A NOTE ON UNIFORM ULTRAFILTERS IN A CHOICELESS CONTEXT

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Abstract. In [4], Hayut and Karagila asked some questions about uniform ultrafilters in a choiceless context. We provide several answers to their questions.

1. Introduction

A proper ultrafilter U over an infinite cardinal κ is said to be *uniform* if every element of U has cardinality κ. Let $\mathcal{U}$ be the class of all infinite cardinals κ which carries a uniform ultrafilter. In ZFC, $\mathcal{U}$ has a trivial structure: It is just the class of all cardinals. However it is not the case if the Axiom of Choice fails.

Theorem 1.1 (Hayut-Kagagila [4]). *Relative to a certain large cardinal assumption, it is consistent that* $\mathsf{ZF}+$ "$\aleph_0, \aleph_\omega \in \mathcal{U}$ *but* $\aleph_{n+1} \notin \mathcal{U}$ *for every* $n < \omega$". *Oppositely, it is also consistent that* $\mathsf{ZF}+$ "$\aleph_\omega \notin \mathcal{U}$ *but* $\aleph_{n+1} \in \mathcal{U}$ *for every* $n < \omega$".

Furthermore, Hayut and Karagila demonstrated that the behavior of $\mathcal{U}$ at the successors of regular cardinals can be manipulated as you like. With these results, they asked the following questions for singular cardinals and its successors.

Question 1.2 ([4]).
(1) Is it consistent for $\aleph_{\omega+1}$ to be the least element of $\mathcal{U}$? More generally, what behavior is consistent at successors of singular cardinals?
(2) Is it consistent for a singular cardinal, and specifically $\aleph_\omega$, to be the least cardinal not in $\mathcal{U}$?
(3) Assume there is a uniform ultrafilter on $\aleph_{\omega_\omega}$, does that imply there is a uniform ultrafilter on $\aleph_\omega$? Or more generally, if $\lambda > \mathrm{cf}(\lambda)$ carries a uniform ultrafilter, does that imply that any other singular cardinal with the same cofinality carries a uniform ultrafilter?

In this paper, we provide several answers to these questions by proving the following theorems:

Theorem 1.3. *Relative to a certain large cardinal assumption, it is consistent that* $\mathsf{ZF}+$ "$\aleph_{\omega+1}$ *is the least element of* $\mathcal{U}$".

Theorem 1.4. *It is consistent that* $\mathsf{ZF}+$ "$\aleph_\omega$ *is the least cardinal not in* $\mathcal{U}$".

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Theorem 1.5. *Relative to a certain large cardinal assumption, it is consistent that* $\mathsf{ZF}+$ "$\aleph_\omega$ *is the least cardinal not in* $\mathcal{U}$"+ "*every singular cardinal* $>\aleph_\omega$ *with countable cofinality is in* $\mathcal{U}$".

In [4], they also asked the following:

Question 1.6 ([4])**.** Is it consistent that κ does not carry a uniform ultrafilter, κ^+ does, but κ^+ is not measurable, and is this possible without using large cardinals? In particular, is it consistent that $\aleph_0$ is the only measurable cardinal, while $\aleph_1 \notin \mathcal{U}$ and $\aleph_2 \in \mathcal{U}$?

While we do not have a full answer to this question, we prove that such a situation has a large cardinal strength.

Theorem 1.7 (In ZF)**.** *If there are cardinals* $\kappa<\lambda$ *with* $\kappa \notin \mathcal{U}$ *but* $\lambda \in \mathcal{U}$*, then there is an inner model of a measurable cardinal.*

This theorem also shows that large cardinal assumptions in Theorems 1.3 and 1.5 cannot be eliminated.

2. Preliminaries

Throughout this paper, we always suppose that every successor cardinal is regular. First we prove basic lemmas which will be used later. The following lemmas follow from the standard arguments, but here we present choiceless proofs for the completeness. Let $\mathbb{P}$ be a poset with maximum element $\mathbb{1}$. For a set x, let $\check{x}$ be a canonical name for x, namely, $\check{x}=\{\langle \check{y}, \mathbb{1}\rangle \mid y \in x\}$.

Lemma 2.1 (In ZF)**.** *Let* $\mathbb{P}$ *be a countable poset. Then* $\mathbb{P}$ *preserves all cofinalities and cardinals.*

Proof. For cofinality, it is enough to show that if κ is regular uncountable, then $\Vdash$"$\check{\kappa}$ is regular". To verify this, take $p \in \mathbb{P}$, $\alpha<\kappa$, and a name $\dot{f}$ such that $p \Vdash \dot{f}: \check{\alpha} \to \check{\kappa}$. For each $\beta<\alpha$, let $A_\beta=\{\xi \mid \exists q \leq p(q \Vdash \dot{f}(\check{\beta})=\check{\xi})\}$. We know $p \Vdash \dot{f}(\check{\beta}) \in \check{A}_\beta$, and since $\mathbb{P}$ is countable, we have that A_β is countable. Let $\gamma=\sup\{\sup A_\beta \mid \beta<\alpha\}$. Since κ is regular uncountable, we have $\gamma<\kappa$ and $p \Vdash \dot{f}``\alpha \subseteq \gamma$. Hence $\dot{f}$ is not forced to be a cofinal map.

