Levels of Information Abstraction: Search for Mathematical Models of Emergence & Reduction of Complexity

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1. Introduction

The main objective of this paper is to explore mathematical models of emergence understood as a characteristic of reality in its ontological, hierarchic architecture that prevents the epistemic reduction of the study of higher levels to the study of those below. The conceptual framework of this work with its foundation in the general concept of information is different from multiple attempts that typically use as fundamental the concepts of property and emergent property. The former has its long-standing tradition going back to Aristotle based on linguistic analysis in which a property is something that can be predicated about something else. The latter makes the distinction of properties of complex collectives that cannot be directly (or even indirectly) derived from the properties of components. As will be explained below, modern physics requires reexamination, reformulation, and reconceptualization of traditional common-sense ideas such as that of property imported from the language of everyday human experience. One of the possible reconceptualizations is in terms of a suitably general concept of information used in this paper. Its advantage is in already developed mathematical formalism for information transcending the limits of its understanding based on language [1].

In addition to the main theme of emergence, the paper explores the conditions for its antinome idea of reduction. Reduction of complexity has been always the main objective of theoretical studies. The fact that there are some limits to the reduction of complexity in transitions between different levels of the ontological hierarchy manifested in the emergence does not eliminate its role as the main intellectual tool. However, we have to be aware of the limitations brought by emergence and explore the conditions that make reduction possible. This paper is the first step in this direction.

There are many different conceptualizations of information and corresponding mathematical formalisms. Those used in this paper are very general and this generality may raise objections of being excessive from some alternative perspectives. However, its advantage is in overarching the majority of alternative studies of information so that the results of the study of emergence presented here can be applied elsewhere. Moreover, the mathematical formalism of information in terms of closure spaces used here allows formalizing emergence and reduction as mathematical concepts and proving or disproving claims about them.

Due to the limited volume of this paper, the proofs of propositions are omitted, but they are rather straightforward once their context is explained. Also, the focus will be only on the existence of emergence (inferred from the necessary conditions for reduction to a lower level), not on sufficient conditions for reduction. The proofs of theorems and more elaborate inquiry into reduction of complexity will be published later in a more extensive paper.

2. The Emergence of Emergentism

The recent fiftieth anniversary of the 1972 publication by Philip W. Anderson of one of the most influential articles of the 20th century with the title *More Is Different: Broken Symmetry and the Nature of the Hierarchical Structure of Science* was not only an opportunity for celebrations but also for recapitulation of the transformation of physics and science in general within the years that followed [2]. Anderson, the 1977 Nobel Prize Winner in Physics, initiated with his *More Is Different* the reversal of the dominating trend of reductionism and promoted the idea of the hierarchical architecture of reality which is reflected in the hierarchical structure of science with the emergence of properties and principles at a higher level that are irreducible to those of lower levels.

The idea of emergentism was not new as we can find its origins (under a different name of "heteropathic laws and effects") in the views of John Stuart Mill in the mid-19th century [3], but it lost its adherents due to its association with the losing scientific status vitalism and indirectly because the scientific progress achieved through the involvement of physics in chemistry and chemistry in biology reinforced the reductionist positions.

The term "emergence" was introduced by George Henry Lewes in 1875 in a philosophical context and did not acquire the status of the name for a commonly recognized ontological view of reality and methodological scientific tool for a long time [4]. More often, the idea of the irreducibility of the inquiry to the lower level components was present in the fringes of science, as a general philosophical position, most frequently in the context of the study of consciousness than as a scientific methodological tool. Probably the best-known expression of such a general philosophical view was the motto "The whole is more than the sum of its parts" for the General System Theory of Ludwig Bertalanffy. There was an increasing interest in emergence in the context of biology in the second half of the 20th century but in physics, the search for the "theory of everything" deriving the entire science from the inquiry of (more and more) elementary particles has continued [5,6]. Anderson's *More is Different* was openly intended as a critique of such a reductionist view of reality and a call for a new emergentist methodology.

The 2022 celebrations of Anderson's paper recognized the triumphal return and blooming of emergentism in the years after its publication. For instance, we can find in the Editorial in *Nature*: "Emergence is now considered one of the hallmarks of complex systems, in which the properties of the whole cannot be directly inferred from the details of the parts but arise from their mutual interactions" [7-9]. This does not mean that the concept of emergence has acquired a clear definition or consistent conceptual framework.

