FORMAL DEGREES AND PARABOLIC INDUCTION: THE MAXIMAL GENERIC CASE

YIYANG WANG

1. Introduction

1.1. **The formal degree conjecture.** Let F be a non-archimedean local field of characteristic 0 and $\mathbf{G}$ a connected reductive group over F with $\mathbf{A}$ the split component of its center. Set $G = \mathbf{G}(F)$ and $A = \mathbf{A}(F)$. For simplicity, throughout this report we shall only consider quasi-split $\mathbf{G}$.

Let (π, V_π) be a discrete series (i.e. irreducible unitary, square-integrable modulo A) of G with a fixed invariant inner product $(\cdot,\cdot)$ on V_π. Let μ be a fixed Haar measure on the quotient group G/A. Recall that the formal degree of π, with respect to the choice of the measure μ, is defined to be the unique positive real number $d(\pi) = d(\pi, \mu) \in \mathbb{R}_{>0}$ such that

$$\int_{G/A} (\pi(g)v, v')\overline{(\pi(g)w, w')}\mu(\dot{g}) = \frac{1}{d(\pi)}(v, w)\overline{(v', w')}, \quad v, v', w, w' \in V_\pi.$$

The formal degree of a discrete series of G is exactly its Plancherel measure in Harish-Chandra's Plancherel formula.

- For real reductive groups, Harish-Chandra exhausted the discrete series and established the explicit Plancherel formula in 1970s, which is widely regarded as one of the highest achievements of representation theory.
- For p-adic groups, there is no explicit classification of discrete series for p-adic groups, due to Galois complexity. It was in [HII08b][HII08a], assuming the local Langlands correspondence (conjectural for general $\mathbf{G}$), that an explicit formula for $d(\pi)$ was conjectured in terms of the adjoint γ-factor.

More precisely, let ψ be a fixed non-trivial additive character of F and $\mu_\psi(g)$ a specific Haar measure on G depending only on ψ (cf. [HII08a]). Assuming the local Langlands correspondence for $\mathbf{G}$, let (φ_π, ρ_π) be the refined Langlands parameter of π. Thus $\varphi_\pi : \mathrm{WD}_F \to {}^LG$ is an admissible homomorphism from the Weil-Deligne group $\mathrm{WD}_F = \mathrm{W}_F \times \mathrm{SL}_2(\mathbb{C})$ into the L-group ${}^LG$, and ρ_π is an irreducible character of a certain component group $\mathcal{S}_\varphi := \pi_0(S_\varphi/Z(\hat{G})^\Gamma)$, where $S_\varphi := Z_{\hat{G}}(\varphi(\mathrm{WD}_F))$ (cf. §2.1). Let

- $\hat{G}$ (resp. $\hat{G}^\natural$) be the dual group of $\mathbf{G}$ (resp. $\mathbf{G}/\mathbf{A}$), thus $\hat{G}^\natural \subset \hat{G}$;
- $S^\natural_{\varphi_\pi} := Z_{\hat{G}^\natural}(\varphi_\pi(\mathrm{WD}_F))$, and $\mathcal{S}^\natural_{\varphi_\pi} := \pi_0(S^\natural_{\varphi_\pi})$ be the corresponding component group.

It was then conjectured that

$$d(\pi) := d(\pi, \mu_\psi) = \frac{\dim \rho_\pi}{|\mathcal{S}^\natural_{\varphi_\pi}|} \cdot |\gamma(0, \pi, \mathrm{Ad}, \psi)|,$$

where Ad is the adjoint representation of ${}^LG$ on $\hat{\mathfrak{g}}/Z(\hat{\mathfrak{g}})^\Gamma$, and $\gamma(s, \pi, \mathrm{Ad}, \psi)$ the corresponding adjoint γ-factor.

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This conjecture gives a beautiful reformulation of Harish-Chandra's work over $\mathbb{R}$, and has been verified for many p-adic groups, such as

- Inner forms of $\mathrm{GL}(n)$: Silberger-Zink, cf. [HII08b, §4];
- Unitary groups $\mathrm{U}(n)$: Beuzart-Plessis [Beu21] and Morimoto [Mor22];
- Odd orthogonal groups $\mathrm{SO}(2n+1)$ and the metaplectic group $\mathrm{Mp}(2n)$: Ichino-Lapid-Mao [ILM17];
- Kaletha's regular and non-singular supercuspidals of tame groups: Schwein [Sch24] and Ohara [Oha23];
- etc.

Let us remark here that the precise definition of the measure μ_ψ is somewhat technical. Though it can be defined quite directly in a formal way, to explain the relation of it to Gross' motive (in [Gro97]) and working properties would be quite time-consuming. However, since the main purpose of this report is to explain the main results and ideas of the computation, we shall not give more precise information on it here, but provide some further remarks in §5.4. Some of the crucial formulas will be given "up to structural constants".

1.2. **The main result of this report.** A natural question related to this conjecture is the reduction to the supercuspidal case.

More precisely, given a discrete series π of G, there exists a parabolic $\mathbf{P} = \mathbf{MN}$, a unitary supercuspidal σ of $M = \mathbf{M}(F)$, and $\lambda \in \mathfrak{a}_{\mathbf{M}}^*$, such that π is a subquotient of the (normalized) parabolic induction $i_P^G(\sigma \otimes \chi_\lambda)$ (where χ_λ is the unramified character of M associated to λ). If we can compute the quotient $d(\pi)/d(\sigma)$, then the conjecture for π would be reduced to that of σ, together with some compatibility relation for L-parameters of π and σ. But the realization of this approach is quite difficult due to at least two reasons:

(1) the cuspidal support of p-adic discrete series is considerably delicate;
(2) the construction of the L-parameter of π in terms of that of σ is also elusive.