For preserving cardinals, take cardinals $\kappa<\lambda$, $p \in \mathbb{P}$, and a name $\dot{f}$ for a function from κ to λ. Define $F: \mathbb{P} \times \kappa \to \lambda$ as follows: If $q \Vdash \dot{f}(\check{\alpha})=\check{\beta}$ for some $\beta<\lambda$, set $F(q,\alpha)=\beta$, here note that such β is unique for q and α. Otherwise, let $F(q,\alpha)=0$. Since $\mathbb{P}$ is countable, we have $|\mathbb{P} \times \kappa|=\kappa$. Hence F cannot be a sujection and we can take $\gamma<\lambda$ with $\gamma \notin \mathrm{range}(F)$. Then we have $p \Vdash \check{\gamma} \notin \mathrm{range}(\dot{f})$, so $\dot{f}$ is forced to be non-surjective. □

For a cardinal κ, an ultrafilter U is *κ-complete* if for every $\alpha<\kappa$ and $f: \alpha \to U$, we have $\bigcap f``\alpha \in U$. U is *σ-complete* if it is ω_1-complete.

Lemma 2.2 (In ZF)**.** *If there is a non-principal non-σ-complete ultrafilter* U *over a set* S*, then* ω *carries a non-principal ultrafilter.*

Proof. Since U is not σ-complete, we can find a function $f: \omega \to U$ such that $\bigcap f``n \in U$ for every $n<\omega$ but $\bigcap f``\omega=\emptyset$. Define $g: S \to \omega$ as follows: For $s \in S$, let $g(s)$ be the least $n<\omega$ with $s \notin \bigcap f``n$. Then the family $g_*(U)=\{X \subseteq \omega \mid g^{-1}(X) \in U\}$ is a non-principal ultrafilter over ω. □

Lemma 2.3 (In ZF). *Suppose the Countable Choice holds. Let $\mathbb{P}$ be a countable poset, κ a cardinal, and $\dot{U}$ a name such that $\Vdash$ "$\dot{U}$ is a σ-complete uniform ultrafilter over $\check{\kappa}$". Then there is $p \in \mathbb{P}$ such that the following hold:*

1. *For every $X \subseteq \kappa$, either $p \Vdash \check{X} \in \dot{U}$ or $p \Vdash \check{\kappa} \setminus \check{X} \in \dot{U}$.*
2. *The set $\{X \subseteq \kappa \mid p \Vdash \check{X} \in \dot{U}\}$ is a σ-complete uniform ultrafilter over κ.*

Proof. (2) is immediate from (1). For (1), suppose not. By the Countable Choice, we can find $\{X_p \mid p \in \mathbb{P}\}$ such that $X_p \subseteq \kappa$, $p \not\Vdash \check{X}_p \in \dot{U}$, and $p \not\Vdash \check{\kappa} \setminus \check{X}_p \in \dot{U}$. Take a name $\dot{Y}$ such that

$$\Vdash \dot{Y} = \bigcap\{\check{X}_p \mid p \in \mathbb{P}, \check{X}_p \in \dot{U}\} \cap \bigcap\{\check{\kappa} \setminus \check{X}_p \mid p \in \mathbb{P}, \check{\kappa} \setminus \check{X}_p \in \dot{U}\}.$$

Since $\mathbb{P}$ is countable, we have $\Vdash \dot{Y} \in \dot{U}$. We also know $\Vdash$"$\dot{Y} \subseteq \check{X}_p$ or $\dot{Y} \cap \check{X}_p = \emptyset$" for every $p \in \mathbb{P}$.

For $p \in \mathbb{P}$, let $Y_p = \{\alpha < \kappa \mid p \Vdash \check{\alpha} \in \dot{Y}\}$. We have $\Vdash \dot{Y} = \bigcup\{\check{Y}_p \mid p \in \dot{G}\}$ where $\dot{G}$ is a canonical name for a generic filter. Again, since $\mathbb{P}$ is countable and $\dot{U}$ is forced to be σ-complete, we can find $p, q \in \mathbb{P}$ such that $q \Vdash \check{p} \in \dot{G} \wedge \check{Y}_p \in \dot{U}$. By extending q, we may assume $q \leq p$. Then $Y_q \supseteq Y_p$, so $q \Vdash \check{Y}_q \in \dot{U}$. Because $q \Vdash \check{Y}_q \subseteq \dot{Y}$, if $Y_q \cap X_q \neq \emptyset$ then $q \Vdash \check{X}_q \in \dot{U}$, this contradicts the choice of X_q. Hence $Y_q \cap X_q = \emptyset$, but then $q \Vdash \check{\kappa} \setminus \check{X}_q \in \dot{U}$, this is also a contradiction. □

We will use *symmetric extensions.* Here we review it, and see Jech [5] for details. We emphasize that we do not need AC for taking symmetric extensions and establishing basic results about it.

Every automorphism π on $\mathbb{P}$ induces the isomorphism π on the $\mathbb{P}$-names, namely, $\pi(\dot{x}) = \{\langle \pi(\dot{y}), \pi(p)\rangle \mid \langle \dot{y}, p\rangle \in \dot{x}\}$ for a $\mathbb{P}$-name $\dot{x}$.

Fact 2.4. *Let $p \in \mathbb{P}$, φ be a formula of set theory, and $\dot{x}_0, \dots, \dot{x}_n$ $\mathbb{P}$-names. Let π be an automorphism on $\mathbb{P}$. Then $p \Vdash \varphi(\dot{x}_0, \dots, \dot{x}_n)$ if and only if $\pi(p) \Vdash \varphi(\pi(\dot{x}_0), \dots, \pi(\dot{x}_n))$.*

Let $\mathcal{G}$ be a subgroup of the automorphism group on $\mathbb{P}$. A non-empty family $\mathcal{F}$ of subgroups of $\mathcal{G}$ is a *normal filter on $\mathcal{G}$* if the following hold:

1. If $H \in \mathcal{F}$ and H' is a subgroup of $\mathcal{G}$ with $H \subseteq H'$, then $H' \in \mathcal{F}$.
2. For $H, H' \in \mathcal{F}$, we have $H \cap H' \in \mathcal{F}$.
3. For every $H \in \mathcal{F}$ and $\pi \in \mathcal{G}$, the set $\pi^{-1} H \pi = \{\pi^{-1} \circ \sigma \circ \pi \mid \sigma \in H\}$ is in $\mathcal{F}$.