Fifty years later, Anderson's original paper remains the best presentation of the concept of emergence with its use as a central concept of broken symmetry. The literature on the emergence and its weaker and stronger versions has grown exponentially in the last two or three decades, but the majority of the works are rather speculative, focusing on the distinctions between the stronger or weaker versions of the claims presented by adherents to emergentism, and with only very few exceptions did not attempt to develop precise methodological tools.

Even in the limits of physics, the attempts to define emergence are usually deficient and superficial producing very vague statements of the type: “An emergent behavior of a physical system is a qualitative property that can only occur in the limit that the number of microscopic constituents tends to infinity” [10]. The problem is not just a logical error of calling an emergent behavior a qualitative property or a naively justified distinction on qualitative properties, but the erroneous focus on the large number of constituents reflecting the false belief that such systems are necessarily complex. In reality, this is frequently exactly the opposite. Systems with a large number of independent elements usually are not complex and their collective, global properties and dynamics can be easily derived from their components [11].

The existing few attempts to provide a more formal description of the emergence have rather narrow contexts that make generalizations difficult or impossible. The main problem is their conceptual poverty due to the restriction to the common-sense ideas at their foundations and the resulting confusion about the meaning of emergence. For instance, Crutchfield in his *The Calculi of Emergence: Computation, Dynamics, and Induction* describes in the abstract of the paper its objectives as follows “Defining structure and detecting the emergence of complexity in nature are inherently subjective, though essential, scientific activities” [12] This may sound reasonable, even if the claim of “inherent subjectivity” is questionable. However, the highlighted passage in the text of the paper “In summary, three notions will be distinguished: (1) The intuitive definition of emergence: ‘something new appears’; (2) Pattern formation: an observer identifies ‘organization’ in a dynamical system; and (3) Intrinsic emergence: the system itself capitalizes on patterns that appear” is disappointing [12]. The paper offers some formalization of these common-sense expressions in a narrow context of computation (including quantification of ‘something new appears’), but the question about its relevance to the study of emergence remains. Is the novelty a key feature of the philosophical, methodological concept of emergence? This shows the urgent need for disambiguation of multiple interpretations of emergence present in the common-sense discourse leading to confusion.

The majority of recent works on emergence use as their fundamental concept “emergent property”. The lessons from the history of set theory (in particular from the need to restrict Comprehension Axiom Schema to avoid paradoxes) and from quantum mechanics in physics (in particular from its quantum logical formulation) show that the seemingly obvious concept of property understood as a predicate distinguishing a set of objects possessing this property loses its common-sense meaning [13,14]. The issue requires an extensive explanation for which there is no space here and which can be found elsewhere with argumentation for replacing the concept of property with the concept of information [1,15].

Risking a gross oversimplification, the transition from the foundations built on the concept of property to those based on the concept of information is achieved by replacing property A with a corresponding elementary instance of information carried by "x is A" with appropriate restriction of its meaning (in general, it may not be and usually is not represented by the simple set-theoretical sentential formula "$x \in A$"). Of course, in some very specific situations, this simple representation is possible, and then the logic of information assumes the structure of a familiar Boolean algebra which can be represented as a Boolean algebra of subsets of a given set in which a property corresponds to a set of elements that posses it. In the case of quantum information systems, this logic of information is not Boolean and has the structure of an irreducible, orthomodular AC lattice $\mathscr{L}_f(H)$ of the closed subspaces of some Hilbert space fitting well as another special case the general formalism of information systems used here [14].

The use of the term "logic" in two different (but related) meanings, its popular and traditional meaning as the fundamental structure of reasoning expressed in a language and as a more general structure of a not-necessarily linguistic information system, may be confusing at first, but this more general terminology is already well established and its avoidance here would be even more confusing considering the loss of consistency with the existing literature of the subject.

In the following, the specifics of conceptual and formal inquiry of information are irrelevant as the discourse will be carried out within the existing mathematical formalism of general algebra. A minimal overview of necessary mathematical concepts will be presented below for the self-sufficiency of the paper and the literature references will be given to more extensive explanations, such as the most comprehensive classical monograph on lattice theory (including closure spaces which are of special importance for his paper) by Garrett Birkoff [16].

3. The Logic of Information Defined in a Closure Space

An alternative approach to the logic of information developed in my publications is focusing on filters defined in closure spaces [17], thus to make this paper self-sufficient an explanation of the most important concepts will follow.

