DIMENSIONS OF PARAMODULAR FORMS WITH INVOLUTION

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1. Introduction

In this paper, we explain explicit dimension formulas of the Atkin-Lehner plus and minus eigenspaces of paramodular forms of degree two of prime level p of weight $det^k Sym(j)$ with $k \geq 3$, where $Sym(j)$ is the symmetric tensor representation of degree j. Since details are in [24], here we give rough outline. In the final section, we add some lengthy numerical examples which could not be included in a submitted paper.

Paramodular forms are interesting objects by several reasons. One of the reasons is related to a generalization of the Shimura-Taniyama conjecture on modularity of elliptic curves over $\mathbb{Q}$. The Blumer and Kramer conjectured in [3] that the L function of any abelian surface defined over $\mathbb{Q}$ with conductor p such that $End(A) \cong \mathbb{Z}$ is given by that of a paramodular form of weight 2 of level p. To check this conjecture for numerical examples, Poor and Yuen gave a lot of experimental results on paramodular forms, even on higher weights. Another interesting point is a theory of new forms by B. Roberts and R. Schmidt, using paramodular groups as levels of new vectors (see [29]).

Also paramodular forms can be regarded as typical explicit examples of Langlands conjecture on correspondence between Siegel modular forms and algebraic modular forms of the compact twist $Sp(2)$ of symplectic group of rank two. Since our calculation heavily depends on such correspondence, we review a brief history of this sort. The prototype of a Jaquet-Langlands correspondence between SL_2 and $SU(2)$ was first given by M. Eichler ([6], [7]) for $\Gamma_0(N)$ and algebraic modular forms, typically with respect to maximal orders of definite quaternion algebras over $\mathbb{Q}$ of prime discriminant $N = p$. (See also [35] and [12]). In 1964 in [25], Ihara raised a similar problem for $Sp(2, \mathbb{R})$ and its compact twist $Sp(2)$, and gave there a lifting theory to the compact twist, a pioneering work before Saito-Kurokawa lift was announced much later in 1978, He himself did not give any concrete conjecture between Siegel modular forms and algebraic modular forms of $Sp(2)$ except for considering what should be the corresponding weights. Later, starting in 1980's, inspired by his thought, we gave several explicit conjectures for several discrete group of parahoric type in [13], [14], [16], [20], [21]. There we gave explicit dimension formula for Siegel modular forms of parahoric levels as well as numerical examples of L functions. Among these, the conjecture in [16] for paramodular forms of prime level p seems simplest, and this case has been proved independently by van Hoften in [39], Rösner and Weissauer in [30] around in 2021, the former using geometric methods and the latter using trace formulas. We review here this simplest case for a while.

Let H_2 be the Siegel upper half space of degree two. For any ring R, we denote by $Sp(2,R)$ the symplectic group of matrix size 4 with coefficients in R. For any positive integer N. we put

$$K(p) = \begin{pmatrix} \mathbb{Z} & N\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & N^{-1}\mathbb{Z} \\ \mathbb{Z} & N\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ N\mathbb{Z} & N\mathbb{Z} & N\mathbb{Z} & \mathbb{Z} \end{pmatrix} \cap Sp(2,\mathbb{Q}).$$

This group is called a paramodular group of level N. This group corresponds with the moduli of abelian surfaces with polarizations of type $(1,N)$. Let $\rho_{k,j} = det^k Sym(j)$ be the irreducible representation of $GL_2(\mathbb{C})$ where det^k is the k-th power of the usual determinant and $Sym(j)$ is the symmetric tensor representation of degree $j \geq 0$ of two variables. We denote by $V_{k,j}$ the representation space of $\rho_{k,j}$. For any $V_{k,j}$-valued function F of H_2 and any $g = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in Sp(2,\mathbb{R})$, we write

$$F|_{k,j}[g] = \rho_{k,j}(CZ+D)^{-1}F(gZ) \qquad Z \in H_2.$$

A $V_{k,j}$-valued holomorphic function F of H_2 such that $F|_{k,j}[\gamma] = F$ for all $\gamma \in K(N)$ is called a paramodular form of weight $\rho_{k,j}$ of level N. We denote the space of such functions by $A_{k,j}(K(N))$. For any function F of H_2, we write

$$\Phi(F) = \lim_{\lambda \to \infty} F\begin{pmatrix} \tau & 0 \\ 0 & i\lambda \end{pmatrix}.$$

A paramodular form such that $\Phi(F|_{k,j}[g]) = 0$ for any $g \in Sp(2,\mathbb{Q})$ is said to be a cusp form. We denote the space of paramodular cusp forms of weight $\rho_{k,j}$ of level N by $S_{k,j}(K(N))$. Since $\rho_{k,j}(-1_2) = (-1)^j$, it is clear that $A_{k,j}(K(N)) = S_{k,j}(K(N)) = 0$ if j is odd. Throuout the paper, we fix a prime p and we mainly treat the case when $N = p$ is a prime.