A triple $\langle \mathbb{P}, \mathcal{G}, \mathcal{F}\rangle$ is called a *symmetric system.*

For a $\mathbb{P}$-name $\dot{x}$, let $\mathrm{sym}(\dot{x}) = \{\pi \in \mathcal{G} \mid \pi(\dot{x}) = \dot{x}\}$, which is a subgroup of $\mathcal{G}$. A name $\dot{x}$ is *symmetric* if $\mathrm{sym}(\dot{x}) \in \mathcal{F}$, and *hereditarily symmetric* if $\dot{x}$ is symmetric and for every $\langle \dot{y}, p\rangle \in \dot{x}$, $\dot{y}$ is hereditarily symmetric.

Fact 2.5. *If $\dot{x}$ is a hereditarily symmetric name and $\pi \in \mathcal{G}$, then $\pi(\dot{x})$ is also hereditarily symmetric.*

Let HS be the class of all hereditarily symmetric names. For a $(V, \mathbb{P})$-generic G, let HS^G be the class of all interpretations of hereditarily symmetric names by G. HS^G is a transitive model of ZF with $V \subseteq \mathrm{HS}^G \subseteq V[G]$. HS^G is called a *symmetric extension of V.*

3. $\aleph_{\omega+1}$ CAN BE THE LEAST ELEMET OF $\mathcal{U}$

We give a proof of Theorem 1.3. For this sake, we use the following Apter and Madigor's theorem. Recall that an uncountable cardinal κ is *measurable* if κ carries a κ-complete non-principal ultrafilter. In ZF, every measurable cardinal is regular.

Theorem 3.1 (Apter [1], Apter-Magidor [2])**.** *Suppose V satisfies $\mathsf{GCH}+$ "$\kappa < \lambda$ are such that κ is supercompact and λ is the least measurable cardinal above κ". Then there is a symmetric extension N such that the following hold in N:*

(1) $\mathsf{DC}_{\aleph_\omega}$ holds in N.
(2) $\lambda = \aleph_{\omega+1}$ is measurable in N.
(3) The cardinal and cofinality structure $\geq \lambda$ is the same as in V.

We start from this Apter and Magidor's model, that is, we work in a model V of $\mathsf{ZF} + \mathsf{DC}_{\aleph_\omega} +$ "$\aleph_{\omega+1}$ is measurable"+"every successor cardinal is regular".

Lemma 3.2. *For $n < \omega$, there is no σ-complete uniform ultrafilter over $\aleph_n$.*

Proof. Suppose not, and take the least $n < \omega$ such that $\aleph_n$ carries a σ-complete uniform ultrafilter U. Clearly $n > 0$. First we show that U is not $\aleph_n$-complete. By $\mathsf{DC}_{\aleph_\omega}$, we can take a 1-1 sequence $\langle r_\alpha \mid \alpha < \aleph_n \rangle$ of subsets of $\aleph_{n-1}$. For $\beta < \aleph_{n-1}$, define $A_\beta \in U$ as follows: If $\{\alpha < \aleph_n \mid \beta \in r_\alpha\} \in U$, then put $A_\beta = \{\alpha < \aleph_n \mid \beta \in r_\alpha\}$. Otherwise, that is, if $\{\alpha < \aleph_n \mid \beta \notin r_\alpha\} \in U$, then $A_\beta = \{\alpha < \aleph_n \mid \beta \notin r_\alpha\}$. If U is $\aleph_n$-complete, we have $\bigcap_{\beta < \aleph_{n-1}} A_\beta \in U$. Pick $\alpha, \alpha' \in \bigcap_{\beta < \aleph_{n-1}} A_\beta$ with $\alpha < \alpha'$. By the choice of the A_β's, we have $r_\alpha = r_{\alpha'}$. This is a contradiction.

Now we know U is not $\aleph_n$-complete. Take the largest $m < n$ such that U is $\aleph_m$-complete. Then we can find $f : \aleph_m \to U$ such that $\bigcap f``\alpha \in U$ for every $\alpha < \aleph_m$ but $\bigcap f``\aleph_m = \emptyset$. Define $g : \aleph_n \to \aleph_m$ as that $g(\beta)$ is the least $\alpha < \aleph_m$ with $\beta \notin \bigcap f``\alpha$. Consider the ultrafilter $g_*(U) = \{X \subseteq \aleph_m \mid g^{-1}(X) \in U\}$. By the choice of g, one can check that $g_*(U)$ is a uniform ultrafilter over $\aleph_m$, moreover it is σ-complete. This contradicts to the minimality of n. $\square$

For a set X, let $\mathrm{Fn}(X, 2)$ be the poset of all finite partial functions from X to 2 with the reverse inclusion. We define a symmetric system $\langle \mathbb{P}, \mathcal{G}, \mathcal{F} \rangle$, which is Feferman's one.