Def. *A closure space $<S,f>$ is a set S with a function $f: 2^S \to 2^S$ on the power set of S called a closure operator that satisfies three conditions: (i) $\forall A \subseteq S$: $A \subseteq f(A)$, (ii) $\forall A, B \subseteq S$: If $A \subseteq B$, then $f(A) \subseteq f(B)$, (iii) $\forall A \subseteq S$: $f(f(A)) = f(A)$.*

Every closure space $<S,f>$ can be defined in an equivalent (cryptomorphic) way by a Moore family of subsets of S, i.e. family closed with respect to arbitrary intersections and including the set S. Every Moore family $\mathscr{M}$ defines a transitive operator: $f(A) = \cap \{M \subseteq \mathscr{M}: A \subseteq M\}$ and in turn, the family f-Cl = $\{M \subseteq S: f(M) = M\}$ is a Moore family. The family f-Cl is a complete lattice $\mathscr{L}_f$ with respect to the set inclusion $\subseteq$.

If needed, the concept of a closure space $<S,f>$ and its lattice of closed elements $\mathscr{L}_f$ can be defined on an arbitrary bounded complete lattice $\mathscr{L}$ instead of the power set 2^S by replacing every occurrence of the set inclusion $\subseteq$ with the symbol of the partial order $\leq$ of $\mathscr{L}$.

The family $\mathfrak{I}$ of elements of a complete lattice $\mathscr{L}$ is called a *filter*, if it satisfies two conditions: (1) $\forall A,B \in \mathscr{L}$: If $A \in \mathfrak{I}$ and $A \leq B$, then $B \in \mathfrak{I}$. (2) $\forall A,B \in \mathscr{L}$: If $A \in \mathfrak{I}$ and $B \in \mathfrak{I}$, then $A \wedge B \in \mathfrak{I}$.

A (proper) filter does not have the leasr element of $\mathscr{L}$ as its element. The maximal (proper) filter on $\mathscr{L}$ is called an *ultrafilter*. All these concepts are the same in the more familiar special case of a complete lattice introduced in the power set 2^S of set S with inclusion $\subseteq$.

With these mathematical preliminaries, we can introduce the basics of the mathematical theory of information used here. We will consider a closure space $<S,f>$ with its corresponding Moore family $\mathscr{M}$ of closed subsets as an **information system**. The specific choice of closure space depends on the choice of the type of information system. For instance, we can consider geometric, topological, logical (linguistic) information, etc. with corresponding lattices of closed subsets defined by geometric, topological, or logical consequence closure operators.

The family of closed subsets $\mathscr{M}$ = f-Cl is equipped with the structure of a complete lattice $\mathscr{L}_f$ which we can consider to be the **logic of information**. It plays a role in the generalization of traditional logic for information systems, although it does not have to be a Boolean algebra.

Encoding of information **(or instance of information)** is a distinction of a subfamily $\mathfrak{I}$ of $\mathscr{M}$, which is a filter in the lattice $\mathscr{L}_f$. The reasons for the association of information with filters are related to the need for semantical analysis of information [1,15], while this very abstract way of conceptualizing it can be better understood in the context of symmetry [1]. It turns out that this abstract form can be easily related to Shannonian information theory within the latter's restricted context [1].

The theory of information presented here includes the case of quantum information for which the closure space is defined by the orthomodular lattice $\mathscr{L}_f$(H) of closed subspaces in a Hilbert space H. Every quantum logic defined by an appropriate set of axioms can always be represented as $\mathscr{L}_f$(H). Moreover, we can identify the information about the system (alternatively called information identifying the system, or the state of the system) with the filter $\mathfrak{I}$ in the quantum logic $\mathscr{L}$(H) consisting of the set of elements of $\mathscr{L}$(H) with the value 1 of probability measure describing the state of the quantum system in the traditional approach [18].

In the general case, the logic $\mathscr{L}(S,f)$ is not necessarily a Boolean lattice or the lattice $\mathscr{L}_f$(H) of closed subspaces of a Hilbert space. Therefore, we have to be cautious not to import into this general theory the facts about the structures (e.g. filters, ultrafilters) from more familiar cases of Boolean or quantum logics which may not be true in general.