The purpose of this report is to discuss the simplest nontrivial case. We shall assume

- π is generic ($\mathbf{G}$ quasi-split), and supported on a maximal Levi $\mathbf{M}$.

Then we can obtain quite complete and satisfactory result (see Theorem 3.1) on the evaluation of $d(\pi)/d(\sigma)$ in this case. As an interesting example or application, we can use it to verify the formal degree conjecture for discrete series of exceptional groups of type G_2 supported on a maximal parabolic.

2. Preliminaries for the main result

For clarity, we first collect some standard notations or conventions for latter use.

- We work over a nonarchimedean local field F of characteristic 0, and fix a nontrivial additive character $\psi : F \to \mathbb{C}^1$.
- Boldfaced letters such as $\mathbf{G}, \mathbf{H}$ will denote algebraic groups over F, and usual letters G, H for their groups of F-points.
- Given an F-group $\mathbf{H}$, we denote by $X^*(\mathbf{H})$ the group of F-rational characters, and $\mathfrak{a}_{\mathbf{H}}^* := X^*(\mathbf{H}) \otimes \mathbb{R}$, $\mathfrak{a}_{\mathbf{H},\mathbb{C}}^* := X^*(\mathbf{H}) \otimes \mathbb{C}$. Denote by $\mathbf{A_H}$ the split component of the center of $\mathbf{H}$.
- Next we recall some standard definitions and notations of unramified characters.

  - For any $\chi \in X^*(\mathbf{H})$, $s \in \mathbb{C}$, we denote by $|\chi|^s$ the unramified character
$$|\chi|^s : H \to \mathbb{C}^\times,\ h \mapsto |\chi(h)|^s.$$
  - Let $H^1 := \bigcap_{\chi \in X^*(\mathbf{H})} \ker |\chi|$ and
$$X^{\mathrm{ur}}(H) := \mathrm{Hom}(H/H^1, \mathbb{C}^\times)$$
be the group of unramified characters of M. We thus have a surjection $\mathfrak{a}^*_{\mathbf{H},\mathbb{C}} \twoheadrightarrow X^{\mathrm{ur}}(H)$ characterized by $\chi \otimes s \mapsto |\chi|^s$, which equipps $X^{\mathrm{ur}}(H)$ with a canonical complex structure. For $\lambda \in \mathfrak{a}^*_{\mathbf{H},\mathbb{C}}$, we denote by $\chi_\lambda \in X^{\mathrm{ur}}(H)$ its image.
  - To simplify the notations, given a smooth representation σ of $H = \mathbf{H}(F)$ and $\lambda \in \mathfrak{a}^*_{\mathbf{H},\mathbb{C}}$, we also denote by σ_λ the representation $\sigma \otimes \chi_\lambda$.
- Now let $\mathbf{G}$ be a connected reductive group over F with a fixed maximal F-split torus $\mathbf{T}_0$. Let $\mathbf{P} = \mathbf{M}\mathbf{N}$ be a parabolic subgroup with $\mathbf{M} \supset \mathbf{T}_0$. We write $\mathfrak{a}^*_0 = \mathfrak{a}^*_{\mathbf{T}_0}$. Here we review some basic facts related to the decomposition of $\mathfrak{a}^*_0$ with respect to $\mathbf{M}$.
  - The inclusion $\mathbf{A_G} \subset \mathbf{G}$ induces the restriction morphism
$$\mathrm{res} : X^*(\mathbf{G}) \to X^*(\mathbf{A_G})$$
and gives an isomorphism $\mathrm{res} \otimes \mathbb{1} : \mathfrak{a}^*_{\mathbf{G}} \xrightarrow{\sim} \mathfrak{a}^*_{\mathbf{A_G}}$ of vectors spaces after $(\cdot) \otimes \mathbb{R}$.
  - The inclusions $\mathbf{A_M} \subset \mathbf{T}_0 \subset \mathbf{M}$ induces
$$X^*(\mathbf{M}) \xrightarrow{\mathrm{res}} X^*(\mathbf{T}_0) \xrightarrow{\mathrm{res}} X^*(\mathbf{A_M})$$
and gives a canonical decomposition
$$\mathfrak{a}^*_0 = \mathfrak{a}^*_{\mathbf{M}} \oplus \mathfrak{a}^{\mathbf{M}*}_0.$$
(Also cf. [Wal03, §I.1].)
  - Similarly, $\mathbf{A_G} \subset \mathbf{A_M} \subset \mathbf{M} \subset \mathbf{G}$ induces a canonical decomposition
$$\mathfrak{a}^*_{\mathbf{M}} = \mathfrak{a}^*_{\mathbf{G}} \oplus \mathfrak{a}^{\mathbf{G}*}_{\mathbf{M}}.$$
It would be useful to identify $\mathfrak{a}^{\mathbf{G}*}_{\mathbf{M}}$ with $X^*(\mathbf{A_M}/\mathbf{A_G}) \otimes \mathbb{R}$.

2.1. **Desiderata of local Langlands correspondence (LLC).** For the convenience of the readers, we recollect some basic facts on the local Langlands correspondence (abbreviated as "LLC" from now on) and fix the notations here. In the rest of this report, F will be p-adic and $\mathbf{G}$ will be quasi-split. Denote by $\Gamma := \Gamma_F$ the absolute Galois group of F.

