On p-adic limits of Siegel-Eisenstein series

by
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Abstract

Recently we developed a structure theory of mod p^m singular holomorphic Siegel modular forms. We use this as a tool in investigating the p-adic limits of Siegel-Eisenstein series (level one). Previous works (Nagaoka, Katsurada and others) already showed in some cases that such p-adic limits may become classical modular forms of level $\Gamma_0^n(p)$. Our method is simpler and covers more general cases; we also get a natural explanation, why such limits become linear combinations of certain genus theta series.
The starting point is the observation that Siegel-Eisenstein series of certain weights are actually mod p^m singular, i.e., all Fourier coefficients of maximal rank are congruent zero mod p^m.

1 Eisenstein series and their p-adic limits

Siegel-Eisenstein series (degree n, level one, even weight $k > n+1$) are defined by

$$E_k^n(Z) = \sum_{C,D} det(CZ+D)^{-k} = \sum_{0\leq T\in\Lambda_n} a_k^n(T)q^T$$

Here Z is in Siegel's upper half space $\mathbb{H}_n$, we denote by Λ_n the set of half-integral symmetric matrices, and we put $q^T := e^{2\pi i tr(TZ)}$ for $T \in \Lambda_n$.
The Fourier coefficients $a_k^n(T)$ are well-known (thanks to Siegel and Katsurada), in particular they are rational with bounded denominators.

Let (k_m) be a sequence of even natural numbers with $k_m \longrightarrow \infty$ which at the same time converges p-adically to the "p-adic weight" (k,a):

$$(k_m) \longrightarrow (k,a) \in \mathbb{Z}_p \times \mathbb{Z}/(p-1)\mathbb{Z}.$$

Under the assumption that the sequence $\left(a_{k_m}^n(T)\right)_m$ converges p-adically (uniformly in T) we may define the formal power series

$$\tilde{E}_{(k,a)}^n := \lim_{m\to\infty} E_{k_m}^n.$$

This formal power series with coefficients in $\mathbb{Q}_p$ is then called p-adic Eisenstein series of weight (k, a). For simplicity of notation, we focus in this text[1] on the case of weight (k, k) with k even and we write

$$k_m = k + a(m)p^{b(m)}$$

where $a(m)$ and $b(m)$ are natural numbers with $(p-1) \mid a(m)$ and $b(m) \longrightarrow \infty$.

Problem: *To determine such limits explicitly and* **to identify them with classical modular forms**.

There are several results on this problem in the literature (starting with Serre, then Nagaoka, Katsurada, Mizuno and the second author), treating rather special cases w.r.t. weight or degree. Most of them use essentially the same strategy: they first compute the p-adic limit of the Fourier coefficients: This is a matter concerning Bernoulli numbers and local singular series. Then they have to identify the limit with a classical modular form; in all known cases, this classical modular form is of level p, more precisely, it is modular for the group $\Gamma_0^n(p) = \{\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in Sp(n, \mathbb{Z}) \mid C \equiv 0 \bmod p\}$.
In such a procedure, it is a nontrivial problem to *guess a candidate* for such a classical modular form.
Mizuno follows a different strategy, employing the technique of Saito-Kurokawa liftings.

- Nagaoka [15] showed that $\tilde{E}^n_{(1, \frac{p+1}{2})}$ is essentially the genus theta series for positive definite binary quadratic forms of discriminant $-p$.
- Katsurada and Nagaoka [9] showed that $\tilde{E}^2_{(k, k+\frac{p-1}{2})}$ is essentially a linear combination of two nebentypus genus theta series and a nebentypus Eisenstein series of level p. Here p is any prime number with $p > 2k$ and $k \equiv \frac{p-1}{2} \bmod 2$.

The case of Katsurada and Nagaoka illustrates the difficulties of finding a candidate whereas Nagaoka's case shows a link to singular modular forms and mod p-power singular forms. It is this link, which we will follow; our method also provides a natural candidate for the limit in question.

