Diophantine problems of the equations yielding generalized Pythagorean triplets

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1 Introduction

For integer $k \ge 2$, consider a set of positive integers $A = \{a_1, \dots, a_k\}$. The number of non-negative integral representations $x_1, x_2, \dots, x_k$, denoted by $d(n; A) = d(n; a_1, a_2, \dots, a_k)$, to $a_1 x_1 + a_2 x_2 + \cdots + a_k x_k = n$ for a given positive integer n is often called the *denumerant* and is equal to the coefficient of x^n in

$$\frac{1}{(1 - x^{a_1})(1 - x^{a_2}) \cdots (1 - x^{a_k})}$$

([22]).

For a non-negative integer p, define S_p and G_p by

$$S_p(A) = \{n \in \mathbb{N}_0 | d(n; A) > p\} \quad \text{and} \quad G_p(A) = \{n \in \mathbb{N}_0 | d(n; A) \le p\}$$

respectively, satisfying $S_p \cup G_p = \mathbb{N}_0$, which is the set of non-negative integers.

We assume that $\gcd(A) = \gcd(a_1, \dots, a_k) = 1$. Then the set $S_p = S_p(A)$ is called *p-numerical semigroup* [17]. $G_p = G_p(A)$ is the set of *p-gaps*. Notice that the set G_p is finite if and only if $\gcd(A) = 1$. Define $g_p(A)$, $n_p(A)$ and $s_p(A)$ by

$$g_p(A) = \max_{n \in G_p(A)} n, \quad n_p(A) = \sum_{n \in G_p(A)} 1, \quad s_p(A) = \sum_{n \in G_p(A)} n\,,$$

respectively, and are called the *p-Frobenius number*, the *p-Sylvester number* (or *p-genus*) and the *p-Sylvester sum*, respectively. When $p = 0$, $g(A) = g_0(A)$, $n(A) = n_0(A)$ and $s(A) = s_0(A)$ are the original Frobenius number, Sylvester number (or genus) and Sylvester sum, respectively.

When $k = 2$, for any non-negative integer p, we have

$$\begin{aligned} g_p(a, b) &= (p + 1)ab - a - b\,, \\ n_p(a, b) &= \frac{1}{2}\bigl((2p + 1)ab - a - b + 1\bigr)\,, \\ s_p(a, b) &= \frac{1}{12}\bigl(2(3p^2 + 3p + 1)a^2 b^2 - 3(2p + 1)ab(a + b) \\ &\qquad + a^2 + b^2 + 3ab - 1\bigr)\,, \end{aligned}$$

each of which is a generalization of the results in [23], [22] and [3], respectively, where $p = 0$. However, when $k \geq 3$, these values cannot be given by any set of closed formulas, which can be reduced to a finite set of certain polynomials ([5]) even if $p = 0$. Only some special cases, explicit closed formulas have been found, including arithmetic, geometric, Fibonacci, Mersenne, repunits and triangular (see, e.g., [20] and references therein). When $p \geq 1$ and $k \geq 3$, the situation becomes even harder, but recently explicit closed formulas have been obtained for the triangular number triplet ([8]), for repunits ([9, 12]), Fibonacci triplet ([15]), Jacobsthal triplets ([13, 11]) and arithmetic triplets ([16]).

In this paper, we consider the triple of the parametrizations yielded from the Diophantine equation $x^2 - y^2 = z^r$ $(r \geq 2)$. For the solution of the Diophantine equation $x^2 - y^2 = z^r$ $(r \geq 2)$, we obtain two kinds of parameterizations [4]. If $s \not\equiv t \pmod 2$, then

$$(x, y, z) = \left(\frac{(s+t)^r + (s-t)^r}{2}, \frac{(s+t)^r - (s-t)^r}{2}, s^2 - t^2\right), \tag{1}$$

where $\gcd(s, t) = 1$. If $2 \nmid t$, then

$$(x, y, z) = \left(2^{r-2}s^r + t^r, 2^{r-2}s^r - t^r, 2st\right), \tag{2}$$

where $\gcd(s, t) = 1$. When $r = 2$, both cases reduce to the Pythagorean triple

$$(x, y, z) = (s^2 + t^2, 2st, s^2 - t^2).$$

One of the main aims of this paper is to give the p-Frobenius numbers of these triples.

Theorem 1. *If $s \not\equiv t \pmod 2$, for a non-negative integer p with $p \leq \lfloor t/(s-t) \rfloor$*

$$\begin{aligned} & g_p\left(\frac{(s+t)^r + (s-t)^r}{2}, \frac{(s+t)^r - (s-t)^r}{2}, s^2 - t^2\right) \\ &= \frac{\big((p+2)s - (p+1)t - 2\big)(s+t)^r + \big(ps - (p-1)t\big)(s-t)^r}{2} \\ &\quad - (s^2 - t^2). \end{aligned}$$

If $2 \nmid t$, then for a non-negative integer p with $p \leq \lfloor (s-t)/t \rfloor$

$$\begin{aligned} & g_p(2^{r-2}s^r + t^r, 2^{r-2}s^r - t^r, 2st) \\ &= 2^{r-2}\big(s + (p+2)t - 2\big)s^r + (pt - s)t^r - 2st. \end{aligned}$$

For the number of pages, the formula for $n_p(A)$ or $s_p(A)$ is not included in this article.