Recall that the L-group of $\mathbf{G}$ is ${}^L G = \hat{G} \rtimes \Gamma$, and an L-parameter of $\mathbf{G}$ is an admissible homomorphism
$$\varphi : \mathrm{WD}_F := \mathrm{W}_F \times \mathrm{SL}_2(\mathbb{C}) \to {}^L G,$$
in the sense that algebraic on $\mathrm{SL}_2(\mathbb{C})$, smooth on W_F and compatible with the natural projection ${}^L G \twoheadrightarrow \Gamma$.

The LLC (which is conjectural for general quasi-split $\mathbf{G}$) is a finite to one surjection
$$\Pi(\mathbf{G}) \twoheadrightarrow \Phi(\mathbf{G}), \quad \pi \mapsto \varphi_\pi,$$
where

- $\Pi(\mathbf{G})$ is the set of equivalence classes of irreducible smooth representations of G;
- $\Phi(\mathbf{G})$ is the set of $\hat{G}$-conjugacy classes of L-parameters of $\mathbf{G}$;
- φ_π is called the L-parameter of π.

This correspondence should satisfy a list of properties which we shall not recall here. Given $\varphi \in \Phi(\mathbf{G})$, we denote its fiber, the corresponding L-packet, by Π_φ.

For quasi-split groups, the refined LLC, which describes the structure of the L-packet Π_φ, can be explained as follows. Throughout this report we only have to work with tempered parameters $\varphi \in \Phi_{\mathrm{temp}}(\mathbf{G})$. We set

$$S_\varphi := Z_{\hat{G}}(\varphi(\mathrm{WD}_F)), \quad \mathcal{S}_\varphi := \pi_0(S_\varphi / Z(\hat{G})^\Gamma).$$

(Here π_0 denotes the component group, and one should pay attention to the quotient by $Z(\hat{G})^\Gamma$ in the definition.) Then the refined LLC for quasi-split groups predicts that for each fixed Whittaker datum $\mathfrak{w} := (\mathbf{B}, \eta)$ of $\mathbf{G}$, there exists a "good" bijection

$$\iota = \iota_{\mathfrak{w}} : \Pi_\varphi \to \mathrm{Irr}(\mathcal{S}_\varphi).$$

This bijection depends on the choice of $\mathfrak{w}$, while for each $\pi \in \Pi_\varphi$, the dimension $\dim \iota_{\mathfrak{w}}(\pi)$ is independent of the choice of $\mathfrak{w}$. Following the convention in endoscopy, we denote it by

$$\langle 1, \pi \rangle := \dim \iota_{\mathfrak{w}}(\pi).$$

A classical conjecture of Shahidi reads

Conjecture 2.1 (Shahidi, generic packet conjecture)**.** *For $\varphi \in \Phi_{\mathrm{temp}}(\mathbf{G})$ and a fixed Whittaker model $\mathfrak{w} = (\mathbf{B}, \eta)$, there exists a unique $\mathfrak{w}$-generic $\pi_{\mathfrak{w}}$ in Π_φ, with $\iota_{\mathfrak{w}}(\pi_{\mathfrak{w}}) = 1$ the trivial representation of $\mathcal{S}_\varphi$.*

In particular, if $\pi \in \Pi_\varphi$ is generic, then $\langle 1, \pi \rangle = 1$.

2.2. **Classification of "maximal generic" discrete series.** As explained in the introduction, we shall compute $d(\pi)/d(\sigma)$ for a generic discrete series π of $G = \mathbf{G}(F)$ supported on a maximal parabolic of $\mathbf{G}$. For such π, there is indeed a classification by Shahidi, using local harmonic analysis. Before stating it, we need some more preparation and notations.

We fix a Borel $\mathbf{B} = \mathbf{TU}$ of our quasi-split group $\mathbf{G}$, and denote by $\mathbf{T}_0$ the maximal split subtorus of $\mathbf{T}$. Let $\mathfrak{a}_0^* := X^*(\mathbf{T}_0) \otimes \mathbb{R}$ be the dual Lie algebra of $\mathbf{T}_0$.

The set of (relative) roots $\Sigma = \Sigma(\mathbf{G}, \mathbf{T}_0)$ is then a subset of $\mathfrak{a}_0^*$, and the choice of $\mathbf{B}$ gives the subset $\Delta \subset \Sigma$ of positive simple roots. Then we have a bijection

$$\{\text{standard maximal parabolics } \mathbf{P} = \mathbf{MN} \subset \mathbf{G}\} \to \{\text{maximal proper subsets of } \Delta\}$$

given by $\mathbf{P} = \mathbf{MN} \mapsto \theta := \Delta - \{\alpha\}$, where $\alpha \in \Delta$ is the unique positive simple root "appearing" in the Lie algebra of $\mathbf{N}$. (Recall that a parabolic $\mathbf{P}$ is said to be standard if it contains the fixed Borel $\mathbf{B}$.)

Now we fix such a maximal standard $\mathbf{P} = \mathbf{MN}$ and its associated $\alpha \in \Delta$. We define the associated fundamental weight to be

$$\tilde{\alpha} := \langle \rho_{\mathbf{P}}, \alpha^\vee \rangle^{-1} \cdot \rho_{\mathbf{P}} \in \mathfrak{a}_{\mathbf{M}}^* \subset \mathfrak{a}_0^*,$$

where $\alpha^\vee$ is the corresponding coroot and $\rho_{\mathbf{P}}$ the half sum of roots in the Lie algebra of $\mathbf{N}$.