[1] see however final remarks

2 Statement of main result

Main Theorem: *For $n \geq 1$, k even, p any regular prime with $p > 2k+1$ the p-adic Eisenstein series is a classical modular form for $\Gamma_0^n(p)$, more precisely we have*

$$\tilde{E}^n_{(k,k)} = \sum_{\mathfrak{S}} a(\mathfrak{S}) \cdot \Theta^n(\mathfrak{S}).$$

Here the summation is over all genera $\mathfrak{S}$ of positive definite quadratic forms S of size $2k$ with level dividing p, trivial nebentypus χ_S and

$$\Theta^n(\mathfrak{S}) = \sum_S \frac{1}{\epsilon(S)} \theta^n_S$$

is an unnormalized genus theta series, i.e., S runs over a set of representatives of $GL(2k, \mathbb{Z})$-classes of quadratic forms in the genus $\mathfrak{S}$ and $\epsilon(S)$ denotes the number of units of S. Furthermore,

$$\theta^n_S(Z) = \sum_{g \in \mathbb{Z}^{(2k,n)}} e^{2\pi i tr(g^t S g \cdot Z)}$$

is the degree n theta series attached to S.
The coefficient $a(\mathfrak{S})$ is explicitly known.

Some Comments

- The genus theta series from above (and hence also $\tilde{E}^n_{(k,k)}$) are in the space of holomorphic Siegel Eisenstein series for $\Gamma_0^n(p)$; in the range of convergence this is assured by the classical theorem of Siegel. It remains true for small k thanks to work of Kudla-Rallis, see [11]) where an appropriate (almost classical, non-adelic) version is explained, as was kindly pointed out to us by Schulze-Pillot [17].

- $\tilde{E}^n_{(k,k)}$ is an eigenform for the Hecke operator $U(p)$ with eigenvalue 1, see Section 5.

- The limit is independent of the choice of the sequence k_m.

- The condition "p a regular prime" can be weakened by requesting a (conjectural) weaker property of Kubota-Leopoldt p-adic L-functions.

3 On mod p^m singular modular forms

There is an elaborated theory of singular modular forms over $\mathbb{C}$, mainly due to E. Freitag [5]. Recently the authors [3] have worked out a theory of mod p^m singular forms, following the lines of Freitag. For simplicity, we consider only the case where the coefficient ring is the ring of p-integral rational numbers $\mathbb{Z}_{(p)}$.

Definition: A Siegel modular form $F = \sum a(T)q^T$ of degree n, level 1, weight k with Fourier coefficients in $\mathbb{Z}_{(p)}$ is called mod p^m singular of rank $r < n$ if $a(T) \equiv 0 \bmod p^m$ for all T with $rank(T) > r$ and there exists a T_0 of rank r with $p \nmid a(T_0)$.

Theorem: *Let p be a prime with $p > r+1$ and let F be a modular form of degree $n \geq 2r$, weight k and mod p^m singular of rank r. Then there exists $e \geq 0$:*

$$F \equiv \sum_S c_S \cdot \theta_S^n \bmod p^m.$$

Here the summation is over all S in $\Lambda_r^+/GL(r,\mathbb{Z})$ with level(S) dividing p^e, Λ_r^+ is the set of positive definite elements of Λ_r, $\chi_S = \chi_p^t$ with $k - \frac{r}{2} = t \cdot \frac{p-1}{2} \cdot p^{m-1}$, and χ_p is the nontrivial quadratic character mod p; we recall that by [2] the number $k - \frac{r}{2}$ is always of the form given above (with $t \in \mathbb{Z}$).

The coefficient c_S is explicitly given as

$$c_S = \frac{1}{\epsilon(S)} a(\begin{pmatrix} S & 0 \\ 0 & 0 \end{pmatrix})^*,$$

where the $*$ indicates that we should take the "primitive" Fourier coefficient (in the sense of [1]). In addition, there is a statement about the mod p^m filtration of θ_S^{n-r} in the sense of Katz.