2 Preliminaries

For a non-negative integer p, we introduce the *p-Apéry set* [1, 10], given by

$$\mathrm{Ap}_p(A) = \mathrm{Ap}_p(a_1, a_2, \dots, a_k) = \{m_0^{(p)}, m_1^{(p)}, \dots, m_{a_1-1}^{(p)}\}.$$

Without loss of generality, we set $a_1 := \min\{A\}$.

Here, for each $0 \le i \le a_1 - 1$, each element $m_i^{(p)}$ satisfies the conditions

$$\text{(i) } m_i^{(p)} \equiv i \pmod{a_1}, \quad \text{(ii) } m_i^{(p)} \in S_p(A), \quad \text{(iii) } m_i^{(p)} - a_1 \notin S_p(A)\,.$$

Note that $m_0^{(0)} = 0$. This set is congruent to the set

$$\{0, 1, \dots, a_1 - 1\} \pmod{a_1}\,.$$

Once knowing the structure of the elements of the p-Apéry set, we can obtain the p-Frobenius number, the p-genus and the p-Sylvester sum ([7, 10]). More general formula including weighted power Sylvester sums can be found in [7, 18, 19].

Lemma 1. *Assume that* $\gcd(A) = \gcd(a_1, \dots, a_k) = 1$. *We have*

$$g_p(A) = \max_{0 \le i \le a_1 - 1} m_i^{(p)} - a_1\,,$$

$$n_p(A) = \frac{1}{a_1} \sum_{i=0}^{a_1-1} m_i^{(p)} - \frac{a_1 - 1}{2}\,,$$

$$s_p(A) = \frac{1}{2a_1} \sum_{i=0}^{a_1-1} \bigl(m_i^{(p)}\bigr)^2 - \frac{1}{2} \sum_{i=0}^{a_1-1} m_i^{(p)} + \frac{a_1^2 - 1}{12}\,.$$

Each formula is a generalization of the classical formula by Brauer and Shockley [2], Selmer [21] and Tripathi [24], where $p = 0$. Note that $m_0^{(0)} = 0$.

3 Frobenius numbers of the triple formed from the Diophantine equations

For $p = 0$ and $r \ge 2$, the Frobenius numbers of the triples of (1) and (2) are given as follows. This is a special case of Theorem 1, but the general p case cannot be obtained directly, but is based upon the case where $p = 0$.

Theorem 2. *If* $s \not\equiv t \pmod 2$*, then*

$$g_0\left(\frac{(s+t)^r + (s-t)^r}{2}, \frac{(s+t)^r - (s-t)^r}{2}, s^2 - t^2\right) = \frac{(2s - t - 2)(s+t)^r + t(s-t)^r}{2} - (s^2 - t^2)\,.$$

If $2 \nmid t$*, then*

$$g_0(2^{r-2}s^r + t^r, 2^{r-2}s^r - t^r, 2st) = 2^{r-2}(s + 2t - 2)s^r - s \cdot t^r - 2st\,.$$

Remark. When $r = 2$, two kinds of parameterizations depend upon which of $s^2 - t^2$ and $2st$ is smaller. Both formulas in Theorem 2 reduce to that by Gil et al. [6]:

$$g_0(s^2 + t^2, 2st, s^2 - t^2) = (s-1)(s^2 - t^2) + (s-1)(2st) - (s^2 + t^2)\,.$$

3.1 When $s \not\equiv t \pmod 2$

For convenience, we put

$$\mathbf{x} := \frac{(s+t)^r + (s-t)^r}{2}, \quad \mathbf{y} := \frac{(s+t)^r - (s-t)^r}{2}, \quad \mathbf{z} := s^2 - t^2 \,. \tag{3}$$

Since $\mathbf{x}, \mathbf{y}, \mathbf{z} > 0$ and $\gcd(\mathbf{x}, \mathbf{y}, \mathbf{z}) = 1$, we see that $s > t$ and $\gcd(s, t) = 1$. Note that $\mathbf{x} > \mathbf{y} > \mathbf{z}$ when $r \ge 3$. When $r = 2$, we assume that $\mathbf{y} > \mathbf{z}$.

The elements of the (0-)Apéry set are given as in Table 1, where each point (Y, X) corresponds to the expression $Y\mathbf{y} + X\mathbf{x}$ and the area of the (0-)Apéry set is equal to $s^2 - t^2$.

$(0,0)$	$\cdots$	$(s-t-1,0)$	$(s-t,0)$	$\cdots$ $\cdots$	$(s-1,0)$
$\vdots$		$\vdots$	$\vdots$		$\vdots$
$(0,s-t-1)$	$\cdots$	$(s-t-1,s-t-1)$	$(s-t,s-t-1)$	$\cdots$ $\cdots$	$(s-1,s-t-1)$
$(0,s-t)$	$\cdots$	$(s-t-1,s-t)$			
$\vdots$		$\vdots$			
$\vdots$		$\vdots$			
$(0,s-1)$	$\cdots$	$(s-t-1,s-1)$			

Table 1: $\mathrm{Ap}_0(\mathbf{x}, \mathbf{y}, \mathbf{z})$ when $s \not\equiv t \pmod 2$

Since

$$s\mathbf{y} - t\mathbf{x} = \mathbf{z}t \sum_{j=1}^{\lfloor r/2 \rfloor} \binom{r-1}{2j-1} s^{r-2j} t^{2j-2} \,, \tag{4}$$

we have $s\mathbf{y} \equiv t\mathbf{x} \pmod{\mathbf{z}}$ and $s\mathbf{y} > t\mathbf{x}$. Therefore, the sequence $\{\ell\mathbf{y} \pmod{\mathbf{z}}\}_{\ell=0}^{\mathbf{z}-1}$ can be arranged as follows.