Next let ${}^LP = {}^LM{}^LN$ be the L-group of $\mathbf{P} = \mathbf{MN}$, which is a (relevant) parabolic subgroup of ${}^LG$. Denote by r the adjoint representation of ${}^LM$ on the Lie algebra ${}^L\mathfrak{n} := \mathrm{Lie}({}^LN)$ of the unipotent radical. By Shahidi's computation (cf. [Sha88, §4]), there exists a $k \in \mathbb{Z}_{\geq 1}$ such that each subspace

$${}^L\mathfrak{n}_i := \{X_{\beta^\vee} \in {}^L\mathfrak{n} \mid \beta \in \Sigma,\ \langle \tilde{\alpha}, \beta^\vee \rangle = i\}, \quad 1 \leq i \leq k$$

is an irreducible (thus nonzero) subrepresentation r_i, and $(r, {}^L\mathfrak{n}) = \oplus_{i=1}^k (r_i, {}^L\mathfrak{n}_i)$ gives the irreducible decomposition of r.

With these preparations, let π be a generic discrete series of G supported on the above fixed maximal standard Levi $\mathbf{M}$ of $\mathbf{P}$, with the associated $\alpha \in \Delta$. Then Shahidi classified cuspidal supports of such π in terms of poles of L-functions $L(s, \sigma, r_i)$ as follows.

Theorem 2.2 (Shahidi, [Sha90])**.** *(Setting as above.) There exists a unique irreducible unitary generic supercuspidal σ of M, and a unique $j \in \{1, 2\}$, such that π is a subrepresentation of $i_P^G \sigma_{\tilde{\alpha}/j}$. Furthermore, this $j \in \{1, 2\}$ is characterized by*

- *$L(s, \sigma, r_j)$ (resp. $\gamma(s, \sigma, r_j, \psi)$) has a simple pole at $s = 0$ (resp. $s = 1$).*

Remark. *To be more precise, L and γ here are the Langlands-Shahidi local factors; they are believed to equal to those obtained from LLC. However, though verified in many cases, this has not been proved yet for general $\mathbf{G}$.*

3. The main result

With the preparation in §2, we can now state our main result. We re-list the notations explained in §2:

- $\mathbf{P} = \mathbf{MN} \subset \mathbf{G}$ a fixed maximal standard parabolic corresponding to $\theta = \Delta - \{\alpha\}$,
- π a generic discrete series of $\mathbf{G}$ supported on $\mathbf{M}$,
- σ the irreducible unitary generic supercuspidal of M, with unique $j \in \{1, 2\}$ such that $\pi \hookrightarrow i_P^G \sigma_{\tilde{\alpha}/j}$.

For convenience, we add the following condition for $\mathbf{G}$:

- assume that the center of $\mathbf{G}$ is anisotropic, and $\mathbf{G}$ is unramified (i.e. splits over a finite unramified extension of F).

The reason for adding this is to make the final formula less technical. They are not essentially used in the proof or computation.

In this setting, the main result can be stated as follows.

Theorem 3.1. *Assuming (refined) LLC for $\mathbf{G}$ and $\mathbf{M}$ (with some natural assumptions), we denote by φ_π (resp. φ_σ) the L-parameter of π (resp. σ). Then under the above setting, we have*

$$\frac{d(\pi)}{d(\sigma)} = j^{-1} \cdot \frac{m}{\langle \chi, \alpha^\vee \rangle} \cdot \frac{|\gamma(0, \pi, \mathrm{Ad}, \psi)|}{|\gamma(0, \sigma, \mathrm{Ad}, \psi)|}, \tag{3.1}$$

where

- *$\chi \in X^*(\mathbf{M}) \simeq \mathbb{Z}$ is the generator with $\langle \chi, \alpha^\vee \rangle > 0$;*
- *m is the index of the restriction morphism $\mathrm{res} : X^*(\mathbf{M}) \hookrightarrow X^*(\mathbf{A_M})$.*

In particular, this result is compatible with the formal degree conjecture if and only if

$$\frac{|\mathcal{S}_{\varphi_\pi}|}{|\mathcal{S}_{\varphi_\sigma}|} = j \cdot \langle \chi, \alpha^\vee \rangle. \tag{3.2}$$

We remark again that $d(\pi)$ and $d(\sigma)$ are formal degrees with respect to proper Haar measures on $G/A_{\mathbf{G}}$ and $M/A_{\mathbf{M}}$. The two measures depend only on the choice of the additive character ψ, and here we shall not give the precise definition of them (cf. §5.4 and [HII08a]).

Here the structural constant $\langle \chi, \alpha^\vee \rangle / m$ is relatively easy to understand or compute. (For many simply connected groups it is just 1.)

In summary, the theorem provides the following reduction: to prove the formal degree conjecture for π, it suffices to verify the conjecture for σ together with the relation (3.2). Thus it would be convenient to apply it when the conjecture is known for $\mathbf{M}$ (such as the split exceptional group G_2). The interesting point here is that the left hand side consists of arithmetic invariants from LLC, while the right hand side consists of spectral data from local harmonic analysis. We shall see later that this gives some clue on the behaviour of $\varphi_\pi|_{\mathrm{SL}_2(\mathbb{C})}$ in terms of the spectral information its cuspidal support.

4. An Example: G_2

As a particularly interesting test example and an application of the main theorem, we could verify the formal degree conjecture for discrete series of split G_2 supported on a maximal Levi.

Let $\mathbf{G} = \mathrm{G}_2$ be the split exceptional group of type G_2 over F. We first recall the classification of discrete series representations of $G = \mathbf{G}(F)$ supported on a maximal Levi, following Muić [Mui97] (also cf. [GS23a, §2]).