4. Emergence and Reduction in Terms of Extensions or Reductions of Closure Spaces

The original concept of emergence as presented by Anderson was based on the recognition of the irreducibility of collective phenomena at a higher level of complexity to the analysis of its individual members at a lower level. In physics, it was understood as the irreducibility of the properties of condensed matter to properties of atomic and subatomic constituents; in biology, the irreducibility between the multiple levels of populations, organisms, organs, tissues, cells,

and molecules. Anderson's "More" is a metaphoric expression not of quantitative but rather qualitative, structural higher level of complexity in collective phenomena.
There are at least two ways we can interpret Anderson's "More" in this context:

- Extension of the closure space $<S,f>$ to a closure space $<T,g>$ such that $S \subseteq T$ & $S \neq T$
- Extension of the closure space $<S,f>$ to a closure space $<T,g>$ such that $T = 2^S$.

It is quite clear that Anderson and the majority of other adherents to emergentism interpret "More" in the second, more specific way. Condensed matter systems in physics are part of what is called "many-body systems" with composite elements and in biology there are multilevel functional hierarchies of organelles, cells, tissues, organs, organisms, etc. based on the inclusion of multiple structured objects of one type in the objects of other type. However, we will consider both cases.

Before presenting the idea of emergence and its opposite reduction formulated in terms of information systems (i.e. closure spaces) let's consider two simple, special examples of a related familiar reduction, that may help understand a more formal description that follows.

The power of so-called linear methods in applications such as geometry, solving systems of linear equations of many variables, solving differential equations, etc. can be identified in the properties of vector spaces, in particular vector spaces of finite dimensions which supply a key to using finitistic methods to infinite sets. Vector spaces V(K) over a field K can be infinite but the methods used in working with vectors can be finitistic as long as the spaces have finite bases, i.e. subsets of vectors that are independent and which generate the entire space (i.e. every vector is a linear combination of the vectors from the base and can be identified with a finite sequence of its coordinates from K). It happens that in this case bases can be equivalently defined as minimal, independent subsets.

This can be expressed in terms of the closure operator f defined on V(K) using linear combinations of vectors from a subset A of V(K) by $f(A) = \{\underline{v} \in V: \underline{v} = \alpha_1\underline{v}_1 + \alpha_2\underline{v}_2 + \ldots + \alpha_k\underline{v}_k$ for $\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_k \in A$ and $\alpha_1, \alpha_2, \ldots, \alpha_k \in K\}$, i.e. $f(A)$ is a vector subspace generated by subset A.

Then A is f-independent *iff* $\forall \underline{v} \in A: \underline{v} \notin f(A \setminus \{\underline{v}\})$ and A f-generates V(K) *iff* $f(A) = V$. From this the concept of a base has been generalized as an f-independent subset of the closure space $<S,f>$ which f-generates S. However, in distinction from the closure in vector spaces in the general case of an arbitrary closure space $<S,f>$ different bases may have different cardinalities or they may not exist at all. Therefore, in the general case, the concept of the dimension may be absent.

The second familiar example is in propositional logic which has the structure of a closure space defined by Tarski using the consequence closure operator Cn on the set of all propositions. In this case, closed subsets are theories, and the central subject of interest is their axiomatization, i.e. a minimal (independent) subset of propositions of a given theory that generates it, i.e. its base or using familiar terminology a set of independent axioms. In this example of propositional logic, bases may have different cardinalities or they may not exist.

The concepts of subspace generation, independence, and bases have been well-known from the beginning of the study of closure spaces. There is another missing property of bases in general closure spaces <S,*f*> for a long time absent in the literature. In vector spaces, their subspaces are generated by appropriate subsets of a base which become bases of these subspaces. This very important property of closures in vector spaces is missing in the general case which means that in general, we cannot reduce the description of the entire space to its base and the concept of a base becomes useless. Very different closure spaces on a set S may have identical bases. Since the term "base" acquired its well-established meaning in the closure space theory a long time ago, I defined another concept of a subset that has this missing property using the name "frame" (at that time not knowing that it is used somewhere else in a very different meaning) [19]. The following is a summary of relevant for this paper basic properties of frames [20].

Def. 4.1: Let <S,*f*> be a closure space and B its subset (not necessarily proper). B is a frame for <S,*f*>, if (*) $\forall A\subseteq S\ \exists B_A\subseteq B: f(A) = f(B_A)$.
(*) Is equivalent to: $\forall A\subseteq S: f(A) = f(B \cap f(A))$

In the following, there is an overview of the most basic features of frames elaborated elsewhere [20]. A frame is proper, if B is a proper subset of $S\setminus f(\varnothing)$; it is a minimal frame if there is no proper subset of B which is a frame. Minimal frames have the missing property of bases considered above (all subspaces are generated by subsets of frames). Frames always exist, but they may be trivial (improper), equal to S or $S\setminus f(\varnothing)$. Obviously, in finite closure spaces, there are always minimal frames which however can be improper. Infinite closure spaces may have many (even infinitely many) frames, but they may have no minimal frames at all. For this paper, the distinction of closure spaces <S,*f*> which have minimal frames is of special importance.