Next we define algebraic modular forms with respect to some open subgroup of the adelization of quaternion hermitian group. Let B be the definite quaternion algebra over $\mathbb{Q}$ ramified at p and infinity. For any place v of $\mathbb{Q}$, we denote by B_v the v-adic completion of B. We define the quaternion hermitian group G of degree two by

$$G = \{g \in M_2(B); gg^* = n(g)1_2 \text{ for some } n(g) \in \mathbb{Q}_+^\times\},$$

where $\mathbb{Q}_+^\times$ is the set of positive rational numbers and we put $g^* = (\overline{g_{ji}})$ for $g = (g_{ij})$ where $\overline{*}$ is the canonical involution. We denote by G_A the adelization of G, by G_v the v component of G_A for any place v of

$\mathbb{Q}$. We put $G^1_\infty = \{g \in G_\infty; n(g) = 1\}$. Then we have $G^1_\infty = Sp(2)$ (the compact twist of $Sp(2,\mathbb{R})$.) To describe open subgroups of G_p in a simpler way, we take $\xi \in GL_2(B_p)$ such that $\xi\xi^* = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and define $G_p^* = \xi G_p \xi^{-1}$. (Actually we may take $\xi \in GL_2(O_p)$.) So we have

$$G_p^* = \left\{ g \in M_2(B_p); g \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} g^* = n(g) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}.$$

Let O be a maximal order of B. For $v \neq p$ and $v < \infty$, we define

$$U_v = GL_2(O_v) \cap G_v.$$

We fix a prime element π of O_p such that $\pi^2 = -p$ and put

$$U_p^* = \begin{pmatrix} O_p & \pi^{-1}O_p \\ \pi O_p & O_p \end{pmatrix}^\times \cap G_p^*,$$

where the superscript $\times$ means the group of the invertible elements of the ring. This is one of the maximal compact subgroups of G_p^* up to conjugation (another one is $GL_2(O_p) \cap G_p^*$). We denote by U_p the subgroup of G_p isomorphic to U_p^* by $G_p \cong G_p^*$. Now we define a subgroup $U_{npg}(p)$ of G_A by

$$U_{npg}(p) = G_\infty U_p \prod_{v \neq p} U_v.$$

This $U_{npg}(p)$ is nothing but the stabilizer of a lattice in the genus of maximal lattices L with $N(L) = \mathfrak{P}$, where $\mathfrak{P}$ is a two sided prime ideal of O over p and $N(L)$ is the two sided ideal of O generated by xy^* for $x, y \in L \subset B^2$. (A lattice L is called maximal if $L \subset M$ with $N(M) = N(L)$ means $L = M$.) This genus is often called non-principal genus (so denoted by U_{npg}, while the genus containing O^2 is called principal. We denote by $(\rho_{f_1,f_2}, \mathfrak{V}_{f_1,f_2})$ the irreducible representation of $Sp(2) = G^1_\infty$ corresponding to the Young diagram (f_1, f_2) $(f_1 \geq f_2 \geq 0)$. We assume that $f_1 \equiv f_2 \bmod 2$. We prolong ρ_{f_1,f_2} to the representation of G_A by

$$G_A \to G_\infty \to G_\infty/center \cong Sp(2)/\{\pm 1_2\} \overset{\rho_{f_1,f_2}}{\to} GL(\mathfrak{V}_{f_1,f_2}).$$

Then the space $\mathfrak{M}_{f_1,f_2}(U_{npg}(p))$ of algebraic modular forms with respect to $U_{npg}(p)$ of weight ρ_{f_1,f_2} is defined by

$$\begin{aligned}\mathfrak{M}_{f_1,f_2}(U_{npg}(p)) =& \{f : G_A \to \mathfrak{V}_{f_1,f_2}; f(uga) = \rho_{f_1,f_2}(u)f(g), \\ & u \in U_{npg}(p), g \in G_A, a \in G\}.\end{aligned}$$

For example, if $(f_1, f_2) = (0, 0)$, then $f(g) \in \mathfrak{M}_{0.0}(U_{npg}(p))$ depends only on $U_{npg}(p)gG$ and $\dim \mathfrak{M}_{0,0}(U_{npg}(p))$ is nothing but the class number of the non-principal genus.

The following theorem has been conjectured in [14], [16] and proved by van Hoften [39], Rösner and Weissauer [30], independently.

Theorem 1.1. *For any integer $k \geq 3$ and any even integer $j \geq 0$, we have an Hecke equivariant isomorphism*

$$\mathfrak{M}^0_{k+j-3,k-3}(U_{npg}) \cong S^0_{k,j}(K(p)),$$

where superscript 0 means a kind of certain new forms, neglecting lifting parts and old form parts.

The exact meaning of the superscript 0 will be explained later.

Soon after this theorem had been proved, Dummigan, Pacetti, Rama and Tornaría generalized this to the case of eigenspaces of Atkin-Lehner involution. So now we will explain the Atkin-Lehner involution. We put

$$\eta = \frac{1}{\sqrt{p}} \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & p & 0 & 0 \\ p & 0 & 0 & 0 \end{pmatrix}.$$

Then we have $\eta^2 = -1_4$ and $F|_{k,j}[\eta^2] = F$, so $A_{k,j}(K(p))$ and $S_{k,j}(K(p))$ are decomposed into the direct sum of eigenspaces $A^{\pm}_{k,j}(K(p))$ and $S^{\pm}_{k,j}(K(p))$ for ± 1 eigenvalues. On the other hand, for a double coset $U_{npg}(p)gU_{npg}(p)$ for $g \in G_A$, we define the action of $U_{npg}(p)gU_{npg}(p)$ on for $\mathfrak{M}_{f_1,f_2}(U_{npg}(p))$ as follows. We write

$$U_{npg}(p)gU_{npg}(p) = \coprod_{j=1}^{d} z_j U_{npg}(p).$$

Then for $f \in \mathfrak{M}_{f_1,f_2}(U_{npg}(p))$ we define

$$([U_{npg}(p)gU_{npg}(p)]f)(x) = \sum_j \rho(z_j) f(z_j^{-1}x), \qquad (x \in G_A).$$

For a prime element π of O_p such that $\pi^2 = -p$, regard π as an element $\pi 1_2$ of $G_p \subset G_A$ and put $R(\pi) = U_{npg}(p)\pi U_{npg}(p) = U_{npg}(p)\pi = \pi U_{npg}(p)$. Then the operator $R(\pi)$ is of order two and we denote the eigenspaces of $R(\pi)$ for ± 1 by $\mathfrak{M}^{\pm}_{f_1,f_2}(U_{npg}(p))$.