[Step 1] After the row of the longer term

$$(0, X), (1, X), \dots, (s-1, X) \quad (0 \le X \le s-t-1)$$

with length s, by increasing by t in the vertical direction, we move to the row

$$(0, X+t), (1, X+t), \dots$$

because $s\mathbf{y} \equiv t\mathbf{x} \pmod{\mathbf{z}}$. If it is still in the longer term, we repeat this [Step 1].

[Step 2] If it reaches the shorter term

$$(0, X'), (1, X'), \dots, (s-t-1, X') \quad (s-t \le X' \le s-1)$$

with length $s-t$, by decreasing by $(s-t)$ in the vertical direction, we move to the row

$$(0, X'-s+t), (1, X'-s+t), \dots$$

because

$$(s-t)\mathbf{y} + (s-t)\mathbf{x} = (s-t)(s+t)^r \equiv 0 \pmod{\mathbf{z}} \,. \tag{5}$$

If it is still in the shorter term, we repeat this [Step 2]. Otherwise, we apply [Step 1]. In fact, after the point $(s-t-1, s-t)$, one moves back to $(0, 0)$.

Since $\gcd(s,t)=1$, all the points inside the area in Table 1 appear in the sequence $\{\ell\mathbf{y} \pmod{\mathbf{z}}\}_{\ell=0}^{\mathbf{z}-1}$ just once. Indeed, this sequence is equivalent to the sequence $\{\ell \pmod{\mathbf{z}}\}_{\ell=0}^{\mathbf{z}-1}$.

It is clear that one of the values at $(s-t-1,s-1)$ or at $(s-1,s-t-1)$ takes the largest element. Since $(s-t-1)\mathbf{y}+(s-1)\mathbf{x}-\big((s-1)\mathbf{y}+(s-t-1)\mathbf{x}=t(s-t)^r>0$, the element at $(s-t-1,s-1)$ is the largest in the Apéry set. Hence, by the first formula of Lemma 1, we have

$$\begin{aligned}
g\big(\mathbf{x},\mathbf{y},\mathbf{z}\big)&=(s-t-1)\mathbf{y}+(s-1)\mathbf{x}-\mathbf{z}\\
&=\frac{(s-t-1)\big((s+t)^r-(s-t)^r\big)}{2}+\frac{(s-1)\big((s+t)^r+(s-t)^r\big)}{2}-(s^2-t^2)\\
&=\frac{(2s-t-2)(s+t)^r+t(s-t)^r}{2}-(s^2-t^2)\,.
\end{aligned}$$

3.2 When $2\nmid t$

For convenience, we put

$$\mathbf{x}':=2^{r-2}s^r+t^r,\quad \mathbf{y}':=2^{r-2}s^r-t^r,\quad \mathbf{z}':=2st\,. \tag{6}$$

Since $\mathbf{x}',\mathbf{y}',\mathbf{z}'>0$ and $\gcd(\mathbf{x}',\mathbf{y}',\mathbf{z}')=1$, we see that $s>\sqrt[r]{4}\,t/2$ and $\gcd(s,t)=1$. Note that $\mathbf{x}'>\mathbf{y}'>\mathbf{z}'$ when $r\ge 3$. When $r=2$, we assume that $\mathbf{y}'=\mathbf{z}>\mathbf{z}'=\mathbf{y}$.

Since $(s+t)\mathbf{x}'-(s-t)\mathbf{y}'=(2^{r-2}s^{r-1}-t^{r-1})\mathbf{z}'>0$, we have $(s+t)\mathbf{x}'\equiv(s-t)\mathbf{y}' \pmod{\mathbf{z}'}$ and $(s+t)\mathbf{x}'>(s-t)\mathbf{y}'$. By a similar way, we know that all the elements of the (0-)Apéry set are given as in Table 2.

$(0,0)$	$\cdots$	$(t-1,0)$	$(t,0)$	$\cdots$	$\cdots$	$(s+t-1,0)$
$\vdots$		$\vdots$	$\vdots$			$\vdots$
$(0,t-1)$	$\cdots$	$(t-1,t-1)$	$(t,t-1)$	$\cdots$	$\cdots$	$(s+t-1,t-1)$
$(0,t)$	$\cdots$	$(t-1,t)$				
$\vdots$		$\vdots$				
$\vdots$		$\vdots$				
$(0,s-1)$	$\cdots$	$(t-1,s-1)$				