Fix a maximal torus $\mathbf{T}$ of $\mathbf{G}$ and a Borel subgroup $\mathbf{B}$ containing $\mathbf{T}$. Let $\Sigma = \Sigma(\mathbf{G}, \mathbf{T})$ be the root system and $\Delta = \{\alpha, \beta\}$ the set of simple roots relative to $\mathbf{B}$, with α short and β long. Then the positive roots can be written as

$$\alpha, \beta, \alpha+\beta, 2\alpha+\beta, 3\alpha+\beta, 3\alpha+2\beta,$$

with corresponding coroots

$$\alpha^\vee, \beta^\vee, \alpha^\vee+3\beta^\vee, 2\alpha^\vee+3\beta^\vee, \alpha^\vee+\beta^\vee, \alpha^\vee+2\beta^\vee,$$

and

$$\langle \alpha, \beta^\vee \rangle = (-1), \langle \beta, \alpha^\vee \rangle = (-3).$$

Let $\mathbf{P}_\alpha = \mathbf{M}_\alpha \mathbf{N}_\alpha$ (resp. $\mathbf{P}_\beta = \mathbf{M}_\beta \mathbf{N}_\beta$) be the maximal parabolic subgroup of $\mathbf{G}$ corresponding to $\{\alpha\}$ (resp. $\{\beta\}$), i.e. $\mathbf{N}_\alpha$ (resp. $\mathbf{N}_\beta$) contains the root subgroup of β (resp. α) and $\mathbf{M}_\alpha$ (resp. $\mathbf{M}_\beta$) contains the root subgroup of α (resp. β).

For convenience of the reader, we list the related normalizations and the decomposition $r = \oplus_i r_i$ (the adjoint representation of ${}^L M$ on ${}^L\mathfrak{n}$) for these two cases as follows.

- $\mathbf{P}_\alpha = \mathbf{M}_\alpha \mathbf{N}_\alpha$,

$$\rho_\alpha = \frac{1}{2}[\beta + (\alpha+\beta) + (2\alpha+\beta) + (3\alpha+\beta) + (3\alpha+2\beta)] = \frac{3}{2}(3\alpha+2\beta),$$
$$\tilde{\beta} = \langle \rho_\alpha, \beta^\vee \rangle^{-1} \rho_\alpha = (3\alpha+2\beta).$$

We may fix an isomorphism $\mathbf{M}_\alpha \simeq \mathrm{GL}_2$ so that the character det of GL_2 corresponds to $3\alpha + 2\beta = \tilde{\beta}$ of $\mathbf{M}_\alpha$, and the modulus character restricted to $\mathbf{M}_\alpha$ is $\delta_\alpha = |\det|^3$.

In this case the representation r of ${}^L M_\alpha$ on ${}^L\mathfrak{n}_\alpha$ decomposes as

$r_1 = \mathrm{std}$	$r_2 = \det$	$r_3 = \mathrm{std} \otimes \det$
$\beta^\vee, (3\alpha+\beta)^\vee$	$(3\alpha+2\beta)^\vee$	$(\alpha+\beta)^\vee, (2\alpha+\beta)^\vee$

- $\mathbf{P}_\beta = \mathbf{M}_\beta \mathbf{N}_\beta$,

$$\rho_\beta = \frac{1}{2}[\alpha + (\alpha+\beta) + (2\alpha+\beta) + (3\alpha+\beta) + (3\alpha+2\beta)] = \frac{5}{2}(2\alpha+\beta),$$
$$\tilde{\alpha} = \langle \rho_\beta, \alpha^\vee \rangle^{-1} \rho_\beta = (2\alpha+\beta).$$

We may fix an isomorphism $\mathbf{M}_\beta \simeq \mathrm{GL}_2$ so that the character det of GL_2 corresponds to $2\alpha+\beta = \tilde{\alpha}$ of $\mathbf{M}_\beta$, and the modulus character restricted to $\mathbf{M}_\beta$ is $\delta_\beta = |\det|^5$.

In this case the representation r of ${}^L M_\beta$ on ${}^L\mathfrak{n}_\beta$ decomposes as

$r_1 = \mathrm{Sym}^3 \otimes \det^{-1}$	$r_2 = \det$
$\alpha^\vee, (\alpha+\beta)^\vee, (3\alpha+\beta)^\vee, (3\alpha+2\beta)^\vee$	$(2\alpha+\beta)^\vee$

Note that in these normalizations, the character det of $\mathbf{M}_\alpha$ (resp. $\mathbf{M}_\beta$) corresponds exactly to the positive root perpendicular to α (resp. β).

Theorem 4.1. *Let τ be an irreducible unitary supercuspidal representation of $\mathrm{GL}_2(F)$ which is self-dual, with central character ω_τ.*

(1) *If $\omega_\tau = 1$, then $i^G_{P_\alpha}(\tau \otimes |\det|^{1/2})$ has a unique irreducible subrepresentation $\pi_\alpha(\tau, 1/2)$.*
(2) *If $\omega_\tau = 1$, then $i^G_{P_\beta}(\tau \otimes |\det|^{1/2})$ has a unique irreducible subrepresentation $\pi_\beta(\tau, 1/2)$.*
(3) *If $\omega_\tau \neq 1$, then $i^G_{P_\beta}(\tau \otimes |\det|)$ has a unique irreducible subrepresentation $\pi_\beta(\tau, 1)$.*

The representations $\pi_\alpha(\tau, 1/2), , \pi_\beta(\tau, 1/2), \pi_\beta(\tau, 1)$ are generic discrete series, and this classifies all the discrete series representations of G supported on a maximal Levi.