Theorem 1.2 ([5])**.** *An essential part of $\mathfrak{M}^{\pm}_{k+j-3,k-3}(U_{npg}(p))$ correspond bijectively to an essential part of $S^{\mp}_{k,j}(K(p))$.*

Here "essential part" roughly means forms that are not obtained by liftings and old forms, but more precise explanation will be given later. By this theorem, we can state the relation between dimensions of $S^{\pm}_{k,j}(K(p))$ and dimensions of $\mathfrak{M}^{\mp}_{f_1,f_2}(U_{npg}(p))$ up to lifting and old forms. In order to explain more precise theorem, we prepare several notation. We write

$$\Gamma_0(p) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}); c \equiv 0 \bmod p \right\}.$$

We write $W = \begin{pmatrix} 0 & -1/\sqrt{p} \\ \sqrt{p} & 0 \end{pmatrix}$. This is the usual Atkin-Lehner involution. We denote by $S_k^{\pm}(\Gamma_0(p))$ the $+1$ and -1 eigenspaces of W in the space of elliptic cusp forms $S_k(\Gamma_0(p))$, respectively. We write $S_k^{new,\pm}(\Gamma_0(p))$ the the space of new forms in each eigenspace in the usual sense. By [39], [30], [5] and by careful use of [31], [32], [33], [34], we obtain the following relations of dimensions, which gives explicit dimension formulas for $S_{k,j}^{\pm}(K(p))$. The notation δ_{ab} is the usual Kronecker symbol that means 1 for $a = b$ and 0 otherwise.

Theorem 1.3 ([24]). *Let p be any prime. For $k \geq 3$ and even $j \geq 0$, we have an explicit formula for $\dim S_{k,j}^{\pm}(K(p))$. It is given by*

$$\dim S_{k,j}^{+}(K(p)) = \dim S_{k,j}(Sp(2,\mathbb{Z})) + \dim \mathfrak{M}_{j+k-3,k-3}^{-}(U_{npg}(p))$$
$$- \dim S_{j+2}^{new,+}(\Gamma_0(p)) \times \dim S_{2k+j-2}(SL_2(\mathbb{Z}))$$
$$\dim S_{k,j}^{-}(K(p)) = \dim S_{k,j}(Sp(2,\mathbb{Z})) - \delta_{j0} \dim S_{2k-2}(SL_2(\mathbb{Z})) - \delta_{j0}\delta_{k3}$$
$$+ \dim \mathfrak{M}_{j+k-3,k-3}^{+}(U_{npg}(p)) - \dim S_{j+2}^{new,-}(\Gamma_0(p)) \times S_{2k+j-2}(SL_2(\mathbb{Z})).$$

Firstly we explain why this is a dimension formula, and secondly we explain the real meaning of the above equality. First, the dimensions of $\mathfrak{M}^{\pm}(U_{npg}(p))$ are newly calculated this time. The formula (which we omit here since it is very lengthy and precisely written in [24]) contains $(p^2 - 1)/5760$ as a main term, and also class numbers of imaginary quadratic field $\mathbb{Q}(\sqrt{-3p})$, $\mathbb{Q}(\sqrt{-2p})$, $\mathbb{Q}(\sqrt{-p})$, as well as the second generalized Bernoulli number associated with quadratic character corresponding to $\mathbb{Q}(\sqrt{p})$. This dimension is obtained by the trace formula of G_A. We have already given the formula for $\dim \mathfrak{M}_{f_1,f_2}(U_{npg}(p))$ for any (f_1, f_2) in [11] II in 1982. So the problem is to obtain $\dim \mathfrak{M}_{f_1,f_2}^{\pm}(U_{npg}(p))$. By [22] and [23], we know that $\mathfrak{M}_{0,0}^{+}(U_{npg}(p))$ is nothing but the class number of some quinary lattices with discriminant $2p$ when $p \neq 2$. Such class number is explicitly calculated by Teruaki Asai in [1] using trace formula. This trace formula is calculated as a sum of contribution of groups elements γ in the automorphism groups of the quinary lattice classes in the genus in question with individual characteristic polynomials of γ. These data can be used to calculate the dimension for general weight by trace formula. Actually we calculated $TrR(\pi) = \dim \mathfrak{M}_{f_1,f_2}^{+}(U_{npg}(p)) - \dim \mathfrak{M}_{f_1,f_2}^{-}(U_{npg}(p))$ by using this. When $p = 2$, we cannot use the comparison with quinary lattices, but we can calculate the trace formula directly from concrete elements of $U_{npg}(2) \cap G$, where the class number of $U_{npg}(2)$ is one. The rest of the dimensions in the right hand side of the formula in the above theorem has been known. For example, formulas for dimension $\dim S_{k,j}(Sp(2,\mathbb{Z})$ are due to [38] for $k \geq 5$ and [27] for $k = 3$, 4, and explicitly written. The formula for $\dim S_{j+2}^{new,+}(\Gamma_0(p))$ is known by Yamauchi and $\dim S_{2k+j-2}(SL_2(\mathbb{Z}))$ is classically known. So for any prime p, we can really write down formulas for $\dim S_{k,j}^{\pm}(K(p))$.