Let $\mathbb{P}$ be the poset $\mathrm{Fn}(\omega \times \omega, 2)$. For a set $A \subseteq \omega \times \omega$, let π_A be the automorphism π_A on $\mathbb{P}$ defined as follows: For $p \in \mathbb{P}$, $\mathrm{dom}(\pi_A(p)) = \mathrm{dom}(p)$, and

$$\pi_A(p)(m,n) = \begin{cases} 1 - p(m,n) & \text{if } \langle m, n \rangle \in A, \\ p(m,n) & \text{if } \langle m, n \rangle \notin A. \end{cases}$$

Let $\mathcal{G}$ be the set $\{\pi_A \mid A \subseteq \omega \times \omega\}$. $\mathcal{G}$ is a subgroup of the automorphism group of $\mathbb{P}$. For $m < \omega$, let $\mathrm{fix}(m) = \{\pi_A \in \mathcal{G} \mid A \cap (m \times \omega) = \emptyset\}$, $\mathrm{fix}(m)$ is a subgroup of $\mathcal{G}$. Let $\mathcal{F} = \{H \subseteq \mathcal{G} \mid H \text{ is a subgroup of } \mathcal{G}, \mathrm{fix}(m) \subseteq H$ for some $m < \omega\}$. It is routine to check that $\mathcal{F}$ is a normal filter on $\mathcal{G}$.

Take a $(V, \mathbb{P})$-generic G. Notice that $\mathbb{P}$ is countable, hence $\mathbb{P}$ preserves all cofinalities and cardinals. In particular every successor cardinal is regular in $V[G]$. Consider a symmetric extension HS^G.

The following Lemmas 3.3–3.6 are known (see [4]), but we present proofs for the completeness. Again, we do not need AC for proving these lemmas.

For $p \in \mathbb{P}$ and $m < \omega$, let $p \restriction m$ be the condition $p \restriction (m \times \omega)$, it is in $\mathrm{Fn}(m \times \omega, 2)$.

Lemma 3.3. *Let $\varphi(v_0, \dots, v_n)$ be a formula of set theory and $\dot{x}_0, \dots \dot{x}_n$ be $\mathbb{P}$-names. Let $m < \omega$, and suppose $\mathrm{fix}(m) \subseteq \mathrm{sym}(\dot{x}_i)$ for every $i \leq n$. If $p \Vdash \varphi(\dot{x}_0, \dots, \dot{x}_n)$, then $p \restriction m \Vdash \varphi(\dot{x}_0, \dots, \dot{x}_n)$.*

Proof. Suppose not, and take $q \leq p \restriction m$ such that $q \Vdash \neg\varphi(\dot{x}_0, \dots, \dot{x}_n)$. Let $A = \{\langle n, i \rangle \in \mathrm{dom}(q) \cap \mathrm{dom}(p) \mid q(n,i) \neq p(n,i)\}$. We know $A \cap (m \times \omega) = \emptyset$, and so $\pi_A \in \mathrm{fix}(m)$ and $\pi_A(\dot{x}_k) = \dot{x}_k$ for $k \leq n$. Moreover $\pi_A(q)$ is compatible with p, but $\pi_A(q) \Vdash \neg\varphi(\dot{x}_1, \dots, \dot{x}_n)$. This is a contradiction. $\square$

For $m < \omega$, let $x_m = \{n < \omega \mid \exists p \in G(p(m,n) = 1)\}$, and $\dot{x}_m$ be the name $\{\langle \check{m}, p \rangle \mid p(m,n) = 1\}$. $\dot{x}_m$ is a canonical hereditarily symmetric name for x_m, so we have $x_m \in \mathrm{HS}^G$.

Lemma 3.4. *In HS^G, there is no non-principal ultrafilter over ω.*

Proof. Suppose there is a non-principal ultrafilter U over ω. Fix a hereditarily symmetric name $\dot{U}$ for U. Fix $m < \omega$ with $\mathrm{fix}(m) \subseteq \mathrm{sym}(\dot{U})$. We prove that both x_m and $\omega \setminus x_m$ are not in U, this is a contradiction.

Suppose $x_m \in U$. Take $p \in G$ such that $p \Vdash \dot{x}_m \in \dot{U}$. Fix a large $n_0 < \omega$ such that $\mathrm{dom}(p) \cap (\{m\} \times \omega) \subseteq \{m\} \times n_0$, and let $A = \{m\} \times [n_0, \omega)$. One can check that $\pi_A(p) = p$, and $\Vdash \pi_A(\dot{x}_m) \cap \dot{x}_m \subseteq \check{n}_0$. Since $p \Vdash \dot{x}_m \in \dot{U}$, we have $p \Vdash \pi_A(\dot{x}_m) \in \pi_A(\dot{U}) = \dot{U}$, hence $p \Vdash \dot{x}_m \cap \pi_A(\dot{x}_m) \subseteq \check{n}_0 \in \dot{U}$. This is a contradiction. The case $\omega \setminus x_m \in U$ follows from a similar argument. $\square$

Since every uniform ultrafilter over $\aleph_\omega$ is not σ-complete, we also have:

Lemma 3.5. *In HS^G, there is no uniform ultrafilter over $\aleph_\omega$.*

Proof. Otherwise, we can take a non-principal ultrafilter over ω by Lemma 2.2, this contradicts to Lemma 3.4. $\square$

Since $\aleph_{\omega+1}$ is measurable in V, we can fix an $\aleph_{\omega+1}$-complete non-principal ultarfilter $U \in V$ over $\aleph_{\omega+1}$, which is a uniform ultrafilter. Since $\mathbb{P}$ is countable and U is σ-complete in V, one can check that:

Lemma 3.6. *In $V[G]$, the set $\{X \subseteq \aleph_{\omega+1} \mid \exists Y \in U(Y \subseteq X)\}$ is a uniform ultrafiler over $\aleph_{\omega+1}$. In particular, in HS^G, U generates a uniform ultrafilter over $\aleph_{\omega+1}$.*

