On the Arakawa lifting

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1 Introduction

For a reductive dual pair (H, G), we have the theta lifting

$$\mathcal{L}\colon \ \{\text{automorphic forms on } H\} \to \{\text{automorphic forms on } G\}$$

defined by

$$\mathcal{L}(f)(g) = \int_{H_{\mathbb{Q}}\backslash H_{\mathbb{A}}} \theta(h,g) f(h) dh \qquad (g \in G_{\mathbb{A}})$$

for an automorphic form f on H, where $\theta(h,g)$ is the theta kernel on $H_{\mathbb{Q}}\backslash H_{\mathbb{A}} \times G_{\mathbb{Q}}\backslash G_{\mathbb{A}}$. One of the fundametal problems is to characterize $\mathrm{Ker}(\mathcal{L})$ and $\mathrm{Im}(\mathcal{L})$ in terms of, say, Fourier expansions or L-values.

The kernel of $\mathcal{L}$ has been studied in many cases. For example, Böcherer and Schulze-Pillot ([BS]) and Hsieh and Namikawa ([HN1],[HN2]) proved the injectivity of the Yoshida lifting. On the other hand, the image of $\mathcal{L}$ is characterized in terms of automorphic L-functions in several cases. As a typical example, we recall the following results on the Saito-Kurokawa lifting due to Evdokimov and Oda.

Theorem 1.1 ([E],[O])**.** *Let F be a Hecke eigenform on $\mathrm{Sp}_2(\mathbb{Z})$ of even weight. Denote by*

$$L^*(s, F; \mathrm{spin}) = (2\pi)^{-s}\Gamma\left(s+k-\frac{3}{2}\right)\Gamma\left(s+\frac{1}{2}\right)\cdot L(s, F; \mathrm{spin})$$

the completed spinor L-function of F with functional equation

$$L^*(s, F; \mathrm{spin}) = L^*(1-s, F; \mathrm{spin}).$$

Then the following three conditions are equivalent.

(1) F is in the Maass space (the image of the Saito-Kurokawa lifting).

(2) $L^*(s, F; \mathrm{spin})$ *has a simple pole at* $s = \dfrac{3}{2}$.

(3) $L(s, F; \mathrm{spin})$ *does not vanish at* $s = -\dfrac{1}{2}$.

In this note, we give a partial analogue of Theorem 1.1 for the *Arakawa lifting*, the theta lifting for the reductive dual pair $(O^*(2,2), \mathrm{Sp}(1,1))$.

2 The Arakawa lifting

Let B be a definite quaternion algebra over $\mathbb{Q}$ of discriminant d_B. For $x \in B$, let $\mathrm{tr}(x) = x + \overline{x}$ and $N(x) = x\overline{x}$ be the reduced trace and the reduced norm of x respectively, where $x \mapsto \overline{x}$ denotes the main involution of B. We fix an identification $B \otimes_{\mathbb{Q}} \mathbb{R} = \mathbb{H}$, where $\mathbb{H}$ is the Hamilton quaternion algebra.

We define two quaternion unitary groups by

$$H = \left\{ h \in \mathrm{GL}_2(B) \colon \overline{{}^t h} J h = J \right\}, \quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$
$$G = \left\{ g \in \mathrm{GL}_2(B) \colon \overline{{}^t g} Q g = Q \right\}, \quad Q = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Then (H, G) forms a dual reductive pair. Note that $H = O^*(2,2)$ is isogeneous to $\mathrm{PGL}_2 \times (B^\times/\mathbb{Q}^\times)$ and that $G = \mathrm{Sp}(1,1)$ is an inner form of Sp_2.

We henceforth fix a maximal order $\mathcal{O}$ of B. For a finite place p of $\mathbb{Q}$, set

$$L_p = \begin{cases} (\mathcal{O}_p, \mathcal{O}_p) & (p \nmid d_B) \\ (\mathcal{O}_p, \mathfrak{P}_p^{-1}) & (p \mid d_B) \end{cases},$$
$$L'_p = \begin{pmatrix} \mathcal{O}_p \\ \mathcal{O}_p \end{pmatrix},$$

where $\mathfrak{P}_p$ is the maximal ideal of $\mathcal{O}_p = \mathcal{O} \otimes_{\mathbb{Z}} \mathbb{Z}_p$. Let

$$K_f = \prod_{p<\infty} K_p, \quad K_p = \{k \in G_p \mid L_p k = L_p\},$$
$$U_f = \prod_{p<\infty} U_p, \quad U_p = \{u \in H_p \mid u L'_p = L'_p\}.$$