The concept of a frame of <S,*f*> is different from the concept of a generating subset (it is more restrictive) or base (generative & independent) in <S,*f*>, however in the closure space defined earlier in a finite-dimensional vector space every base is a minimal frame. So in this very special case, the concept of a base and minimal frame coincide.

Lemma 4.1: *B is a frame for <S,f>, if either one of the equivalent conditions is satisfied:*

- *(Def.)* $\forall A\subseteq S\ \exists B_A\subseteq B: f(A) = f(B_A)$.
- $\forall A\subseteq S: f(A) = f(B \cap f(A))$
- $\forall C,D\subseteq X: f(C)\cap B = f(D)\cap B \Rightarrow f(C) = f(D)$

We can see that the action of the closure operator *f* is determined by the action of *f* on subsets of frame B.
With these basic properties of frames established earlier [20], we can return to our study of emergence and reduction.

Def. 4.2: *We can consider now a reversed process where we start from a closure space <B,g> and a set S, such that* $B \subseteq S$ *and B is a frame in <S,f>. Then, we call <S,f> a frame extension of <B,g>, if the action of g on B is a restriction of the action of f to subsets of B.*

It turns out that a closure space <B,*g*> can have multiple frame extensions defined on the same set S, but with different closure operators *f*. These extensions have isomorphic logics ($\mathcal{L}_f$), but their closure operators may be different.

Thm. 4.2: *Let f_1 and f_2 be different closure operators defined on the same set S, g_1 and g_2 be their respective restrictions to a subset B of S, which is a frame for both closure spaces <S,f_1> and <S,f_2>. Then, from the equality of the restrictions $g_1 = g_2$ follows the isomorphism of the lattice of f_1-closed subsets of S and the lattice of f_2-closed subsets of S, i.e. isomorphism of their logics.*

This means that when we consider the reverse process of frame-generated extensions of the first type when the closure space <*S,f*> is extended to a closure space <*T,g*> such that $S \subseteq T$ & $S \neq T$ we may have different frame-generated extensions, however, they always have isomorphic logics. This can be considered as emergence through ramification. The lower-level information system does not determine the higher-level extended information system uniquely, but its multiple extensions have isomorphic logics. This opens a new, additional direction of study (not pursued in this paper) of the role of symmetry and its breaking in the identification of the selected extension [1].

We also considered a second, closer to the original intention interpretation of Anderson's "More": Extension of the closure space <*S,f*> to a closure space <*T,g*> such that $T = 2^S$.

In this case, we have a more clear meaning of "more" as an increase in complexity expressed by the cardinal exponent. Here too, we can use several results from my earlier work [21].

Let <S, *f*> be a closure space and $T= 2^S$. Define a binary relation R between S and $T= 2^S$ by: $\forall x\in S\ \forall A\subseteq S$: xRA *iff* $x\in f(A)$. Then the relation R defines a Galois connection (polarity) between the Boolean algebra of subsets of S and Boolean algebra of subsets of T and it turns out that Galois closures $f(A)$ and $g(\beta)$ are:

- $\forall A\subseteq S$: $f(A) = R^{*a}R^{a}(A)$ and
- $\forall \beta \subseteq T = 2^S$: $g(\beta) = R^{a}R^{*a}(\beta) = \{A\subseteq S: \cap\{f(B): B\in\beta\} \subseteq f(A)\}$,

Then in the latter case, we have $\forall\beta\subseteq 2^S$: $\beta\in \mathcal{L}_g$ *iff* $[\forall A\subseteq S: \cap\{f(B): B\in\beta\} \subseteq f(A) \Rightarrow A\in\beta\}$ and the lattices of closed subsets $\mathcal{L}_g$ and $\mathcal{L}_f$ are dually isomorphic.