The full explicit local Langlands correspondence for G_2 has been established in [GS23b] based on a series work in exceptional theta correspondence. The conjectural L-parameters for non-supercuspidal discretes series have already been classified in [GS23a, §3.5], which we now recall.

We identify the L-group of (split) G_2 with the complex simple Lie group $\mathrm{G}_2(\mathbb{C})$ with trivial center; also, we identify the roots of $\mathrm{G}_2(\mathbb{C})$ with the coroots of $\mathbf{G} = \mathrm{G}_2$. Given a root $\gamma^\vee \in \Sigma^\vee$, we denote by $\mathrm{SL}_{2,\gamma^\vee}$ the SL_2 generated by the root subgroups of $\pm\gamma^\vee$. Now on the dual side $\alpha^\vee$ (resp. $\beta^\vee$) is the long (resp. short) root, and

$$Z_{\mathrm{G}_2}(\mathrm{SL}_{2,\alpha^\vee}) = \mathrm{SL}_{2,(\alpha^\vee+2\beta^\vee)},\ Z_{\mathrm{G}_2}(\mathrm{SL}_{2,\beta^\vee}) = \mathrm{SL}_{2,(2\alpha^\vee+3\beta^\vee)}.$$

Here $(\alpha^\vee+2\beta^\vee)$ (resp. $(2\alpha^\vee+3\beta^\vee)$) is also the positive root perpendicular to $\alpha^\vee$ (resp. $\beta^\vee$). Finally, the long root subgroups of $\mathrm{G}_2(\mathbb{C})$ generate an SL_3.

Now let π be an irreducible discrete series representation of G with L-parameter $\varphi = \varphi_\pi : \mathrm{W}_F \times \mathrm{SL}_2(\mathbb{C}) \to \mathrm{G}_2(\mathbb{C})$. We then have

π	$\varphi(\mathrm{SL}_2)$	$\varphi(\mathrm{W}_F)$	$\mathcal{S}_\varphi$
$\pi_\alpha(\tau, 1/2)$	$= \mathrm{SL}_{2,(\alpha^\vee+2\beta^\vee)}$	$\subset \mathrm{SL}_{2,\alpha^\vee} \subset {}^L M_\alpha$	$\mathbb{Z}/2\mathbb{Z}$
$\pi_\beta(\tau, 1/2)$	$= \mathrm{SL}_{2,(2\alpha^\vee+3\beta^\vee)}$	$\subset \mathrm{SL}_{2,\beta^\vee} \subset {}^L M_\beta$	$\mathbb{Z}/2\mathbb{Z}$
$\pi_\beta(\tau, 1)$	$= \mathrm{SO}_3 \subset \mathrm{SL}_3$	$= Z_{\mathrm{G}_2}(\mathrm{SO}_3) \simeq S_3$	1
		$\subset \mathrm{SL}_{2,\beta^\vee} \subset {}^L M_\beta$	

by [GS23a, §3.5]. (Note that the representations $\pi_\alpha(\tau, 1/2)$, $\pi_\beta(\tau, 1/2)$, $\pi_\beta(\tau, 1)$ above are denoted by $\delta_P(\tau)$, $\delta_Q(\tau)$, $\pi_{\mathrm{gen}}[\tau]$, respectively, in [GS23a].)

Clearly, the orders of the component groups are compatible with the formal degree conjecture. The compatibility of Langlands-Shahidi γ-factors and those from LLC for split G_2 has been verified in [Sha89]. Finally, the formal degree conjecture is known for $\mathbf{M}_\alpha, \mathbf{M}_\beta \simeq \mathrm{GL}_2$;

thus as an application (cf. explanation in §5.4) we have verified the formal degree conjecture for discrete series of G_2 supported on maximal Levi subgroups.

5. Idea of the proof

Now we sketch the main ideas of the proof. The computation uses three tools, and the most difficult part is the evaluation of certain structrual constants.

For simplicity, we shall continue assuming $\mathbf{G}$ to be unramified (quasi-split) and $\mathbf{A_G} = 1$. We fix

$$\mathbf{P} = \mathbf{MN},\ \theta = \Delta - \{\alpha\},\ \tilde{\alpha} \in \mathfrak{a}^*_{\mathbf{M}},\ \pi, \sigma, j \in \{1, 2\} \tag{5.1}$$

as before, cf. the beginning of §3.

5.1. **Heiermann's formula.** The starting point of the computation is the following Heiermann's result. To explain it and for clarity of notations, we temporarily let

- $\mathbf{H}$ be a general connnected reductive group over F,
- $\mathbf{Q} = \mathbf{LU}$ be an arbitrary F-parabolic of $\mathbf{H}$,
- $\mu : \mathfrak{a}^*_{\mathbf{L},\mathbb{C}} \to \mathbb{C}$ be the corresponding Harish-Chandra μ-function (or the Plancherel density function) defined on a dense open subset of $\mathfrak{a}^*_{\mathbf{L},\mathbb{C}}$.

