Superprime, fuly weakly prime, and fully weakly semiprime Modules

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This is a report on the author's presentation at the 2024 RIMS workshop "Group, Algebra, Language and Related Areas in Computer Science entitled "Superprime modules". The talk presented was a preliminary report and an introduction to the subject noted as the title above. More detailed and completed version of this manuscript is expected to be submited elsewhere.

It is a joint work with John . A. Beachy.

We consider module theoretic generalization of rings in which every ideal is weakly prime studied in [3], rings in which every nonzero ideal is prime introduced in [4], and super prime rings that was introduced at the 2019 RIMS workshop by the author.

We assume *R* is an associate ring with identity, and *M* is a fixed nonzero left *R*-module. A left *R*-module *X* in a category of unital left *R*-module is said to be *M*-generated if there exists an *R*-epimorphism from a direct sum of copies of *M* onto *X*.

Several definitions were first noted:

For submodules *N, L of M*, we define a product $N_M L = \sum \{ f(N) \mid f \in Hom_R(M,\ L) \}$. A submodule *Q* of *M* is called a fully invariant submodule if $f(Q) \subseteq Q$ for every $f \in Hom_R(M, L)$.

A proper fully invariant submodule *Q* of *M* is called a prime submodule if $N_M L \subseteq Q$ implies $N \subseteq Q$ or $L \subseteq Q$ for all fully invariant submodules *N*, *L* of *M*. It is called a semiprime module if $N_M N \subseteq Q$ implies $N \subseteq Q$ for all fully invariant submodules *N* of *M*. *Q* is called a weakly prime submodule if $N_M L \subseteq Q$ implies $N \subseteq Q$ or $L \subseteq Q$ for all fully invariant submodules N, L of M such that $N_M L \neq 0$. It is called a weakly semiprime submodule if $N_M N \subseteq Q$ implies $N \subseteq Q$ for all fully invariant submodules N of M such that $N_M N \neq 0$. M is said to be fully prime (fully semiprime) module if every proper fully invariant submodule is prime (semiprime). A prime submodule of Q is called superprime if $\bigcap_{i \in I} J_i \subseteq Q$ implies $J_i \subseteq Q$ for some i, where J_i is an fully invariant submodule of M.

It was then noted that a left ideal of R is fully invariant as a submodule if and only if it is an ideal of R, and the product $I_R J$ of two ideals I, J of R is the usual product IJ, and hence it follows that the defnitions above generalize the corresponding definitions of ideals of a ring R. It was also noted the following difference: For a ring R with identity, we have $RI = I$ for all ideals I. For a submodule $N \subseteq M$, we have an analogous condition that holds only in certain cases: $M_M N = M$ if and only if N is M-generated if and only if $\sum \{ f(M) \mid f \in Hom_R(M, N) \} = N$.

Although the author did not allocate sufficient time to delve into superprime modules and thereby omitting this topic on this report, its underlying motivation was noted as follows:

A ring in which every ideal is an idempotent is called a fully idempotent ring. If a ring R is fully idempotent, then for any ideals J, K of R, $J \cap K = JK$. Hence an ideal P in a fully idempotent ring R is prime if and only if $J \cap K \subseteq P$ implies $J \subseteq P$, or $K \subseteq P$. We define a prime ideal P in an arbitry ring R to be superprime if $\bigcap_{i \in I} J_i \subseteq P \Rightarrow J_i \subseteq P$ for some i, where J_i is an ideal of R. A ring in which 0 is superprime will be called a superprime ring. Thus, a superprime ring is primitive if and only if it is semiprimitive. If a ring is commutative, then a superprime ideal is maximal. A few other propositions on superprime rings were introduced during the 2019 RIMS workshop by the author.

Rings in which every nonzero proper ideal is prime is called an almost fully prime ring and it was studied in [4]. Rings in which every ideal is weakly prime was studied in [3]. These classes of rings have at most two maximal ideals and if they have two maximal ideals, they are the same class and it is a direct sum of two simple rings. A module theoretic generalization of these classes were considered next. A left *R*-module *M* is said to be almost fully prime if every nonzero proper fully invariant submodule is prime; and is called a fully weakly prime module if every proper fully invariant submodule is a weakly prime submodule. It is said to be a fully weakly semiprime module if every proper fully invariant submodule is a weakly semiprime submodule.

Some of the propositions that generalize those appear in [1] and [3] were stated and the ideals of proofs were explained next.

<u>Proposition</u> Let *M* be a left *R*-module. The following conditions are equivalent:
(1) *M* is a fully weakly semiprime module;
(2) for any proper fully invariant submodules *N, L* of *M*, we have $N_M L = N \cap L$ whenever $(N \cap L)_M (N \cap L) \neq 0$;
(3) for any proper fully invariant submodule *N* of *M*, either $N_M N = N$ or $N_M N = 0$.

<u>Proposition</u> Let *M* be a fully weakly semiprime module. Then the direct sum $M^{(I)}$ is fully weakly semiprime for any index set *I*.

<u>Proposition</u> Let *M* be a left *R*-module. Then *M* is a fully weakly prime module if and only if for any fully invariant submodules *N, L* of *M,* one of the following occurs:
$N_M L = N,\ N_M L = L,$ or $N_M L = 0.$

The presentation concluded with the following example:

<u>Example</u> Let *N* be a fully invariant submodule of M. Proposition 3.2 of [1] shows that if *M* is a fully prime module, then so is *N,* while Corollary 2.2 of [1] shows that if *M* is fully semiprime module, then so is *N*. Both results fail for fully weakly prime (semiprime) modules. For example, the Prüfer group $\mathbf{Z}_{2^\infty}$ is a fully weakly prime module. The submodule generated by the congruence class of 8 is fully invariant but not weakly semiprime since it is isomorphic to the cyclic group of order 8, whose maximal submodule is neither idempotent

nor squre zero.

References

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