We fix an embedding $\mathbb{H}$ into $\mathrm{M}_2(\mathbb{C})$. Let

$$K_\infty = \left\{ \begin{pmatrix} a & b \\ b & a \end{pmatrix} \mid a, b \in \mathbb{H}, a \pm b \in \mathbb{H}^1 \right\} \subset G_\infty,$$
$$U_\infty = \mathbb{H}^1 \cdot \mathrm{SO}(2, \mathbb{R}) \subset H_\infty,$$

where $\mathbb{H}^1 = \{x \in \mathbb{H} \mid \bar{x}x = 1\}$.

We henceforth suppose that $\kappa > 4$. Let σ_κ be the symmetric tensor representation of $\mathrm{M}_2(\mathbb{C})$ on V_κ, the space of homogeneous polynomials in X and Y of total degree κ.

Let S_κ^G be the space of smooth functions $F\colon G_\mathbb{Q} \backslash G_\mathbb{A} \to V_\kappa$ satisfying

(1) $F(gk_f k_\infty) = \sigma_\kappa(a+b)^{-1} F(g) \quad (g \in G_\mathbb{A}, k_f \in K_f, k_\infty = \begin{pmatrix} a & b \\ b & a \end{pmatrix} \in K_\infty).$

(2) $F(g) = \dfrac{\kappa(\kappa-1)}{8\pi^2} \displaystyle\int_{G_\infty} \Omega(x_\infty) F(g x_\infty^{-1}) dx_\infty$, where $\Omega\colon G_\infty \to \mathrm{End}(V_\kappa)$ is the spherical function defined by $\Omega(g) = \sigma_\kappa(\Delta_g)^{-1} N(\Delta_g)^{-1}$ with $\Delta_g = \dfrac{1}{2}(a+b+c+d) \in \mathbb{H}^\times$ for $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G_\infty$.

(3) F is bounded.

Next let S_κ^H be the space of smooth functions $f\colon H_\mathbb{Q} \backslash H_\mathbb{A} \to V_\kappa$ satisfying

(1) $f(hu_f u_\infty) = \sigma_\kappa(c + \sqrt{-1}d)^{-1} f(h) \quad (h \in H_\mathbb{A}, u_f \in U_f, u_\infty = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in U_\infty).$

(2) $f(h) = \dfrac{\kappa-1}{2\pi} \displaystyle\int_{H_\infty} \omega(x_\infty) f(h x_\infty^{-1}) dx_\infty$, where $\omega\colon H_\infty \to \mathrm{End}(V_\kappa)$ is the spherical function defined by $\omega(h) = \sigma_\kappa(\delta_h)^{-1}$ with $\delta_h = \dfrac{1}{2}(a + d + \sqrt{-1}(-b+c))$ for $h = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in H_\infty$.

(3) f is bounded.

To define the theta kernel, we define the Weil representation r of $H_{\mathbb{A}} \times G_{\mathbb{A}}$ on $\mathscr{S}(B_{\mathbb{A}}^2) \otimes_{\mathbb{C}} \mathrm{End}(V_\kappa)$ by

$$\begin{aligned} r(h,1)\varphi(X) &= \varphi({}^t\overline{h}X) \qquad (h \in H_{\mathbb{A}}), \\ r\left(1, \begin{pmatrix} \alpha & 0 \\ 0 & {}^t\overline{\alpha}^{-1} \end{pmatrix}\right)\varphi(X) &= |N(\alpha)|^2\,\varphi(X\alpha) \quad (\alpha \in B_{\mathbb{A}}^\times), \\ r\left(1, \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix}\right)\varphi(X) &= \psi\left(-\mathrm{tr}(\beta\,{}^t\overline{X}JX)\right)\varphi(X) \quad (\beta \in B_{\mathbb{A}}, \beta + \overline{\beta} = 0), \end{aligned}$$

where ψ is the nontrivial additive character of $\mathbb{A}/\mathbb{Q}$ with $\psi(x_\infty) = \mathbf{e}(x_\infty) := e^{2\pi i x_\infty}$ for $x_\infty \in \mathbb{R}$, and $\mathscr{S}(B_{\mathbb{A}}^2)$ is the space of Schwartz-Bruhat functions on $B_{\mathbb{A}}^2$. The theta kernel $\theta\colon H_{\mathbb{Q}}\backslash H_{\mathbb{A}} \times G_{\mathbb{Q}}\backslash G_{\mathbb{A}} \to \mathrm{End}(V_\kappa)$ is defined by