Table 2: $\mathrm{Ap}_0(\mathbf{x}',\mathbf{y}',\mathbf{z}')$ when $2\nmid t$

Compare the elements at $(t-1,s-1)$ and $(s+t-1,t-1)$, which take possible maximal values. Since

$$(s+t-1)\mathbf{y}'+(t-1)\mathbf{x}'-\big((t-1)\mathbf{y}'+(s-1)\mathbf{x}'\big)=2st(2^{r-3}s^{r-1}-t^{r-1})+t^{r+1}>0\,,$$

we find that the element at $(s+t-1,t-1)$ is the largest in the Apéry set. By the first formula of Lemma 1, we have

$$\begin{aligned}
g\big(\mathbf{x}',\mathbf{y}',\mathbf{z}'\big)&=(s+t-1)\mathbf{y}'+(t-1)\mathbf{x}'-\mathbf{z}'\\
&=(s+t-1)(2^{r-2}s^r-t^r)+(t-1)(2^{r-2}s^r+t^r)-2st\\
&=2^{r-2}(s+2t-2)s^r-s\cdot t^r-2st\,.
\end{aligned}$$

4 $p>0$

When $p>0$, we need the discussion from $\mathrm{Ap}_{p-1}(A)$ to $\mathrm{Ap}_p(A)$.

4.1 When $s \not\equiv t \pmod 2$

• $p = 1$

All elements of $\mathrm{Ap}_1(A)$ are arranged in the form of shifting elements of $\mathrm{Ap}_0(A)$ whose remainders modulo $\mathbf{z}$ are equal.

Assume that $s < 2t$. See Table 3.

Since $(s-t)\mathbf{y} + (s-t)\mathbf{x} \equiv 0 \pmod{\mathbf{z}}$, each value at (Y, X) is equivalent to the value at $(Y+s-t, X+s-t)$. In addition, by $s\mathbf{y} \equiv t\mathbf{x} \pmod{\mathbf{z}}$, the elements of the first t rows in $\mathrm{Ap}_0(A)$ are shifted by $(Y, X) \to (Y+s-t, X+s-t)$ (to the lower right direction) as the elements of $\mathrm{Ap}_1(A)$. However, as the column width of the element in the first $(s-t)$ rows is s, if it is transferred as it is, there will be a part that protrudes sideways, and such a part is located below the lower left area of $\mathrm{Ap}_0(A)$ (this position is reasonable because $s\mathbf{y} \equiv t\mathbf{x} \pmod{\mathbf{z}}$).

Finally, all elements other than the elements in the first t rows move directly to the side of the area of $\mathrm{Ap}_0(A)$ in the upper right (this position is also reasonable because $s\mathbf{y} \equiv t\mathbf{x} \pmod{\mathbf{z}}$).

From this arrangement, $\mathrm{Ap}_1(A)$ also forms a complete residue system modulo $\mathbf{z}$.

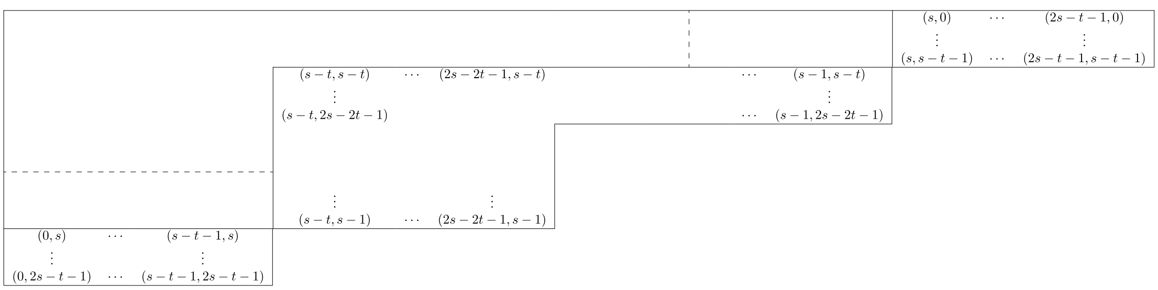


Table 3: $\mathrm{Ap}_1(\mathbf{x}, \mathbf{y}, \mathbf{z})$ when $s \not\equiv t \pmod 2$

Now, we shall show that each element has at least two different representations. For the $(s-t) \times (s-t)$ area at the bottom left of Table 3, by

$$t\mathbf{y} - s\mathbf{x} = \mathbf{z} \sum_{j=0}^{\lfloor (r-1)/2 \rfloor} \binom{r-1}{2j} s^{r-2j-1} t^{2j},$$

we have for $0 \le Y \le s-t-1$ and $0 \le X \le s-t-1$

$$0\mathbf{z} + Y\mathbf{y} + (X+s)\mathbf{x} = \left(\sum_{j=0}^{\lfloor (r-1)/2 \rfloor} \binom{r-1}{2j} s^{r-2j-1} t^{2j} \right) \mathbf{z} + (Y+t)\mathbf{y} + X\mathbf{x}.$$

For the $(s-t) \times (s-t)$ area at the top right of Table 3, by (4), we have for $0 \le Y \le s-t-1$ and $0 \le X \le s-t-1$

$$0\mathbf{z} + (Y+s)\mathbf{y} + X\mathbf{x} = \left(t \sum_{j=1}^{\lfloor r/2 \rfloor} \binom{r-1}{2j-1} s^{r-2j} t^{2j-2} \right) \mathbf{z} + Y\mathbf{y} + (X+t)\mathbf{x}.$$