Proof. It is enough to check that for every $X \subseteq \aleph_{\omega+1}$, there is $Y \in U$ such that $Y \subseteq X$ or $X \cap Y = \emptyset$. Take $p \in \mathbb{P}$ and a name $\dot{X}$ for a subset of $\aleph_{\omega+1}$. For $q \leq p$, let $Y_q = \{\alpha < \aleph_{\omega+1} \mid q \Vdash \check{\alpha} \in \dot{X}\}$. If $Y_q \in U$ for some $q \leq p$, then $q \Vdash \check{Y}_q \subseteq \dot{X}$. If $Y_q \notin U$ for every $q \leq p$, let $Y = \bigcap_{q \leq p}(\aleph_{\omega+1} \setminus Y_q)$. We have $Y \in U$ since U is σ-complete and $\mathbb{P}$ is countable. In addition we have $p \Vdash \check{Y} \cap \dot{X} = \emptyset$. $\square$

For $m < \omega$, let $G_m = G \cap \mathrm{Fn}(m \times \omega, 2)$. G_m is $(V, \mathrm{Fn}(m \times \omega, 2))$-generic. The name $\{\langle (p \restriction m)\check{}, p\rangle \mid p \in \mathbb{P}\}$ is a canonical hereditarily symmetric name for G_m. Hence we have:

Lemma 3.7. *$V[G_m] \subseteq \mathrm{HS}^G$ for every $m < \omega$.*

Finally we prove that $\aleph_n$ does not carry a uniform ultrafilter over $\aleph_n$ for every $n < \omega$ in HS^G.

Lemma 3.8. *In HS^G, for every $n < \omega$, there is no uniform ultrafilter over $\aleph_n$.*

Proof. Suppose not, and let $n < \omega$ and W a uniform ultrafilter over $\aleph_n$. We know $n > 0$ by Lemma 3.4. If W is not σ-complete in HS^G, then we can derive a non-principal ultrafilter over ω by Lemma 2.2, this is a contradiction. Hence W is σ-complete.

Take a hereditarily symmetric name $\dot{W}$ for W and $m < \omega$ such that $\mathrm{fix}(m) \subseteq \mathrm{sym}(\dot{W})$. For each $X \in \mathcal{P}(\aleph_n) \cap V[G_m]$, there is a hereditarily symmetric name $\dot{X}$ for X with $\mathrm{fix}(m) \subseteq \mathrm{sym}(\dot{X})$. Hence by Lemma 3.3, we have:

$$X \in W \iff p \Vdash \dot{X} \in \dot{W} \text{ for some } p \in G_m.$$

Thus $W' = W \cap V[G_m] \in V[G_m]$, which is a σ-complete ultrafilter over $\aleph_n$ in $V[G_m]$. Then by Lemma 2.3, we can find a σ-complete uniform ultrafilter over $\aleph_n$ in V, this contradicts to Lemma 3.2. □

4. $\aleph_\omega$ can be the least cardinal not in $\mathcal{U}$

We start the proof of Theorem 1.4. For a set X and a cardinal κ, let $\mathrm{Fn}(X, 2, {<}\kappa)$ be the poset of all partial functions $p : X \to 2$ with size $< \kappa$. The following lemma is well-known:

Lemma 4.1. *Let κ be a regular uncountable cardinal and $\mathbb{P}, \mathbb{Q}$ posets. If $\mathbb{P}$ satisfies the κ-c.c. and $\mathbb{Q}$ is κ-closed, then $\Vdash_{\mathbb{P}}$ "$\check{\mathbb{Q}}$ is κ-Baire".*

Suppose GCH. For $n < \omega$, let $\mathbb{Q}_n = \mathrm{Fn}(\aleph_\omega, 2, {<}\aleph_n)$. Let $\mathbb{P}$ be the full support product of the $\mathbb{Q}_n$'s, that is, $\mathbb{P} = \prod_{n<\omega} \mathbb{Q}_n$, and $p \le q \iff p(n) \le q(n)$ in $\mathbb{Q}_n$ for every $n < \omega$. For simplicity, we denote $p(n)(\alpha)$ as $p(n, \alpha)$.

Let $n < \omega$ and $\kappa = \aleph_{n+1}$. The poset $\mathbb{P}$ can be factored as the product $(\prod_{m \le n} \mathbb{Q}_n) \times (\prod_{n<m<\omega} \mathbb{Q}_m)$. The poest $\prod_{n<m<\omega} \mathbb{Q}_m$ is κ-closed, and, by GCH, $\prod_{m \le n} \mathbb{Q}_n$ satisfies the κ-c.c. Thus $\prod_{m\le n} \mathbb{Q}_n$ forces that $\prod_{n<m<\omega} \mathbb{Q}_m$ is κ-Baire. By using this observation, one can check that $\mathbb{P}$ preserves all cofinalities and cardinals.

For $A \subseteq \omega \times \aleph_\omega$, define the automorphism π_A on $\mathbb{P}$ as follows: $\mathrm{dom}(\pi_A(p)(n)) = \mathrm{dom}(p(n))$ for every $n < \omega$, and

$$\pi_A(p)(n, \alpha) = \begin{cases} 1 - p(n, \alpha) & \text{if } \langle n, \alpha\rangle \in A, \\ p(n, \alpha) & \text{if } \langle n, \alpha\rangle \notin A. \end{cases}$$

Let $\mathcal{G} = \{\pi_A \mid A \subseteq \omega \times \aleph_\omega\}$, this is a subgroup of the automorphism group of $\mathbb{P}$. For $n < \omega$, let $\mathrm{fix}(n) = \{\pi_A \mid (n \times \aleph_\omega) \cap A = \emptyset\}$. $\mathrm{fix}(n)$ is a subgroup of $\mathcal{G}$. Let $\mathcal{F} = \{H \subseteq \mathcal{G} \mid H \text{ is a subgroup with } \mathrm{fix}(n) \subseteq H \text{ for some } n < \omega\}$. One can check that $\mathcal{F}$ is a normal filter on $\mathcal{G}$.