For convenience of the readers, we recall the definition of μ here. Let τ be an irreducible supercuspidal of L. The intertwining operator $J_{\bar{Q}|Q}(\tau) : i^H_Q \sigma \to i^H_{\bar{Q}} \sigma$ is defined to be

$$J_{\bar{Q}|Q}(\tau) f(g) := \int_{\bar{U}} f(\bar{u}h) d\bar{u}, \quad (f \in i^H_Q \tau, h \in H),$$

and the Harish-Chandra μ-function is a rational function μ defined on $\mathcal{O} = \mathcal{O}_\sigma = \{\tau \otimes \chi_\lambda \mid \lambda \in \mathfrak{a}^*_{\mathbf{L},\mathbb{C}}\}$ by

$$J_{Q|\bar{Q}}(\tau') \circ J_{\bar{Q}|Q}(\tau') = \mu(\tau')^{-1}$$

for τ' in a Zariski open subset of $\mathcal{O}$. Note that the definition here is compatible with [Hei04, §1.5] but differs from that in [Sha90][Wal03]. When $\mathbf{Q} = \mathbf{LU}$ is maximal, the μ-function here is equal to $\gamma(\mathbf{H}/\mathbf{L})\mu$ defined in [Sha90][Wal03].

In local harmonic analysis, we always fix a special maximal compact subgroup K of $H = \mathbf{H}(F)$ such that the Iwasawa decomposition $G = QK$ holds, and for each closed subgroup $H' \subset H$, we equip H' with the measure such that $H' \cap K$ has volume one. The notation $\deg(\cdot)$ below means the formal degrees defined by this choice of measures.

Theorem 5.1 (Heiermann [Hei04])**.** *Let τ be an irreducible supercuspidal of $L = \mathbf{L}(F)$. Then for $\lambda_0 \in \mathfrak{a}^*_{\mathbf{L},\mathbb{C}}$, the parabolic induction $i^H_Q \sigma_{\lambda_0}$ has a discrete series subquotient if and only if the Harish-Chandra μ function has a pole of maximal order at $\lambda = \lambda_0$.*

Moreover, in this case, one can compute $\deg(\pi)/\deg(\sigma)$ *in terms of a "multiple residue" of the μ-function at $\lambda = \lambda_0$. Although we shall not give the general version, in our setting* (5.1) *above, we have*

$$\deg(\pi) \sim \deg(\sigma) \cdot \mathrm{Res}_{s=1/j}\, \mu(\sigma \otimes \chi_{s\tilde{\alpha}}), \tag{5.2}$$

where "∼" means up to some explicit structural constant.

In summary, the cuspidal support of a p-adic discrete series can be characterized by a pole of maximal order for the Harish-Chandra μ-function, and the multiple residue here, up to some structural constants, is $\deg(\pi)/\deg(\sigma)$.

5.2. **Langlands-Shahidi method.** The next tool is a fundamental result of the Langlands-Shahidi method, which expresses the μ-function in terms of γ-factors. Here the adjoint representation r of ${}^LM$ on ${}^L\mathfrak{n}$ and its irreducible components r_i would come into play (cf. notations and review of §2.2). The main theorem together with (3.12) of [Sha90] can then be formulated as

Theorem 5.2. *Let σ be an irreducible generic unitary supercuspidal representation of M. Then*

$$\mu(\sigma \otimes \chi_{s\tilde{\alpha}}) = \prod_{i=1}^{m} \gamma^{\mathrm{Sh}}(is, \sigma, r_i, \bar{\psi})\gamma^{\mathrm{Sh}}(-is, \tilde{\sigma}, r_i, \psi), \tag{5.3}$$

where γ^{Sh} are Shahidi's γ-factors characterized in [Sha90, Theorem 3.5].

Remark. *Assuming the L-parameter φ_σ of σ (conjectural in general), it is believed that these $\gamma^{\mathrm{Sh}}(s, \sigma, r_i, \psi)$ should coincide with corresponding Artin γ-factors $\gamma(s, r_i \circ \varphi_\sigma, \psi)$ obtained from the L-parameter. Though verified in many cases, this has not yet been established in full generality. Thus we have to assume this for our computation.*

As we can see immediately, if we combine (5.2) with (5.3), then we obtain a relation of the form

$$\frac{\deg(\pi)}{\deg(\sigma)} \sim \operatorname*{Res}_{s=1/j} \gamma^{\mathrm{Sh}}(is, \sigma, r_i, \bar{\psi})\gamma^{\mathrm{Sh}}(-is, \tilde{\sigma}, r_i, \psi), \tag{5.4}$$

where the (simple) appears in the term $\gamma^{\mathrm{Sh}}(js, \sigma, r_j, \overline{\psi})$ for variable js, $s = 1/j$.

To connect the right hand side to the adjoint γ-factors of π and σ, we need a conditional construction of the parameter φ_π in terms of φ_σ.

5.3. **Construction of discrete parameters.** We shall be brief in this part. A very subtle but fundamental problem in LLC for p-adic groups, is the construction of discrete L-parameters in terms of those for supercuspidal representations. (On contrary, the Langlands classification, and the classification of tempered parameters in terms of discrete ones, are very clear and explicit.)

We shall not discuss any more related detail here, but only want to point out that Heiermann posed a conditional construction for this in [Hei06]. One of the main input for this construction is also Theorem 5.1. We will not need any detail for this construction, but only have to know that it predicts

$$\varphi_\pi(w, 1) = \iota_P \circ \varphi_\sigma(w, 1) \text{ for } w \in \mathrm{W}_F \quad \text{and} \quad \varphi_\pi(\begin{bmatrix} q^{1/2} & \\ & q^{-1/2} \end{bmatrix}) = s_{\sigma,\lambda}, \tag{*}$$

where $\iota_P : {}^LP \hookrightarrow {}^LG$ is the embedding and $s_{\sigma,\lambda}$ is an explicit element in ${}^LM$. Furthermore, when φ_σ is trivial on Deligne SL_2, the element $s_{\sigma,\lambda}$ would lie in the center of ${}^LM$.

For irreducible generic supercuspidals, it is a well-known conjecture that the parameters of such representations are SL_2-trivial (cf. for example, Conjecture 7.1 of [GR10]).