Secondly we see that two relations above are explained as correspondences between new forms that are not obtained by liftings. First we explain the paramodular old forms. In case of elliptic modular forms, the old forms of $\Gamma_0(N)$ are roughly defined as forms coming from $\Gamma_0(N_0)$ (by several ways) such that $N_0|N$. This time, we have $\Gamma_0(N) \subset \Gamma_0(N)$. For paramodular groups, we have no inclusion relation between $K(N)$ and $K(N_0)$, but still we want to subtract forms from $K(N_0)$ by the several trace maps from $K(N) \cap K(N_0)$ to $K(N)$. Or representation theoretically speaking, we consider an automorphic representation associated with $K(N)$ which has $K(N_0)$ fixed vectors. For example, for $K(p)$, we have $K(1) = Sp(2,\mathbb{Z})$, and we may consider two groups $Sp(2,\mathbb{Z})$ and $\eta Sp(2,\mathbb{Z})\eta^{-1}$ as local standard maximal compact subgroups in the Tits building at p. The paramodular forms obtained from $S_{k,j}(Sp(2,\mathbb{Z}))$ and $S_{k,j}(\eta Sp(2,Z)\eta^{-1})$ by taking the trace are regarded as old forms. Such explanation first appeared in [14]. It is more clearly explained by Ralf Schmidt in [31], and there it was proved that when a form in $S_{k,0}(Sp(2,\mathbb{Z}))$ is a Saito-Kurokawa lift, the both trace images coincide. So the dimension of new forms of $S_{k,j}(K(p))$ in the above sense are $\dim S_{k,j}(K(p)) - 2\dim S_{k,j}(Sp(2,\mathbb{Z})) + \delta_{even,k}\dim S_{2k-2}(SL_2(\mathbb{Z}))$, where $\delta_{even,k} = 1$ if k is even and 0 otherwise. The Saito Kurokawa lifts are known to go to $S^{+}_{k,0}(Sp(2,\mathbb{Z}))$, so for each Atkin-Lehner sign, and except for Saito-Kurokawa lift, for each element of $S_{k,j}(Sp(2,\mathbb{Z}))$, there is one old form in $S^{+}_{k,j}(K(p))$ and one in $S^{-}_{k,j}(K(p))$. So the dimension of new forms are

$$\begin{aligned}\dim S^{new,+}_{k,j}(K(p)) &= \dim S_{k,j}(K(p)) - \dim S_{k,j}(Sp(2,\mathbb{Z}))\\ \dim S^{new,-}_{k,j}(K(p)) &= \dim S_{k,j}(K(p)) - \dim S_{k,j}(Sp(2,\mathbb{Z}))\\ &\quad + \delta_{j0}\delta_{k,even}\dim S_{2k-2}(SL_2(\mathbb{Z})).\end{aligned}$$

Next we must subtract some parts obtained by liftings. Both $S_{k,j}(K(p))$ and $\mathfrak{M}_{k+j-3,k-3}(U_{npg}(p))$ contain the same Saito-Kurokawa lifts with level p for some elliptic cusp forms. We cannot see this part from the comparison of dimensions since their contribution is the same dimension. We omit this part from our consideration. Now we have Saito-Kurokawa lift from $S_{2k-2}(SL_2(\mathbb{Z}))$ for odd k to $\mathfrak{M}_{k-3,k-3}(U_{npg}(p))$. This part is in $\mathfrak{M}^{+}_{k-3,k-3}(U_{npg}(p))$ by [5]. There is no such lift to paramodular forms. So this part is subtracted in the second relations in the above from $\dim \mathfrak{M}^{+}_{k-3,k-3}(U_{npg}(p))$. (The part when k is even is already explained by lift to $S_{k,j}(K(p))$). We also have the Yoshida type lift from $S^{new}_{j+2}(\Gamma_0(p)) \times S_{2k+j-2}(SL_2(\mathbb{Z}))$ to $\mathfrak{M}_{k+j-3,k-3}(U_{npg}(p))$ (first constructed in [25]). For these forms, by virtue of [5], we see that pairs of forms in $S^{new,+}_{j+2}(\Gamma_0(p)) \times S_{2k+j-2}(SL_2(\mathbb{Z}))$ are lifted into $\mathfrak{M}^{-}_{k+j-3,k-3}(U_{npg}(p))$ and pairs in $S^{new,-}_{j+2}(\Gamma_0(p)) \times S_{2k+j-2}(SL_2(\mathbb{Z}))$ into $\mathfrak{M}^{+}_{k+j-3,k-3}(H_{npg}(p))$. This explains the subtraction of this part in the above relations. Finally, the part $\delta_{j0}\delta_{k3}$ is subtracted from $\mathfrak{M}^{+}_{0,0}(U_{npg}(p))$

since the space $\mathfrak{M}^+_{0,0}(U_{npg}(p))$ contains a constant function which does not correspond with any paramodular cusp forms.

2. Non existence of weight 3

$S_3^+(K(p))$ gives the canonical divisor of the moduli of Kummer varieties with $(1,p)$ polarization ([8]). In [9], it is asked when $S_3^+(K(N))=0$ and they gave several concrete examples of such N. When N is a prime p, we can determine exactly when $S_3^+(K(p))=0$ by evaluating the dimension formula.

Proposition 2.1. *Let p be a prime. Then we have*

$$\dim S_3^+(K(p))=0$$

if and only if p is any prime such that $p\leq 163$ or $p=179$, 181, 191, 193, 199, 211, 229, 241.

By the way, we have $\dim S_3^+(K(p))=1$ if $p=167$, 173, 197, 223, 233, 239, 251, 271, 277, 281, 313, 331, 337 and $\dim S_3^+(K(p))=2$ if $p=227$, 257, 263, 269, 283, 349, 379. 409, 421.

The proof is given in the following way. Roughly speaking, the dimension formula contains polynomials in p, class numbers and the Bernoulli numbers, and we evaluate these numbers carefully as order of p. Then we can give a rough estimate that $\dim S_3^+(K(p))>0$ for big p. Then we calculate dimensions directly for finitely many remaining p.

3. Bias of dimensions of plus and minus eigenspaces

K. Martin considered in [26] dimensional difference between the space of the Atkin-Lehner plus and minus in the case of elliptic modular forms.. Here we can consider the same question for our case. In spite of the fact that our dimension formulas are much more complicated than in the case of elliptic modular forms, we can prove the same sort of bias of dimensions.