Take a $(V, \mathbb{P})$-generic G. For $n < \omega$, let $G_n = \{p \upharpoonright n \mid p \in G\}$ which is $(V, \prod_{m<n} \mathbb{Q}_m)$-generic. Let $\dot{G}_n = \{\langle (p \upharpoonright n)\check{}, p\rangle \mid p \in \mathbb{P}\}$. $\dot{G}_n$ is a hereditarily symmetric name for G_n with $\mathrm{fix}(n) \subseteq \mathrm{sym}(\dot{G}_n)$. Hence we have:

Lemma 4.2. *$G_n \in \mathrm{HS}^G$, in particular $V[G_n] \subseteq \mathrm{HS}^G$ for every $n < \omega$.*

Lemma 4.3. *For every $n < \omega$, $\aleph_n$ carries a uniform ultrafilter over $\aleph_n$ in HS^G.*

Proof. Fix $n < \omega$. $\mathbb{P}$ can be factored as the product $(\prod_{m \leq n} \mathbb{Q}_n) \times (\prod_{n<m<\omega} \mathbb{Q}_m)$, and, in $V[G_{n+1}]$, the poset $(\prod_{n<m<\omega} \mathbb{Q}_m)$ is $\aleph_{n+1}$-Baire. Hence $\mathcal{P}(\aleph_n)^{V[G]} = \mathcal{P}(\aleph_n)^{V[G_{n+1}]}$. Because $V[G_{n+1}]$ satisfies AC, we can find a uniform ultrafilter over U in $V[G_{n+1}]$, and U remains an ultrafilter in $V[G]$. Since $V[G_{n+1}] \subseteq \mathrm{HS}^G \subseteq V[G]$, we have $U \in \mathrm{HS}^G$ is a uniform ultrafilter in HS^G. □

For each $n < \omega$, let $X_n = \{\alpha < \aleph_\omega \mid p(n, \alpha) = 1 \text{ for some } p \in G\}$, and $\dot{X}_n$ be the $\mathbb{P}$-name $\{\langle \check{\alpha}, p\rangle \mid p \in \mathbb{P}, p(n, \alpha) = 1\}$. $\dot{X}_n$ is a canonical hereditarily symmetric name for X_n with $\mathrm{fix}(n+1) \subseteq \mathrm{sym}(\dot{X}_n)$. Hence $X_n \in \mathrm{HS}^G$. It is routine to check the following:

Lemma 4.4. *Let $n < \omega$ and $A \subseteq \omega \times \aleph_\omega$. Let $x = \{\alpha < \aleph_\omega \mid \langle n, \alpha\rangle \notin A\}$. Then $\Vdash \dot{X}_n \cap \pi_A(\dot{X}_n) \subseteq \check{x}$.*

Lemma 4.5. *There is no uniform ultrafilter over $\aleph_\omega$ in HS^G.*

Proof. Suppose not. Let $U \in \mathrm{HS}^G$ be a uniform ultrafilter over $\aleph_\omega$, and $\dot{U}$ be a hereditarily symmetric name for U. Take $n < \omega$ with $\mathrm{fix}(n) \subseteq \mathrm{sym}(\dot{U})$. We show that both X_n and $\aleph_\omega \setminus X_n$ are not in U.

First suppose to the contrary that $X_n \in U$. Take $p \in G$ such that $p \Vdash \dot{X}_n \in \dot{U}$. Let $d = \mathrm{dom}(p(n))$, and $A = \{\langle n, \alpha\rangle \in \omega \times \aleph_\omega \mid \alpha \notin d\}$. We have $\pi_A \in \mathrm{fix}(n)$ and $\pi_A(p) = p$. Moreover $\Vdash \dot{X}_n \cap \pi_A(\dot{X}_n) \subseteq \check{d}$. Since $\pi_A \in \mathrm{fix}(n) \subseteq \mathrm{sym}(\dot{U})$, we have $p \Vdash \pi_A(\dot{X}_n) \in \pi_A(\dot{U}) = \dot{U}$, so we have $p \Vdash \dot{X}_n \cap \pi_A(\dot{X}_n) \subseteq \check{d} \in \dot{U}$. However this is impossible since $|d| < \aleph_n$. The case that $\aleph_\omega \setminus X_n \in U$ follows from a similar argument. □

This completes the proof of Theorem 1.4. Our proof is flexible; We can prove the following by a similar argument. The proof is left to the reader.

Theorem 4.6. *Suppose GCH. Let α be a limit ordinal. Then there is a cardinal preserving symmetric extension in which $\aleph_\alpha$ is the least cardinal not in $\mathcal{U}$.*

Let us say that a cardinal κ is *strong limit* if for every $\alpha < \kappa$, there is no surjection from $\mathcal{P}(\alpha)$ onto κ. In the resulting model of Theorem 1.4, $\aleph_\omega$ is not strong limit.

Question 4.7. Is it consistent that $\aleph_\omega$ is strong limit and the least cardinal not in $\mathcal{U}$?

5. $\aleph_\omega \notin \mathcal{U}$ but $\lambda \in \mathcal{U}$ for every $\lambda > \aleph_\omega$ with countable cofinality

In this section we give a proof of Theorem 1.5.

Suppose GCH, and there is a strongly compact cardinal κ. Every regular cardinal $\geq \kappa$ carries a κ-complete uniform ultrafilter.