For the middle area of $\mathrm{Ap}_1(A)$, by (5), we have for $0 \le Y \le s-1$ and $0 \le X \le s-1$

$$0\mathbf{z} + (Y+s-t)\mathbf{y} + (X+s-t)\mathbf{x} = (s+t)^{r-1}\mathbf{z} + Y\mathbf{y} + X\mathbf{x}\,.$$

There are four candidates at

$$\begin{array}{ll}(s-t-1, 2s-t-1), & (s-1, 2s-2t-1),\\ (2s-2t-1, s-1), & (2s-t-1, s-t-1)\end{array}$$

to take the largest value in $\mathrm{Ap}_1(A)$. Since $t\mathbf{x} > t\mathbf{y}$, the first one and the third one are larger than the second one and the fourth one, respectively. Since $(s-t)\mathbf{x} > (s-t)\mathbf{y}$, the first one is bigger than the third one. Hence, by the first formula of Lemma 1

$$\begin{aligned} g_1(\mathbf{x},\mathbf{y},\mathbf{z}) &= (s-t-1)\mathbf{y} + (2s-t-1)\mathbf{x} - \mathbf{z}\\ &= \frac{(3s-2t-2)(s+t)^r + s(s-t)^r}{2} - (s^2-t^2)\,. \end{aligned}$$

4.2 $p \ge 2$

When $p \ge 2$, it continues until $p \le \lfloor t/(s-t) \rfloor$, the area of $\mathrm{Ap}_1(A)$ moves to the area of $\mathrm{Ap}_2(A)$, which moves to the area of $\mathrm{Ap}_3(A)$, and so on, in the correspondence relation modulo $(\mathbf{z})$. Table 4 shows the areas of the $\mathrm{Ap}_p(A)$ $(p = 0, 1, 2, 3)$ for the case where $3 \le \lfloor t/(s-t) \rfloor < 4$. In Table 4, the area of $\mathrm{Ap}_0(A)$ is marked as 0 (including 0_a and 0_b); that of $\mathrm{Ap}_1(A)$ is marked as 1 (including 1_c and 1_d) with 1_a and 1_b; that of $\mathrm{Ap}_2(A)$ is marked as 2 (including 2_e and 2_f) with 2_a, 2_b, 2_c and 2_d; that of $\mathrm{Ap}_3(A)$ is marked as 3 with 3_a, 3_b, 3_c, 3_d, 3_e and 3_f. The areas having the same residue modulo $(\mathbf{z})$ are determined as

$$\begin{aligned} 0_a \Rightarrow 1_a \Rightarrow &2_a \Rightarrow 3_a\,,\\ 0_b \Rightarrow 1_b \Rightarrow &2_b \Rightarrow 3_b\,,\\ 1_c \Rightarrow &2_c \Rightarrow 3_c\,,\\ 1_d \Rightarrow &2_d \Rightarrow 3_d\,,\\ &2_e \Rightarrow 3_e\,,\\ &2_f \Rightarrow 3_f\,, \end{aligned}$$

and the main parts are as

$$\begin{aligned} &0 \text{ (excluding } 0_a \text{ and } 0_b) \Rightarrow 1 \text{ (including } 1_a \text{ and } 1_b)\,,\\ &1 \text{ (excluding } 1_c \text{ and } 1_d) \Rightarrow 2 \text{ (including } 2_e \text{ and } 2_f)\,,\\ &2 \text{ (excluding } 2_e \text{ and } 2_f) \Rightarrow 3\,. \end{aligned}$$

That is, the elements of the area of the lower left stair portions in $\mathrm{Ap}_p(A)$ correspond to the elements of the area of the upper right stair portion in $\mathrm{Ap}_{p+1}(A)$, and are aligned from the upper right row to the lower left. The elements of the area of the upper right stair portion in $\mathrm{Ap}_p(A)$ correspond to the elements of the area of the lower left stair portion in $\mathrm{Ap}_{p+1}(A)$, respectively, and line up in the upper-right direction from the lowest left column. The elements of the area of $\mathrm{Ap}_p(A)$ in the center portion, except for the $(s-t)\times(s-t)$ area in the lower left and the $(s-t)\times(s-t)$ area in the upper right, correspond to the elements of the area of $\mathrm{Ap}_{p+1}(A)$ in the lower right diagonal direction.

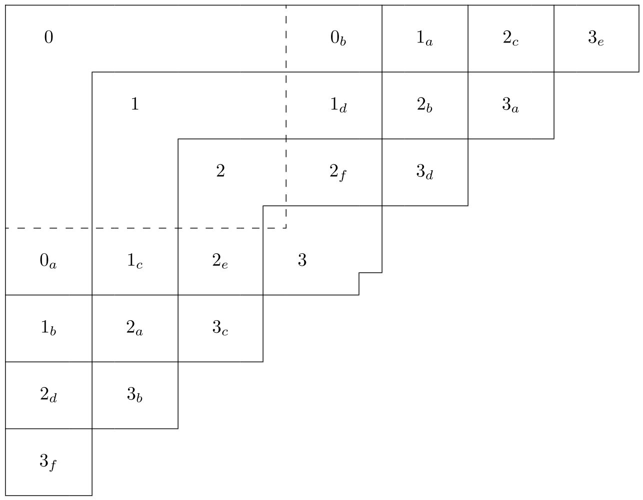


Table 4: $\mathrm{Ap}_p(\mathbf{x},\mathbf{y},\mathbf{z})$ $(p=0,1,2,3)$ when $s\not\equiv t \pmod 2$