Combining the above, it is technical but not very difficult to obtain

Theorem 5.3. *In our setting (π, $\mathbf{P} = \mathbf{MN}$, α, σ, j, r_i, etc.), assuming*

- *LLC for $\mathbf{G}, \mathbf{M}$ with the validity of $\varphi_\sigma|_{\mathrm{SL}_2(\mathbb{C})} = 1$ (σ irreducible generic supercuspidal) and (*),*

- *the Langlands-Shahidi local factors γ^{Sh} do coincide with γ-factors from LLC,*

we have

$$\gamma(s,\pi,\mathrm{Ad},\psi)=\left(\frac{1-q^{-s}}{1-q^{s-1}}\right)\cdot\gamma(s,\sigma,\mathrm{Ad},\psi)\cdot\prod_{i=1}^{m}\gamma(s+is_0,\sigma,r_i,\psi)\gamma(s-is_0,\tilde{\sigma},r_i,\psi).$$

(We have to emphasize the working hypotheses here, which are not accessible for general quasi-split groups.)

As explained in Lemma 1.2 of [HII08b], $\gamma(0,\pi,\mathrm{Ad},\psi),\gamma(0,\sigma,\mathrm{Ad},\psi)$ are holomorphic and nonzero at $s=0$. Since when $s\to 0$

$$\frac{1-q^{-s}}{1-q^{s-1}}\sim\frac{s\cdot\log q}{1-q^{-1}},$$

we have

Corollary 5.4. *Following the setting at the beginning of this subsection,*

$$\frac{\gamma(0,\pi,\mathrm{Ad},\psi)}{\gamma(0,\sigma,\mathrm{Ad},\psi)}=\left(\frac{\log q}{1-q^{-1}}\right)\cdot\operatorname*{Res}_{s=0}\prod_{i=1}^{m}\gamma(s+is_0,\sigma,r_i,\psi)\gamma(s-is_0,\tilde{\sigma},r_i,\psi).$$

5.4. **Completion of the proof.** Now it is quite clear that if we combine corollary 5.4 with (5.4) (obtained by Heiermann's formula together with Langlands-Shahidi method), then we can finally obtain a formula of the form

$$\frac{d(\pi)}{d(\sigma)}\sim\frac{\deg(\pi)}{\deg(\sigma)}\sim\frac{|\gamma(0,\pi,\mathrm{Ad},\psi)|}{|\gamma(0,\sigma,\mathrm{Ad},\psi)|}\cdot j^{-1}.$$

Here "$\sim$" also means up to some structural constant, and j is the main term of interest. It becomes transparent now how this j appears:

- in corollary 5.4, the variable for the residue is $s+1$ (the j-th term in the right product), taken at $s=0$, while
- in (5.4), the variable for the residue is js, taken at $s=1/j$.

This gives a clear explanation on how the cuspidal support could be reflected in arithmetic invariants of L-parameters.

Finally, let us give some remarks on the evaluation of "structural constants" mentioned above. There are essentially two constants to determine, for the final formula, which are indeed the most time consuming steps of the computation.

(1) In (5.2) there is an explicit constant related to the orbit

$$\mathcal{O}=\mathcal{O}_\sigma:=\{\sigma\otimes\chi_\lambda\mid\lambda\in\mathfrak{a}^*_{\mathbf{M},\mathbb{C}}\}$$

(in the sense of isomorphism classes). For a maximal Levi, though technical, this can be computed directly; but it seems quite tough to generalize it to general Levis.

(2) We have to evaluate the quotient

$$\left(\frac{\deg(\pi)}{\deg(\sigma)}\right)\cdot\left(\frac{d(\pi)}{d(\sigma)}\right)^{-1}$$

via comparison of measures. This would be reduced to a technical exercise in Bruhat-Tits theory (through Gross' motive for reductive groups).

That is the reason for adding the condition "(quasi-split) unramified" for our group $\mathbf{G}$: in this case one can choose a hyperspecial maximal compact group and the computation becomes simpler. For general cases, in principle there is no essential obstruction, but the general formula becomes considerably complicated. The readers interested in this may take a look at the list of explicit examples in Appendix E of [GI14].

As we can see, the result reflects some very rigid restrictions of construction of discrete parameters. As explained in §5.1, if a discrete series π has cuspidal support $(\mathbf{M}, \sigma \otimes \chi)$ where σ is an irreducible unitary cuspidal of $M = \mathbf{M}(F)$ and χ an unramified character of M, then $\varphi_\pi|_{\mathrm{W}_F}$ is just the composition of φ_σ and the natural embedding ${}^LM \hookrightarrow {}^LG$. However, the behaviour of $\varphi_\pi|_{\mathrm{SL}_2(\mathbb{C})}$ is quite mysterious; we do not have more explicit characterization of it, but it dominates the centralizer of the parameter. From the result explained above, we see that how this subtle question is related to harmonic analysis and cuspidal support.

It is a natural question to ask whether we can do similar computation for non-generic discrete series supported on non-maximal Levis. We give a final remark that there are at least three new problems to resolve.

(1) When the Levi is not maximal, the "multiple residue" and the orbit constant in (5.2) would become quite tough to evaluate.
(2) For non-generic σ, $\varphi_\sigma|_{\mathrm{SL}_2(\mathbb{C})}$ may not be trivial, then a general version of corollary 5.4 would become more complicated (due to the nontrivial unipotent class).
(3) For non-generic σ, the Langlands-Shahidi method is inaccessible. This is quite essential, and one needs endoscopy to recover (5.3).

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