$$\theta(h,g) = \sum_{X \in B^2} r(h,g)\varphi_0(X),$$

where

$$\varphi_0(X_f X_\infty) = \mathrm{char}_{L_f}(X_f)\exp(-2\pi\,\overline{{}^tX_\infty}X_\infty)\sigma_\kappa((1, -\sqrt{-1})\overline{X_\infty})$$

for $X_f \in B_{\mathbb{A}_f}^2$ and $X_\infty \in \mathbb{H}^2$ and $L_f = \prod_{p<\infty} L_p$.

For $f \in \mathcal{S}_\kappa^H$, set

$$\mathcal{L}(f)(g) = \int_{H_{\mathbb{Q}}\backslash H_{\mathbb{A}}} \theta(h,g)f(h)dh \qquad (g \in G_{\mathbb{A}}).$$

In [Ar], Arakawa proved the following result, which was later generalized to the case of $\mathrm{Sp}(1,q)$ $(q \geq 1)$ by the second named author ([Na]).

Theorem 2.1. *We have $\mathcal{L}(f) \in \mathcal{S}_\kappa^G$.*

We call $\mathcal{L}\colon S_\kappa^H \to S_\kappa^G$ the *Arakawa lifting*. The authors of this note studied arithmetic properties of $\mathcal{L}$ in [MN1],[MN2] and [MN3].

3 The main results

Let $\mathcal{L}^*\colon \mathcal{S}_\kappa^G \to \mathcal{S}_\kappa^H$ be the adjoint of $\mathcal{L}$ with respect to the Petersson inner products on $\mathcal{S}_\kappa^H$ and $\mathcal{S}_\kappa^G$. The main results of this note are given as follows.

Theorem 3.1 (Murase-Narita). *Suppose that $F \in S_\kappa^G$ is a Hecke eigenform. Then $f = \mathcal{L}^*(F) \in \mathcal{S}_\kappa^H$ is a Hecke eigenform and we have*

$$L(F,s;\mathrm{std}) = \zeta(s)L(f,s;\mathrm{std}).$$

Here $L(F,s;\mathrm{std})$ is the standard L-function of F of degree 5 and $L(f,s;\mathrm{std})$ is the standard L-function of f of degree 4, and $\zeta(s)$ is the Riemann zeta function.

Theorem 3.2 (Murase-Narita). *Suppose that d_B is a prime number and let $F \in S_\kappa^G$ be a Hecke eigenform.*

(1) We have

$$\mathcal{L}(\mathcal{L}^*F)(g) = c\, L(F,0;\mathrm{std})\, F(g) \quad (g \in G_{\mathbb{A}}), \tag{3.1}$$

where c is an elementary constant.

(2) We have an inner product formula:

$$||\mathcal{L}^*(F)||^2 = c\, L(F,0;\mathrm{std}) \cdot ||F||^2,$$

and hence $\mathcal{L}^*(F) = 0 \iff L(F,0;\mathrm{std}) = 0$.

(3) If $L(F,0;\mathrm{std}) \neq 0$, then $f := \mathcal{L}^*(F)$ does not vanish and F is the Arakawa lift of $(c \cdot L(F,0;\mathrm{std}))^{-1} f$. In particular $F \in \mathrm{Im}(\mathcal{L})$.

Remarks:

1. The proof of Theorem 3.2 is based on
   - the doubling method,
   - the Siegel-Weil formula for quaternion unitary groups due to Yamana ([Ya]),

– a trick using the spherical function Ω.

2. The assumption on d_B comes from this Siegel-Weil formula. When d_B is a composite, we need a modification of the assertion of Theorem 3.2 (we omit the detail in this note).

3. We have a formula for $\mathcal{L}^*(\mathcal{L}(f))$ similar to (3.1) for a Hecke eigenform f on H.

4. Let $\mathcal{L}' \colon$ {Jacobi forms} $\to$ {automorphic forms on $O(2,m)$} be a generalization of Maass lift studied independently by Gritsenko ([G]) and Sugano ([Su]). In [Su], Sugano showed that the lift is related to the theta lift for the reductice dual pair $(\mathrm{SL}_2, O(2,m))$ studied by Oda and Rallis-Schiffmann, and proved a formula for $(\mathcal{L}')^* \circ \mathcal{L}'$ similar to (3.1) by a method different from ours.

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