More generally and more precisely, for $1\le l\le p$, each element of the l-th $(s-t)\times(s-t)$ block from the left in the area of the lower left stair portions in $\mathrm{Ap}_p(A)$ is expressed by

$$\big((l-1)s-(l-1)t+i,(p-l+1)s-(p-l)t+j\big)\qquad (0\le i\le s-t-1,\ 0\le j\le s-t-1)\,, \tag{7}$$

and for $1\le l'\le p$, each element of the l'-th $(s-t)\times(s-t)$ block from the right in the area of the upper right stair portions in $\mathrm{Ap}_{p'}(A)$ is expressed by

$$\big((p'-l'+1)s-(p'-l')t+i,(l'-1)s-(l'-1)t+j\big)\qquad (0\le i\le s-t-1,\ 0\le j\le s-t-1)\,. \tag{8}$$

Then by $s\mathbf{y}\equiv t\mathbf{x} \pmod{\mathbf{z}}$, we have the congruent relation for $p'=p+1$ and $l'=p'-l+1=p-l+2$

$$\begin{aligned}&\big((l-1)s-(l-1)t+i\big)\mathbf{y}+\big((p-l+1)s-(p-l)t+j\big)\mathbf{x}\\ &\quad\equiv\big((p'-l'+1)s-(p'-l')t+i\big)\mathbf{y}+\big((l'-1)s-(l'-1)t+j\big)\mathbf{x}\pmod{\mathbf{z}}\,,\end{aligned}$$

as well as for $p=p'+1$ and $l=p-l'+1=p'-l'+2$.

For simplicity, denote by (Z,Y,X) the value of $Z\mathbf{z}+Y\mathbf{y}+X\mathbf{x}$. Each element of the leftmost $(s-t)\times(s-t)$ area of $\mathrm{Ap}_p(A)$ $(p\ge 1)$ has exactly $(p+1)$ representations, because

$$\big(0,0,ps-(p-1)t\big)=\big(js+(j-1)t,jt-(j-1)s,(p-j)s-(p-j)t\big)\qquad (j=1,2,\dots,p)\,.$$

Note that $ps \le (p+1)t$ since $p \le \lfloor t/(s-t) \rfloor$.

Each element of the second from the left $(s-t)\times(s-t)$ area of $\mathrm{Ap}_p(A)$ $(p \ge 2)$ has exactly $(p+1)$ representations, because

$$\begin{aligned}&\bigl(0, s-t, (p-1)s-(p-2)t\bigr) = \bigl(s+t, 0, (p-2)s-(p-3)t\bigr)\\&= \bigl(js+(j-1)t, (j-1)t-(j-2)s, (p-j-1)s-(p-j-1)t\bigr)\\&\qquad (j=1,2,\dots,p-1)\,.\end{aligned}$$

Each element of the third from the left $(s-t)\times(s-t)$ area of $\mathrm{Ap}_p(A)$ $(p \ge 3)$ has exactly $(p+1)$ representations, because

$$\begin{aligned}&\bigl(0, 2s-2t, (p-2)s-(p-3)t\bigr) = \bigl(s+t, s-t, (p-3)s-(p-4)t\bigr)\\&= \bigl(2s+2t, 0, (p-4)s-(p-5)t\bigr)\\&= \bigl(js+(j-1)t, (j-2)t-(j-3)s, (p-j-2)s-(p-j-2)t\bigr)\\&\qquad (j=1,2,\dots,p-2)\,.\end{aligned}$$

In general, each element of the l-th $(1 \le l \le \lfloor t/(s-t) \rfloor)$ from the left $(s-t)\times(s-t)$ area of $\mathrm{Ap}_p(A)$ $(p \ge l)$ has exactly $(p+1)$ representations, because

$$\begin{aligned}&\bigl(0, (l-1)s-(l-1)t, (p-l+1)s-(p-l)t\bigr)\\&= \bigl(i(s+t), (l-i-1)(s-t), (p-l-i+1)s-(p-l-i)t\bigr)\\&\qquad (i=1,2,\dots,l-1)\\&= \bigl(js+(j-1)t, (j-l+1)t-(j-l)s, (p-l-j+1)(s-t)\bigr)\\&\qquad (j=1,2,\dots,p-l+1)\,.\end{aligned}$$

Similarly, each element of the l'-th $(1 \le l' \le \lfloor t/(s-t) \rfloor)$ from the top right $(s-t)\times(s-t)$ area of $\mathrm{Ap}_p(A)$ $(p \ge l')$ has exactly $(p+1)$ representations, because

$$\begin{aligned}&\bigl(0, (p-l'+1)s-(p-l')t, (l'-1)s-(l'-1)t\bigr)\\&= \bigl(i(s+t), (p-l'-i+1)s-(p-l'-i)t, (l'-i-1)(s-t)\bigr)\\&\qquad (i=1,2,\dots,l'-1)\\&= \bigl((j-1)s+jt, (p-l'-j+1)(s-t), (j-l'+1)t-(j-l')s\bigr)\\&\qquad (j=1,2,\dots,p-l'+1)\,.\end{aligned}$$

Concerning the central portion of $\mathrm{Ap}_p(A)$, it is easy to see that each element is expressed by

$$\bigl(0, p(s-t)+i, p(s-t)+j\bigr) \tag{9}$$

$$\begin{aligned}&(0 \le i \le s-t-1, 0 \le j \le pt-(p-1)s-1;\\&\qquad s-t \le i \le pt-(p-1)s-1, 0 \le j \le s-t-1)\,,\end{aligned} \tag{10}$$

and all elements have exactly $(p+1)$ representations, because

$$\bigl(0, p(s-t), p(s-t)\bigr) = \bigl(j(s+t), (p-j)(s-t), (p-j)(s-t)\bigr)$$

$(j = 1, 2, \dots, p)$.