Proposition 3.1. *For any integer $k\geq 3$ and any prime p, we have*

$$(-1)^k\big(\dim S_k^+(K(p))-\dim S_k^-(K(p))\big)\geq 0,$$

The list of $k\geq 3$ and p such that $\dim S_k^+(K(p))=\dim S_k^-(K(p))$ is given as follows.

$$\begin{aligned}(p,k)=&(2,3),(2,4),(2,5),(2,6),(2,7),(2,9),(2,13)\\ &(3,3),(3,4),(3,5),(3,7),(5,3),(5,4),(7,3),(11,3).uadratic\end{aligned}$$

These are exactly the cases such that $S_k(K(p))=0$.

We note that asymptotically, the main term of both $\dim S_k^+(K(p))$ and $\dim S_k^-(K(p))$ are $k^3p^2/5760$ and the difference occurs from lower terms.

4. Examples of dimensions of low levels and palindromic property

In this section, we give examples of generating functions of dimensions of paramodular forms for small prime levels and $j = 0$. It often happens for small p that the generating function is *palindromic* (a some kind of symmetry). We will explain the meaning of this after giving concrete examples. For $p \leq 7$, the examples given below has been known before (see [18], [15], [4], [19], [40]). Below we omit the case that are not palindromic except for $A(K(11))$.

$$\sum_{k=0}^{\infty} \dim A_k(Sp(2,\mathbb{Z}))t^k = \frac{1+t^{35}}{(1-t^4)(1-t^6)(1-t^{10})(1-t^{12})}$$

$$\sum_{k=0}^{\infty} \dim S_k^+(K(2))t^k = \frac{t^8+t^{10}+t^{12}-t^{20}+t^{23}+t^{33}}{(1-t^4)(1-t^6)(1-t^8)(1-t^{12})}$$

$$\sum_{k=0}^{\infty} \dim A_k^+(K(2))t^k = \frac{1+t^{10}+t^{23}+t^{33}}{(1-t^4)(1-t^6)(1-t^8)(1-t^{12})}$$

$$\sum_{k=0} \dim S_k^-(K(2))t^k = \frac{t^{11}+t^{20}+t^{21}+t^{22}+t^{24}-t^{32}}{(1-t^4)(1-t^6)(1-t^8)(1-t^{12})}$$

$$\sum_{k=0} \dim A_k^-(K(2))t^k = \frac{t^{11}+t^{12}+t^{21}+t^{22}}{(1-t^4)(1-t^6)(1-t^8)(1-t^{12})}$$

$$\sum_{k=0}^{\infty} \dim S_k^+(K(3))t^k = \frac{t^6+t^8+t^{10}+t^{12}-t^{18}+t^{21}+t^{23}+t^{31}}{(1-t^4)(1-t^6)^2(1-t^{12})}$$

$$\sum_{k=0}^{\infty} \dim A_k^+(K(3))t^k = \frac{1+t^8+t^{10}+t^{21}+t^{23}+t^{31}}{(1-t^4)(1-t^6)^2(1-t^{12})}$$

$$\sum_{k=0}^{\infty} \dim S_k^-(K(3))t^k = \frac{t^9+t^{11}+t^{18}+t^{19}+t^{20}+t^{22}+t^{24}-t^{30}}{(1-t^4)(1-t^6)^2(1-t^{12})}$$

$$\sum_{k=0}^{\infty} \dim A_k^-(K(3))t^k = \frac{t^9+t^{11}+t^{12}+t^{19}+t^{20}+t^{22}}{(1-t^4)(1-t^6)^2(1-t^{12})}$$

$$\sum_{k=0}^{\infty} \dim S_k^+(K(5))t^k = \frac{Q_+^{(5)}(t)}{(1-t^4)(1-t^6)(1-t^{10})(1-t^{12})},$$

$$\sum_{k=0}^{\infty} \dim S_k^-(K(5))t^k = \frac{Q_-^{(5)}(t)}{(1-t^4)(1-t^6)(1-t^{10})(1-t^{12})},$$

where

$$\begin{aligned}
Q_+^{(5)}(t) =& t^6 + 2t^8 + 3t^{10} + 3t^{12} + 2t^{14} + 2t^{16} + t^{17} + t^{18} \\
&+ 2t^{19} + 2t^{21} - t^{22} + 2t^{23} + 2t^{25} + 2t^{27} + t^{29} + t^{35}, \\
Q_-^{(5)}(t) =& t^5 + t^7 + t^9 + 2t^{11} + 2t^{13} + t^{14} + 2t^{15} + t^{16} + t^{17} + t^{18} \\
&+ t^{19} + 2t^{20} + t^{21} + 3t^{22} + t^{23} + 3t^{24} + t^{26} + t^{28} + t^{30} - t^{34}.
\end{aligned}$$

$$\begin{aligned}
\sum_{k=0}^{\infty} \dim A_k^+(K(5)) t^k &= \frac{P_+^{(5)}(t)}{(1-t^4)(1-t^6)(1-t^{10})(1-t^{12})}, \\
\sum_{k=0}^{\infty} \dim A_k^-(K(5)) t^k &= \frac{P_-^{(5)}(t)}{(1-t^4)(1-t^6)(1-t^{10})(1-t^{12})}, \\
\sum_{k=0}^{\infty} \dim A_k(K(5)) t^k &= \frac{P^{(5)}(t)}{(1-t^4)(1-t^5)(1-t^6)(1-t^{12})},
\end{aligned}$$