4 The link from $\tilde{E}^n_{(k,k)}$ to mod p^m singular forms

Let ν_p be the standard p-valuation of $\mathbb{Q}$ normalized by $\nu_p(p) = 1$.
For a degree n modular form F with Fourier coefficients in $\mathbb{Z}_{(p)}$ and any j with $0 \leq j \leq n$ we define

$$\nu_p^{(j)}(F) := \inf\{\nu_p(a(T)) \mid T \in \Lambda_n, rank(T) = j\}.$$

Proposition: For $n \geq 4k$, and a regular prime p with $p > 2k+1$ we have

- For all r with $2k < r \leq n$ there exists C_r with

$$\nu_p^{(r)}(E^n_{k_m}) \geq C_r + b(m).$$

- For $\lambda_m := \nu_p^{(2k)}(E^n_{k_m})$ and all r with $0 \leq r < 2k$ we have

$$\nu_p^{(r)}(E^n_{k_m}) \geq \lambda_m.$$

- For all m we have $\lambda_m \leq 0$ and λ_m becomes constant for large m.

Proof: We recall that (for $T \in \Lambda_r^+$ with r odd) the T Fourier coefficient is of the form

$$a^r_{k_m}(T) = \frac{k_m}{B_{k_m}} \cdot \prod_{i=1}^{[\frac{r}{2}]} \frac{2k_m - 2i}{B_{2k_m-2i}} \times d$$

with some integer d; the case of r even is similar.
The crucial factor in the formula above is for $i = k$ (this factor always appears!). We observe that (by Clausen-von Staudt)

$$\nu_p(\frac{2k_m - 2k}{B_{2k_m-2k}}) = \nu_p(\frac{2a(m)p^{b_m}}{B_{2a(m)p^{b_m}}}) \geq b(m) + 1$$

The factors for $i < k$ are harmless (by Kummer type congruences), but the factors for $i > k$ are delicate: Indeed, we need a condition to show that they are p-adically bounded; the regularity of p is a sufficient condition. □

Corollary: *Let p, k and n be as above. We put $c(m) := b(m) + C_r - \lambda_m$. Then $p^{-\lambda_m} \cdot E^n_{k_m}$ is mod $p^{c(m)}$ singular of rank r. Moreover, there is $e_m \geq 0$ with*

$$p^{-\lambda_m} \cdot E^n_{k_m} \equiv \sum_S c_S^{(m)} \Theta^n_S \bmod p^{c(m)}.$$

Here S runs over representatives of $\Lambda^+_{2k}/GL(2k,\mathbb{Z})$ such that the level(S) divides p^{e_m} and χ_S is trival. The factor $c_S^{(m)}$ is explictly given by

$$c_S^{(m)} = p^{-\lambda_m} \frac{1}{\epsilon(S)} a^{(2k)}_{k(m)}(S)^* \bmod p^{c(m)}.$$

Using Siegel's ϕ-operator several times, we see that the corollary also holds for smaller n.
Furthermore, by applying the Hecke operator $T(p)$ and by the considerations of the next section, we can see that only $e_m \in \{0,1\}$ occurs.
Finally to get the theorem, we observe that the coefficient $c_S^{(m)}$ depends only on the genus of S and we may do a resummation as in the theorem.

5 How to use the Hecke operators $T(p)$ and $U(p)$

Here we just collect some properties of $T(p)$ and $U(p)$, which should be well-known. Let M_k^n (and $M_k^n(\Gamma_0^n(p))$ repectively) denote the space of all modular forms of degree n, weight k and level one (level $\Gamma_0^n(p)$ respectively).

Lemma: *Let n, k be positive integers with $k \geq n$ and p a prime. We may decompose the Hecke operator $T(p) : M_k^n \longmapsto M_k^n$ as*

$$T(p) = U(p) + p^{k-n}\tilde{V}(p)$$

with an endomorphism $\tilde{V}(p)$ of $M_k^n(\Gamma_0(p))$, which preserves integrality of Fourier coefficients.