Finally, the candidates to take the largest value in $\mathrm{Ap}_p(A)$ are clearly scattered in the lower right corners:

$$
\begin{aligned}
&\bigl(0, l(s-t)-1, (p+2-l)s-(p+1-l)t-1\bigr) \quad (l = 1, 2, \dots, p),\\
&\bigl(0, (p+1)(s-t)-1, s-1\bigr), \quad \bigl(0, s-1, (p+1)(s-t)-1\bigr),\\
&\bigl(0, (p+2-l')s-(p+1-l')t-1, l'(s-t)-1\bigr) \quad (l' = 1, 2, \dots, p).
\end{aligned}
$$

By comparing these values, we can find that $\bigl(0, s-t-1, (p+1)s-pt-1\bigr)$ is the largest. Hence, by the first formula of Lemma 1

$$
\begin{aligned}
g_p(\mathbf{x}, \mathbf{y}, \mathbf{z}) &= (s-t-1)\mathbf{y} + \bigl((p+1)s-pt-1\bigr)\mathbf{x} - \mathbf{z}\\
&= \frac{\bigl((p+2)s-(p+1)t-2\bigr)(s+t)^r + \bigl(ps-(p-1)t\bigr)(s-t)^r}{2} - (s^2-t^2).
\end{aligned}
$$

In addition, Theorem 1 does not hold for $p > \lfloor t/(s-t) \rfloor$. As can be seen from the example in Table 4, the elements of the central area of $\mathrm{Ap}_4(A)$ corresponding to the elements of the central area of $\mathrm{Ap}_3(A)$ are not all left, and there will be elements corresponding to another location. Due to the deviation, the place where the maximum value is taken also changes from $(0, s-t-1, (p+1)s-pt-1)$ in $\mathrm{Ap}_p(A)$ for $p > \lfloor t/(s-t) \rfloor$. In the case of the example in Table 5, for $p = 4$, the elements in the area of the stair part on both sides still regularly move to the opposite side, but in the main central part, some surplus elements move to the lower left ($3_i \Rightarrow 4_i$) and some to the upper-right ($3_k \Rightarrow 4_k$). In this case, in general, $(0, 2s-2t-1, (p+1)s-pt-1)$ takes the largest value. It is as shown in Table 5. At $p = 5$, the place where the largest value is taken becomes more complicated since the corresponding residue part is further displaced.

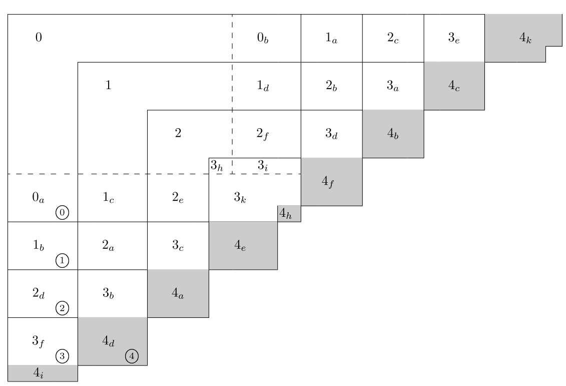


Table 5: $\mathrm{Ap}_p(\mathbf{x}, \mathbf{y}, \mathbf{z})$ $(p = 4)$ when $s \not\equiv t \pmod 2$

In the table, ⓝ denotes the position of the largest element in $\mathrm{Ap}_n(A)$. Note that the area 3_h (and so, 4_h) does not exist if $t/(s-t)$ is an integer.

4.3 When $2 \nmid t$

When $p \ge 1$, the situation is somewhat similar to that of the case where $s \not\equiv t \pmod 2$, but the roles of $\mathbf{z}' = 2st$ and $\mathbf{z} = s^2 - t^2$ are interchanged. Namely, the role of $(s-t)$ and

t is interchanged.

Table 6 shows the case where $3 < \lfloor (s-t)/t \rfloor < 4$. The numbers $0, 1, 2, 3$ indicate the area of $\mathrm{Ap}_p(A)$ for $p = 0, 1, 2, 3$.