Let's consider a simple example in the familiar case of a trivial closure space in which every subset is closed and its logic is the Boolean algebra 2^S of all subsets of the set S. This means $\forall A \subseteq S$: $A = f(A)$ and we have $\forall x\in S \forall A\subseteq S$: xRA *iff* $x\in A$. Then the closure operator on 2^S $g(\beta) = \{A\subseteq S: \cap \{f(B): B\in\beta\} \subseteq f(A)\}$ simplifies to: $g(\beta) = \{A\subseteq S: \cap \{B: B\in\beta\} \subseteq A\}$.

In this very special case we can use the traditional concept of property and xRA describes the relationship "x has all properties defining set A" and $g(\beta)$ is the principal filter in 2^S generated by the intersection of all subsets in β, i.e. it is the family of all subsets which include this intersection. The key point is in simplification of "xRA *iff* $x\in f(A)$" to "xRA *iff* $x\in A$". In the former, general case, we have an instance of elementary general information, and in the latter a set-theoretical interpretation of the instance of information about x that it has property A.

Now we have the central mathematical result of this paper.

Thm. 4.3: *Let (as above, g be the closure operator on 2^S generated by a closure operator f on S)* $\forall\beta \subseteq T = 2^S$: $g(\beta) = \{A \subseteq S: \cap\{f(B): B \in \beta\} \subseteq f(A)\}$.

Then the family f-Cl $\subseteq T = 2^S$ *of f-closed subsets of S (i.e. f-Cl* $=\{A \subseteq S: A = f(A)\}$*) is a frame for the closure space* $<2^S, g>$*, i.e.* $\forall\beta \subseteq T = 2^S\ \exists\beta_A \subseteq f\text{-}Cl \subseteq 2^S$: $g(\beta) = g(\beta_A)$.

Now, using similar reasoning as before for frame extensions we have a mechanism of emergence related to the second type of extension to a power set. Here we have that emergent logic may be a logic that cannot be generated from the lower level. In other words, the logic cannot be frame-generated at a lower level.

The main consequence of this theorem regarding emergence consists of the two sufficient conditions for the information logic $\mathscr{L}_g$ that cannot be frame-generated from the lower-level logic $\mathscr{L}_f$. We can identify two properties of all logics $\mathscr{L}_g$ always frame-generated by any lower-level logic $\mathscr{L}_f$. The absence of these two properties in $\mathscr{L}_g$ indicates that this logic cannot be frame-generated from any logic $\mathscr{L}_f$ at the lower level. This means that $\mathscr{L}_g$ is an emergent logic of information or in other words that the information system defined by closure *g* is emergent.

The two necessary conditions for the reduction, i.e. for the existence of an information system $<S, f>$ which frame-generates an information system $<2^S, g>$ follow from the following theorem.

Thm. 4.4: *Let (as above, g be the closure operator on 2^S generated by a closure operator f on S)* $\forall\beta \subseteq T = 2^S$: $g(\beta) = \{A \subseteq S: \cap\{f(B): B \in \beta\} \subseteq f(A)\}$. *Then:*

- *(i)* *(quite obviously)* $g(\varnothing) \neq \varnothing$. *Always* $g(\varnothing) = \{S\} \neq \varnothing$.
- *(ii)* *The subset of closed subsets f-Cl* $\subseteq 2^S$ *is not only a frame for* $<2^S, g>$ *but always a minimal frame.*

Our main task to formalize emergence in terms of information defined in closure spaces and prove the existence of emergent (non-reducible to lower-level) information systems is complete. Closure spaces $<2^S, g>$ such that $g(\varnothing) = \varnothing$ or that do not have a minimal frame cannot be derived from lower-level spaces $<S, f>$ and the examples of such spaces are well-known [20]. However, there is a legitimate question of whether such emergent (irreducible to lower-level) closure spaces are of any importance and interest or just pathological curiosities.

The answer to this question comes from the earlier results in my paper on meta-closures [22]. We can consider a closure space $<2^S, g>$ defined by the Moore family of all Moore families of closed subsets for each of the closure operators on S (obviously, 2^S is a Moore family for the trivial closure operator with all subsets closed, an arbitrary intersection of Moore families on S is itself a Moore family, with the intersection of every empty family being 2^S). Then this meta-closure space $<2^S, g>$ is defined by: $\forall\mathscr{B} \subseteq 2^S$: $g(\mathscr{B}) = \{B \subseteq S: \exists\mathfrak{C} \subseteq \mathscr{B} : B = \cap\,\mathfrak{C}\}$. If S is infinite, then its meta-closure space does not have minimal frames. This gives us a highly non-trivial example of an emergent closure space, i.e. emergent information system.

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