

Theorem 21. Let κ be an inaccessible cardinal. Then $\mathrm{KT}^{\aleph_0}_\kappa(\kappa)$ implies $\mathfrak{v}^\forall_\kappa \leq \mathfrak{d}_\kappa$. $\square$

Here, for a cardinal κ and $c, h \in \kappa^\kappa$, letting $\prod c = \prod_{\alpha<\kappa} c(\alpha)$ and $S(c,h) = \prod_{\alpha<\kappa}[c(\alpha)]^{<h(\alpha)}$, we define

$$\begin{aligned}\mathfrak{v}^\forall_{\kappa,c,h} = \min\{|X| : X \subseteq \prod c, (\forall \varphi \in S(c,h))(\exists x \in X)\\ (\forall \alpha < \kappa)(\exists \beta \in [\alpha, \kappa))(x(\alpha) \notin \varphi(\alpha))\}.\end{aligned}$$

Also, we define $\mathfrak{v}^\forall_\kappa = \min\{\mathfrak{v}^\forall_{\kappa,c,h} : c, h \in \kappa^\kappa, \text{ and } h \text{ diverges to } \infty\}$.

However, for an inaccessible cardinal κ, the consistency of $\mathfrak{d}_\kappa < \mathfrak{v}^\forall_\kappa$ is not currently known. The situation differs from cardinal invariants at ω in that forcing notions such as random forcing are not known for higher cardinals, nor are good generalizations of properties such as ω^ω-bounding proper forcing.

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