Let $\mathbb{P}$ be the poset from the previous section, and let $\mathrm{Col}(\aleph_{\omega+1}, <\kappa)$ be the standard $\aleph_{\omega+1}$-closed Levy collapsing poset which force κ to be $\aleph_{\omega+2}$. Take a $(V, \mathrm{Col}(\aleph_{\omega+1}, <\kappa))$-generic H. In $V[H]$, by Hayut-Karagila's symmetric collapse argument ([4]), we can find a symmetric extension M of V in which the following hold:

(1) $\kappa = \aleph_{\omega+2}$.
(2) For every regular cardinal $\lambda \geq \kappa$, every κ-complete uniform ultrafilter U over λ in V generates a κ-complete uniform ultrafilter.
(3) $\mathcal{P}(\aleph_\omega)^M = \mathcal{P}(\aleph_\omega)^V$.

Next take a $(V[H], \mathbb{P})$-generic G. Since $\mathrm{Col}(\aleph_{\omega+1}, <\kappa)$ is $\aleph_{\omega+1}$-closed, $\mathbb{P}$ preserves all cardinals between $V[H]$ and $V[H][G]$, in particular $\kappa = \aleph_{\omega+2}$ in $V[G][H]$. Then, take a symmetric extension N of $M[G]$ via $\mathbb{P}$ as in the previous section. We show that N is a required model.

By the argument before, we can show that in N, $\aleph_\omega$ is the least cardinal not in $\mathcal{U}$. In addition, since $\mathbb{P}$ has cardinality $\aleph_{\omega+1}$ in V (and so in M), for every regular cardinal $\lambda \geq \kappa$, a κ-complete uniform ultrafilter over λ in M generates an ultrafilter in N.

Lemma 5.1. *In N, every singular cardinal $\lambda > \aleph_\omega$ with countable cofinality carries a uniform ultrafilter.*

Proof. Fix a singular cardinal $\lambda > \aleph_\omega$ with countable cofinality. Note that $\mathrm{cf}(\lambda) = \omega$ in V. In V, fix an increasing sequence $\langle \lambda_n \mid n < \omega \rangle$ of regular cardinals with limit λ, and fix also $\langle U_n \mid n < \omega \rangle$ such that each U_n is a κ-complete uniform ultrafilter over λ_n.

In N, for each $n < \omega$, U_n generates an ultrafilter. Fix a non-principal ultrafilter U over ω, and define a filter W over λ as:

$$X \in W \iff \{n < \omega \mid X \cap \lambda_n \in U_n\} \in U.$$

It is easy to check that W is a uniform ultrafilter over λ. □

By a similar argument, one can prove that every cardinal $> \aleph_{\omega+1}$ with cofinality not equal to $\aleph_{\omega+1}$ carries a uniform ultrafilter in N, however $\aleph_{\omega+1}$ would not.

Question 5.2. Is it consistent that $\aleph_\omega$ is the unique cardinal not in $\mathcal{U}$?

6. Consistency strength about $\mathcal{U}$

To prove Theorem 1.7, we need the notions of indecomposable ultrafilter and regular ultrafilter.

Definition 6.1. Let U be an ultrafilter over a set S.

(1) Let κ be a cardinal. U is said to be *κ-indecomposable* if for every $f : S \to \kappa$, there is $X \in [\kappa]^{<\kappa}$ such that $f^{-1}(X) \in U$.
(2) For cardinals $\kappa \leq \lambda$, U is said to be *(κ, λ)-regular* if there is a family $\{A_\alpha \mid \alpha < \lambda\} \subseteq U$ such that $\bigcap_{\alpha \in x} A_\alpha = \emptyset$ for every $x \in [\lambda]^\kappa$.

The existence of non-regular ultrafilters is a large cardinal property.

Theorem 6.2 (Donder [3]). *If there are cardinals $\kappa < \lambda$ such that λ carries a uniform ultrafilter which is not (ω, κ)-regular, then there is an inner model of a measurable cardinal.*

The following lemmas are kind of folklore.

Lemma 6.3. *Let U be an ultrafilter over a set S and κ a cardinal. Then the following are equivalent:*

(1) U is (ω, κ)-regular.
(2) There is a family $\{B_s \mid s \in S\}$ such that $B_s \in [\kappa]^{<\omega}$ and $\{s \in S \mid \alpha \in B_s\} \in U$ for every $\alpha < \kappa$.

Proof. (1) $\Rightarrow$ (2). Fix a family $\{A_\alpha \mid \alpha < \kappa\} \subseteq U$ witnessing that U is (ω, κ)-regular. For each $s \in S$, let $B_s = \{\alpha < \kappa \mid s \in A_\alpha\}$. By the choice of the A_α's, we have that B_s is finite. Moreover, for $\alpha < \kappa$, we have $\{s \in S \mid \alpha \in B_s\} = A_\alpha \in U$.

(2) $\Rightarrow$ (1). Let $A_\alpha = \{s \in S \mid \alpha \in B_s\} \in U$ for $\alpha < \kappa$. For $x \in [\kappa]^\omega$, if there is $s \in \bigcap_{\alpha \in x} A_\alpha$, then B_s is infinite, this is a contradiction. □

Lemma 6.4. *Let U be an ultrafilter over a set S and κ a cardinal. If U is κ-indecomposable, then U is not (ω, κ)-regular.*

Proof. Suppose not. By the previous lemma, we can find a family $\{B_s \mid s \in S\}$ such that $B_s \in [\kappa]^{<\omega}$ and $\{s \in S \mid \alpha \in B_s\} \in U$ for every $\alpha < \kappa$. Since U is κ-indecomposable and $\kappa^{<\omega} = \kappa$, there is $X \subseteq [\kappa]^{<\omega}$ such that $|X| < \kappa$ and $\{s \in S \mid B_s \in X\} \in U$. Because $|\bigcup X| < \kappa$, we can pick $\alpha \in \kappa \setminus \bigcup X$. Then there must be $s \in S$ such that $B_s \in X$ but $\alpha \in B_s$, this is a contradiction. □

By this lemma and Donder's theorem, we have:

Proposition 6.5. *If there are cardinals $\kappa < \lambda$ such that λ carries a κ-indecomposable uniform ultrafilter, then there is an inner model of a measurable cardinal.*

We start the proof of Theorem 1.7.