Lemma: *Let λ_{k_m} be the eigenvalue of $T(p)$ for $E_{k_m}^n$. Then*

$$\lambda_{k_m} \equiv 1 \bmod p^{c(m)}.$$

6 Final remarks

Remark 1: For simplicity of notation, we treated only the case of p-adic weight (k,k). Along the same lines, one can study the case of p-adic weight $(k, k+\frac{p-1}{2})$. Here, under similar conditions, the p-adic limit is then in the space of genus theta series for $\Gamma_0^n(p)$ with nontrivial quadratic character.

Remark 2: As mentioned before, it is nice feature in our treatment that we do not need to "guess" a candidate for the p-adic limit. By our method, we get a natural linear combination of level p genus theta series. We should mention, that there is another possibility to identify a candidate (not using theta series!): By our statement about the $U(p)$-eigenvalue being 1, there is such a candidate. The Eisenstein series for $\Gamma_0^n(p)$ with $U(p)$ - eigenvalue 1 (and its Fourier expansion) has been calculated explicitly in various ways. It is possible to use this approach to determine - under conditions somewhat different from the ones of the present work - the p-adic limit (this is work in progress together with K. Gunji).

References

[1] S. Böcherer, Über die Fourierkoeffizienten der Siegelschen Eisensteinreihen. (German) [On the Fourier coefficients of Siegel's Eisenstein series] Manuscripta Math. 45 (1984), no. 3, 273-288.

[2] S. Böcherer, T. Kikuta, On mod p singular modular forms. Forum Math. 28 (2016), no. 6, 1051-1065.

[3] S. Böcherer, T. Kikuta, Structure theorem for mod p^m singular forms. arXiv:2302.00309 [math.NT].

[4] E. Freitag, Siegelsche Modulfunktionen. (German) [Siegel modular functions] Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 254. Springer-Verlag, Berlin, 1983. x+341 pp. ISBN: 3-540-11661-3

[5] E. Freitag, Siegel Eisenstein series of arbitrary level and theta series. Abh. Math. Sem. Univ. Hamburg 66 (1996), 229-247.

[6] K. Gunji, On the functional equations of the Siegel Eisenstein series of an odd prime level p. arXiv:2306.10696

[7] H. Katsurada, An explicit formula for Siegel series. Amer. J. Math. 121 (1999), no. 2, 415-452.

[8] H. Katsurada, S. Nagaoka, On some p-adic properties of Siegel-Eisenstein series. J. Number Theory 104 (2004), no. 1, 100-117.

[9] H. Katsurada, S. Nagaoka, On p-adic Siegel Eisenstein series. J. Number Theory 251 (2023), 3-30.

[10] T. Kikuta, S. Nagaoka, On a correspondence between p-adic Siegel-Eisenstein series and genus theta series. Acta Arith. 134 (2008), no. 2, 111-126.

[11] S. Kudla, Some extensions of the Siegel-Weil formula. Eisenstein series and applications, 205-237, Progr. Math., 258, Birkhäuser Boston, Boston, MA, 2008.

[12] Y. Mizuno, A p-adic limit of Siegel-Eisenstein series of prime level q. Int. J. Number Theory 4 (2008), no. 5, 735-746.

[13] Y. Mizuno, On p-adic Siegel-Eisenstein series of weight k. Acta Arith. 131 (2008), no. 2, 193-199.

[14] Y. Mizuno, S. Nagaoka, Some congruences for Saito-Kurokawa lifts. Abh. Math. Semin. Univ. Hambg. 80 (2010), no. 1, 9-23.

[15] S. Nagaoka, A remark on Serre's example of p-adic Eisenstein series. Math. Z. 235 (2000), no. 2, 227-250.

[16] J.-P. Serre, Formes modulaires et fonctions zêta p-adiques. Modular functions of one variable, III (Proc. Internat. Summer School, Univ. Antwerp, Antwerp, 1972), pp. 191-268. Lecture Notes in Math., Vol. 350, Springer, Berlin, 1973.

[17] R. Schulze-Pillot, Letter March 2023.

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