where

$$\begin{aligned}
P_+^{(5)}(t) =& 1 + t^6 + 2t^8 + 2t^{10} + 2t^{12} + 2t^{14} + 2t^{16} + t^{17} + t^{18} + \\
& 2t^{19} + 2t^{21} + 2t^{23} + 2t^{25} + 2t^{27} + t^{29} + t^{35}, \\
P_-^{(5)}(t) =& t^5 + t^7 + t^9 + 2t^{11} + t^{12} + 2t^{13} + t^{14} + 2t^{15} + t^{16} + t^{17} + t^{18} \\
&+ t^{19} + 2t^{20} + t^{21} + 2t^{22} + t^{23} + 2t^{24} + t^{26} + t^{28} + t^{30}, \\
P^{(5)}(t) =& 1 + t^6 + t^7 + 2t^8 + t^9 + 2t^{10} + t^{11} + 2t^{12} + 2t^{14} + 2t^{16} + 2t^{18} \\
&+ t^{19} + 2t^{20} + t^{21} + 2t^{22} + t^{23} + t^{24} + t^{30}.
\end{aligned}$$

$$\begin{aligned}
\sum_{k=0}^{\infty} \dim S_k^+(K(7)) t^k &= \frac{Q_+^{(7)}(t)}{(1-t^4)^2(1-t^6)(1-t^{12})}, \\
\sum_{k=0}^{\infty} \dim S_k^-(K(7)) t^k &= \frac{Q_-^{(7)}(t)}{(1-t^4)^2(1-t^6)(1-t^{12})},
\end{aligned}$$

where

$$\begin{aligned}
Q_+^{(7)}(t) =& t^4 + 2t^6 + 2t^8 + 2t^{10} + 2t^{12} + t^{13} + t^{14} \\
&+ t^{15} + t^{17} + 2t^{19} + 2t^{21} + 2t^{23} + t^{29}, \\
Q_-^{(7)}(t) =& t^5 + 2t^7 + 2t^9 + 2t^{11} + t^{13} + t^{14} + t^{15} \\
&+ 2t^{16} + t^{17} + 2t^{18} + 2t^{20} + 2t^{22} + 2t^{24} - t^{28}.
\end{aligned}$$

$$\sum_{k=0}^{\infty} \dim A_k^+(K(7))t^k = \frac{P_+^{(7)}(t)}{(1-t^4)^2(1-t^6)(1-t^{12})},$$
$$\sum_{k=0}^{\infty} \dim A_k^-(K(7))t^k = \frac{P_-^{(7)}(t)}{(1-t^4)^2(1-t^6)(1-t^{12})},$$
$$\sum_{k=0}^{\infty} \dim A_k(K(7))t^k = \frac{P^{(7)}(t)}{(1-t^4)^2(1-t^6)(1-t^{12})}.$$

$$\begin{aligned} P_+^{(7)}(t) =& 1 + 2t^6 + 2t^8 + 2t^{10} + t^{12} + t^{13} + t^{14} + t^{15} + t^{16} \\ &+ t^{17} + 2t^{19} + 2t^{21} + 2t^{23} + t^{29}, \\ P_-^{(7)}(t) =& t^5 + 2t^7 + 2t^9 + 2t^{11} + t^{12} + t^{13} + t^{14} + t^{15} + t^{16} \\ &+ t^{17} + 2t^{18} + 2t^{20} + 2t^{22} + t^{24}, \\ P^{(7)}(t) =& 1 + t^5 + 2t^6 + 2t^7 + 2t^8 + 2t^9 + 2t^{10} + 2t^{11} + 2t^{12} + 2t^{13} \\ &+ 2t^{14} + 2t^{15} + 2t^{16} + 2t^{17} + 2t^{18} + 2t^{19} + 2t^{20} + 2t^{21} \\ &+ 2t^{22} + 2t^{23} + t^{24} + t^{29}. \end{aligned}$$

$$\sum_{k=0}^{\infty} \dim A_k^+(K(11))t^k = \frac{P_+^{(11)}(t)}{(1-t^4)(1-t^6)(1-t^{10})(1-t^{12})}$$
$$\sum_{k=0}^{\infty} \dim A_k(K(11))t^k = \frac{P^{(11)}(t)}{(1-t^4)(1-t^5)(1-t^6)(1-t^{12})}$$

$$\begin{aligned} P_+^{(11)}(t) =& 1 + t^4 + 3t^6 + 5t^8 + 7t^{10} + t^{11} + 9t^{12} + 3t^{13} \\ &+ 10t^{14} + 5t^{15} + 9t^{16} + 7t^{17} + 7t^{18} + 9t^{19} + 5t^{20} \\ &+ 10t^{21} + 3t^{22} + 9t^{23} + t^{24} + 7t^{25} + 5t^{27} + 3t^{29} + t^{31} + t^{35} \\ P^{(11)}(t) =& 1 + t^4 + t^5 + 3t^6 + 3t^7 + 5t^8 + 4t^9 + 6t^{10} + 6t^{11} + 8t^{12} + 7t^{13} \\ &+ 10t^{14} + 9t^{15} + 9t^{16} + 8t^{17} + 8t^{18} + 7t^{19} + 7t^{20} \\ &+ 6t^{21} + 6t^{22} + 4t^{23} + 3t^{24} + t^{25} - t^{27} - t^{29} + t^{30} \end{aligned}$$

$$\sum_{k=0}^{\infty} \dim A_k^+(K(13))t^k = \frac{P_+^{(13)}(t)}{(1-t^4)^2(1-t^6)(1-t^{12})}$$
$$\sum_{k=0}^{\infty} \dim A_k(K(13))t^k = \frac{P^{(13)}(t)}{(1-t^4)^2(1-t^6)(1-t^{12})}$$