Theorem 6.6 (In ZF). *If there are cardinals $\kappa < \lambda$ with $\kappa \notin \mathcal{U}$ but $\lambda \in \mathcal{U}$, then there is an inner model of a measurable cardinal.*

Proof. Fix a uniform ultrafilter U over λ. Then U must be κ-indecomposable; If not, there is $f : \lambda \to \kappa$ such that $f^{-1}(X) \notin U$ for every $X \in [\kappa]^{<\kappa}$. Then the ultrafilter $f_*(U) = \{X \subseteq \kappa \mid f^{-1}(X) \in U\}$ forms a uniform ultrafilter over κ, so $\kappa \in \mathcal{U}$. This is a contradiction.

We note that, we cannot apply Donder's theorem at the moment, since V may not satisfy the Axiom of Choice. To settle this matter, we have to take an inner model of ZFC and work in it.

Here we recall some definition. Let $\mathrm{OD}[U]$ be the class of all sets which are definable with parameters U and ordinals, and $\mathrm{HOD}[U]$ be of all x with $\mathrm{trcl}(\{x\}) \subseteq \mathrm{OD}[U]$. $\mathrm{HOD}[U]$ is a transitive model of ZFC with $U' = U \cap \mathrm{HOD}[U] \in \mathrm{HOD}[U]$. U' is a uniform ultrafilter over λ in $\mathrm{HOD}[U]$. We use the following Vopěnka's theorem: For every set X of ordinals, there is a poset $\mathbb{P} \in \mathrm{HOD}[U]$ and a $(\mathrm{HOD}[U], \mathbb{P})$-generic $G \in V$ with $X \in \mathrm{HOD}[U][G]$.

See, e.g., Woodin-Davis-Rodriguez [7] for the proof. Note that, for proving Vopěnka's theorem, V does not need to satisfy AC.

Case 1: κ is regular. We see that U' is κ-indecomposable in $\mathrm{HOD}[U]$. To do this, take $f : \lambda \to \kappa$ with $f \in \mathrm{HOD}[U]$. By the assumption and the regularity of κ, there is $\alpha < \kappa$ such that $f^{-1}(\alpha) \in U$, hence $f^{-1}(\alpha) \in U'$.

We have that U' is κ-indecomposable in $\mathrm{HOD}[U]$. Because $\mathrm{HOD}[U]$ is a model of ZFC, $\mathrm{HOD}[U]$ has an inner model of a measurable cardinal by Proposition 6.5.

Case 2: κ is singular. Now suppose to the contrary that there is no inner model of a measurable cardinal. In this case, we use Dodd-Jensen core model $K^{\mathrm{HOD}[U]}$ of $\mathrm{HOD}[U]$, which is a forcing invariant definable transitive model of ZFC. By the assumption that no inner model of a measurable cardinal, $K^{\mathrm{HOD}[U]}$ satisfies the covering theorem for $\mathrm{HOD}[U]$, that is, for every set $x \in \mathrm{HOD}[U]$ of ordinals, there is $y \in K^{\mathrm{HOD}[U]}$ such that $x \subseteq y$ and $|y|^{\mathrm{HOD}[U]} \leq \max(\aleph_1^{\mathrm{HOD}[U]}, |x|^{\mathrm{HOD}[U]})$. For details of Dodd-Jensen core model, see Jech [6].

Again, we see that U' is κ-indecomposable in $\mathrm{HOD}[U]$. Take a function $f \in \mathrm{HOD}[U]$ from λ to κ. By the assumption, there is $X \in [\kappa]^{<\kappa}$ such that $f^{-1}(X) \in U$. By Vopěnka's theorem, we can find a poset $\mathbb{P} \in \mathrm{HOD}[U]$ and a $(\mathrm{HOD}[U], \mathbb{P})$-generic $G \in V$ with $X \in \mathrm{HOD}[U][G]$. Since $\mathrm{HOD}[U][G]$ does not have an inner model of a measurable cardinal, $K^{\mathrm{HOD}[U][G]}$ satisfies the covering theorem for $\mathrm{HOD}[U][G]$. So there is $Y \in K^{\mathrm{HOD}[U][G]}$ such that $X \subseteq Y \subseteq \kappa$ and $|Y|^{\mathrm{HOD}[U][G]} \leq \max(\aleph_1^{\mathrm{HOD}[U][G]}, |X|^{\mathrm{HOD}[U][G]})$. We know $K^{\mathrm{HOD}[U]} = K^{\mathrm{HOD}[U][G]}$, so $Y \in K^{\mathrm{HOD}[U]} \subseteq \mathrm{HOD}[U]$. Clearly $f^{-1}(Y) \in U$, and we have to check $|Y|^{\mathrm{HOD}[U]} < \kappa$. Since $\mathrm{HOD}[U][G]$ satisfies AC, we have that $\kappa > \aleph_1^{\mathrm{HOD}[U][G]}$. Thus we have $|Y| < \kappa$ in $\mathrm{HOD}[U]$.

Now we know that U' is a κ-indecomposable uniform ultrafilter over λ in $\mathrm{HOD}[U]$. Since $\mathrm{HOD}[U]$ is a model of ZFC, we have that there is an inner model of a measurable cardinal by Proposition 6.5. This is a contradiction. $\square$

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