$$
\begin{aligned}
P_+^{(13)}(t) =& 1 + t^4 + 5t^6 + 6t^8 + 6t^{10} + t^{11} + 5t^{12} + 4t^{13} + 5t^{14} \\
&+ 5t^{15} + 4t^{16} + 5t^{17} + t^{18} + 6t^{19} + 6t^{21} + 5t^{23} + t^{25} + t^{29} \\
P^{(13)}(t) =& 1 + t^3 + t^4 + 3t^5 + 5t^6 + 5t^7 + 6t^8 + 6t^9 + 7t^{10} + 7t^{11} + 8t^{12} \\
&+ 9t^{13} + 9t^{14} + 9t^{15} + 9t^{16} + 8t^{17} + 7t^{18} + 7t^{19} + 6t^{20} \\
&+ 6t^{21} + 5t^{22} + 5t^{23} + 3t^{24} + t^{25} + t^{26} + t^{29}
\end{aligned}
$$

$$
\sum_{k=0}^{\infty} \dim A_k^+(K(17))t^k = \frac{P_+^{(17)}(t)}{(1-t^4)^2(1-t^6)(1-t^{12})}
$$

$$
\begin{aligned}
P_+^{(17)}(t) =& 1 + t^4 + 6t^6 + 9t^8 + t^9 + 10t^{10} + 4t^{11} + 9t^{12} + 8t^{13} + 9t^{14} + 9t^{15} \\
&+ 8t^{16} + 9t^{17} + 4t^{18} + 10t^{19} + t^{20} + 9t^{21} + 6t^{23} + t^{25} + t^{29}
\end{aligned}
$$

$$
\sum_{k=0}^{\infty} \dim A_k^+(K(19))t^k = \frac{P_+^{(19)}(t)}{(1-t^4)(1-t^6)(1-t^{10})(1-t^{12})}
$$

$$
\begin{aligned}
P_+^{(19)}(t) =& 1 + 3t^4 + 8t^6 + 14t^8 + t^9 + 20t^{10} + 4t^{11} + 26t^{12} + 10t^{13} + 29t^{14} \\
&+ 16t^{15} + 27t^{16} + 22t^{17} + 22t^{18} + 27t^{19} + 16t^{20} + 29t^{21} + 10t^{22} \\
&+ 26t^{23} + 4t^{24} + 20t^{25} + t^{26} + 14t^{27} + 8t^{29} + 3t^{31} + t^{35},
\end{aligned}
$$

$$
\sum_{k=0}^{\infty} \dim A_k^+(K(23))t^k = \frac{P_+^{(23)}(t)}{(1-t^4)^2(1-t^6)(1-t^{12})}
$$

$$
\begin{aligned}
P_+^{(23)}(t) =& 1 + 2t^4 + 9t^6 + t^7 + 14t^8 + 4t^9 + 17t^{10} + 9t^{11} + 17t^{12} \\
&+ 15t^{13} + 17t^{14} + 17t^{15} + 15t^{16} + 17t^{17} + 9t^{18} + 17t^{19} \\
&+ 4t^{20} + 14t^{21} + t^{22} + 9t^{23} + 2t^{25} + t^{29} - 2t^{28} + t^{29}
\end{aligned}
$$

$$
\sum_{k=0}^{\infty} \dim A_k^+(K(29))t^k = \frac{P_+^{(29)}(t)}{(1-t^4)(1-t^6)(1-t^{10})(1-t^{12})}
$$

$$
\begin{aligned}
P_+^{(29)}(t) =& 1 + 4t^4 + 14t^6 + t^7 + 27t^8 + 5t^9 + 41t^{10} + 15t^{11} \\
&+ 55t^{12} + 29t^{13} + 65t^{14} + 43t^{15} + 65t^{16} + 56t^{17} \\
&+ 56t^{18} + 65t^{19} + 43t^{20} + 65t^{21} + 29t^{22} + 55t^{23} \\
&+ 15t^{24} + 41t^{25} + 5t^{26} + 27t^{27} + t^{28} + 14t^{29} + 4t^{31} + t^{35}.
\end{aligned}
$$

$$
\sum_{k=0}^{\infty} \dim A_k^+(K(31))t^k = \frac{P_+^{(31)}(t)}{(1-t^4)(1-t^6)(1-t^{10})(1-t^{12})}
$$

$$
\begin{aligned}
P_+^{(31)}(t) =&1 + 6t^4 + 17t^6 + t^7 + 32t^8 + 6t^9 + 48t^{10} + 16t^{11} \\
&+ 64t^{12} + 32t^{13} + 74t^{14} + 48t^{15} + 73t^{16} + 63t^{17} \\
&+ 63t^{18} + 73t^{19} + 48t^{20} + 74t^{21} + 32t^{22} + 64t^{23} \\
&+ 16t^{24} + 48t^{25} + 6t^{26} + 32t^{27} + t^{28} + 17t^{29} + 6t^{31} + t^{35}.
\end{aligned}
$$

$$
\sum_{k=0}^{\infty} \dim A_k^+(K(41))t^k = \frac{P_+^{(41)}(t)}{(1-t^4)(1-t^6)(1-t^{10})(1-t^{12})}
$$

$$
\begin{aligned}
P_+^{(41)}(t) = &1 + 7t^4 + 24t^6 + 3t^7 + 49t^8 + 14t^9 + 77t^{10} + 35t^{11} + 105t^{12} \\
&+ 63t^{13} + 126t^{14} + 91t^{15} + 130t^{16} + 116t^{17} + 116t^{18} + 130t^{19} \\
&+ 91t^{20} + 126t^{21} + 63t^{22} + 105t^{23} + 35t^{24} + 77t^{25} + 14t^{26} \\
&+ 49t^{27} + 3t^{28} + 24t^{29} + 7t^{31} + t^{35}.
\end{aligned}
$$

$$
\sum_{k=0}^{\infty} \dim A_k^+(K(47))t^k = \frac{P_+^{(47)}(t)}{(1-t^4)^2(1-t^6)(1-t^{12})}
$$

$$
\begin{aligned}
P_+^{(47)}(t) = &1 + 7t^4 + t^5 + 27t^6 + 8t^7 + 49t^8 + 25t^9 + 66t^{10} + 47t^{11} \\
&+ 72t^{12} + 66t^{13} + 73t^{14} + 73t^{15} + 66t^{16} + 72t^{17} + 47t^{18} \\
&+ 66t^{19} + 25t^{20} + 49t^{21} + 8t^{22} + 27t^{23} + t^{24} + 7t^{25} + t^{29}
\end{aligned}
$$

$$
\sum_{k=0}^{\infty} \dim A_k^+(K(59))t^k = \frac{P_+^{(59)}(t)}{(1-t^4)(1-t^5)(1-t^6)(1-t^{12})}
$$

$$
\begin{aligned}
P_+^{(59)}(t) =&1 + 11t^4 + 40t^6 + 12t^7 + 87t^8 + 30t^9 + 144t^{10} + 48t^{11} + 190t^{12} \\
&+ 59t^{13} + 219t^{14} + 59t^{15} + 219t^{16} + 59t^{17} + 190t^{18} \\
&+ 48t^{19} + 144t^{20} + 30t^{21} + 87t^{22} + 12t^{23} + 40t^{24} + 11t^{26} + t^{30}
\end{aligned}
$$

$$
\sum_{k=0}^{\infty} \dim A_k^+(K(71))t^k = \frac{P_+^{(71)}(t)}{(1-t^4)(1-t^5)(1-t^6)(1-t^{12})}
$$

$$
\begin{aligned}
P_+^{(71)}(t) =&1 + 15t^4 + t^5 + 56t^6 + 19t^7 + 123t^8 + 47t^9 + 204t^{10} \\
&+ 75t^{11} + 270t^{12} + 92t^{13} + 311t^{14} + 93t^{15} + 311t^{16} \\
&+ 92t^{17} + 270t^{18} + 75t^{19} + 204t^{20} + 47t^{21} + 123t^{22} + 19t^{23} \\
&+ 56t^{24} + t^{25} + 15t^{26} + t^{30}
\end{aligned}
$$

Now we give some theoretical explanation. We consider a graded $\mathbb{C}$ algebra $A = \oplus_{k=0}^{\infty} A_k$ with $A_0 = \mathbb{C}$ that is finitely generated over

$\mathbb{C}$. Assume that A is of Krull dimension m (maximal number of elements of A which are algebraically independent over $\mathbb{C}$). We define the generating function of dimensions of homogeneous part of degree k by

$$F(A,t) = \oplus_{k=0}^{\infty} \dim A_k t^k.$$

This is a rational function of t ([37] p. 479). We say that $F(A,t)$ is palindromic if $F(A,1/t) = (-1)^m t^\ell F(A,t)$ for some integer ℓ. A graded ring A is said to be Cohen-Macaulay if there exists algebraically independent homogeneous elements $\theta_1, \ldots, \theta_m$ such that A is free over $\mathbb{C}[\theta_1, \ldots, \theta_m]$ ([37] Proposition 3.1). Now there exists a polynomial ring $\mathfrak{A}$ such that $A = \mathfrak{A}/I$ for some homogeneous ideal I ([37] p. 495). Take a finite free resolution of A as $\mathfrak{A}$ module, that is, an exact sequence of $\mathfrak{A}$ free modules

$$0 \to M_h \to \cdots \to M_1 \to M_0 \to A \to 0.$$

Take the dual sequence, that is,

$$0 \to M_0^* \to \cdots \to M_h^*$$

where $M_h^* = Hom_{\mathfrak{A}}(M, \mathfrak{A})$ and define $\Omega_{\mathfrak{A}}(A)$ as $M_h^*/Im(M_{h-1}^*)$. Here $\Omega_{\mathfrak{A}}(A)$ has an A module structure and this depends only on A and not on $\mathfrak{A}$, When A is Cohen-Macaulay, the graded ring A is Gorenstein if and only if $\Omega(A) \cong A$ ([37] p.502). The following theorem was informed to me by Ralf Schmidt, who is interested in several palindromic generating functions of dimensions.

Theorem 4.1 ([36] Theorem 4.4)**.** *Assume that A is a Cohen-Macaulay graded ring. Then $F(A,t)$ is palindromic if and only if A is Gorenstein.*

We calculated $F(A(K(p)),t)$ and $F(A^+(K(p)))$ for primes $p < 100$ and could observe the following results.

Proposition 4.2. *For primes $p < 100$. we have the following results.*
(i) $F(A(K(p)),t)$ is palindromic if and only if $p = 2$, 3, 5, 7, 13.
(ii) $F(A^+(K(p)),t)$ is palindromic if and only if $p = 2$, 3, 5, 11, 13, 17, 19, 23, 29, 31, 41, 47, 59, 71.
(iii) The primes in (1) are exactly those such that $S_2(\Gamma_0(p)) = 0$. The primes in (ii) are exactly those such that $J_{2,p} = 0$, where $J_{2,p}$ is the space of Jacobi forms of $SL_2(\mathbb{Z})$ of weight 2 of index p.

I do not understand any intrinsic meaning of (iii). By seeing $A(K(11),t)$, maybe $A(K(11))$ is not Cohen-Macaulay. By [15] and [4], $A(2)$, $A^+(2)$, $A(3)$, $A^+(3)$ are Cohen-Macaulay and Gorenstein. Brandon Williams informed me that $A(K(p))$ and $A^+(K(p))$ are Cohen-Macaulay for $p = 5$ and 7 by using his result in [40]. So these are also Gorenstein. We may ask if the list in the above Proposition would be Gorenstein and if those $p < 100$ not listed in the above would not be Gorenstein.

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