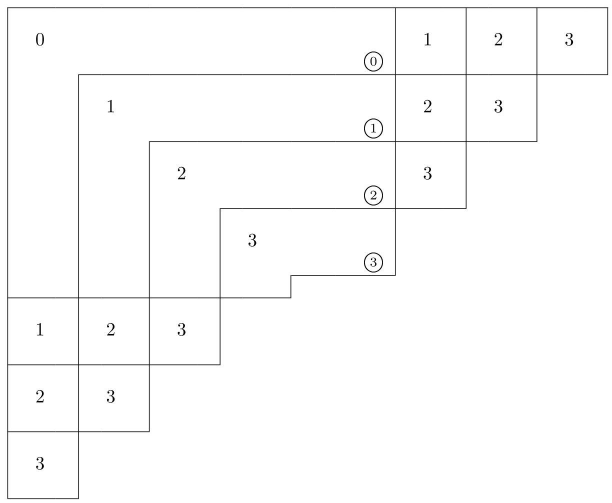


Table 6: $\mathrm{Ap}_p(\mathbf{x}', \mathbf{y}', \mathbf{z}')$ $(p = 0, 1, 2, 3)$ when $2 \nmid t$

In Table 6, the position to take the largest value of each $\mathrm{Ap}_p(A)$ is indicated by ⓟ $(p = 0, 1, 2, 3)$. Hence, by the first formula of Lemma 1

$$\begin{aligned} g_p(\mathbf{x}', \mathbf{y}', \mathbf{z}') &= (s+t-1)\mathbf{y}' + \bigl((p+1)t - 1\bigr)\mathbf{x}' - \mathbf{z}' \\ &= 2^{r-2}\bigl(s + (p+2)t - 2\bigr)s^r + (pt - s)t^r - 2st\,. \end{aligned}$$

5 More variations

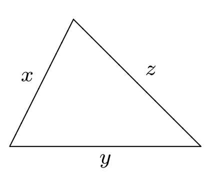
Almost the same results do not only hold for a 90 degree triangle, but also for a 60 degree or 120 degree triangle ([14]).

$x^2 - xy + y^2 = z^2$ (60 degree) has the parameterization

$$(x, y, z) = (s^2 - 3t^2 + 2st, 4st, s^2 + 3t^2)$$

or

$$(x, y, z) = (s^2 - t^2, 2st - t^2, s^2 - st + t^2).$$



Theorem 3. *Let s and t be positive integers with $s > t$, $\gcd(s, t) = 1$, $s \not\equiv t \pmod 2$ and $3 \nmid s$. When $s < 3t$, for a nonnegative integer p with $p \le \lfloor (2t)/(s-t) \rfloor$, we have*

$$g_p(s^2 - 3t^2 + 2st, 4st, s^2 + 3t^2)$$

$$= (s-t-1)(s^2+3t^2) + \big((p+1)s-(p-1)t-1\big)(4st) - (s^2-3t^2+2st)\,.$$

When $s > 3t$, for a nonnegative integer p with $p \le \lfloor (s-t)/(2t) \rfloor$, we have

$$\begin{aligned} & g_p(s^2-3t^2+2st, 4st, s^2+3t^2) \\ & = (2t-1)(s^2+3t^2) + \big(s+(2p+1)t-1\big)(s^2-3t^2+2st) - 4st\,. \end{aligned}$$

Theorem 4. *Let s and t be positive integers with $s > t$, $\gcd(s,t) = 1$ and $3 \nmid (s+t)$. When $s < 2t$, for a nonnegative integer p with $p \le \lfloor t/(s-t) \rfloor$, we have*

$$\begin{aligned} & g_p(s^2-t^2, 2st-t^2, s^2-st+t^2) \\ & = (s-t-1)(s^2-st+t^2) + \big(s+p(s-t)-1\big)(2st-t^2) - (s^2-t^2)\,. \end{aligned}$$

When $s > 2t$, for a nonnegative integer p with $p \le \lfloor (s-t)/t \rfloor$, we have

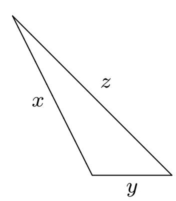
$$\begin{aligned} & g_0(s^2-t^2, 2st-t^2, s^2-st+t^2) \\ & = (t-1)(s^2-st+t^2) + (s+pt-1)(s^2-t^2) - (2st-t^2)\,. \end{aligned}$$

$x^2+xy+y^2 = z^2$ (120 degree) has the parameterization

$$(x,y,z) = (s^2-3t^2-2st, 4st, s^2+3t^2)$$

or

$$(x,y,z) = (s^2-t^2, 2st+t^2, s^2+st+t^2).$$



Theorem 5. *Let s and t be positive integers with $s > 3t$, $\gcd(s,t) = 1$, $s \not\equiv t \pmod 2$ and $3 \nmid s$. When $s < (3+2\sqrt{3})t$, for a nonnegative integer p with $p \le \lfloor (2t)/(s-3t) \rfloor$, we have*

$$\begin{aligned} & g_p(s^2-3t^2-2st, 4st, s^2+3t^2) \\ & = (s-3t-1)(4st) + \big(s-t+p(s-3t)-1\big)(s^2+3t^2) - (s^2-3t^2-2st)\,. \end{aligned}$$

When $s > (3+2\sqrt{3})t$, for a nonnegative integer p with $p \le \lfloor (s-3t)/(2t) \rfloor$, we have

$$\begin{aligned} & g_p(s^2-3t^2-2st, 4st, s^2+3t^2) \\ & = (s+3t-1)(s^2-3t^2-2st) + \big(2(p+1)t-1\big)(s^2+3t^2) - 4st\,. \end{aligned}$$

Theorem 6. *Let s and t be positive integers with $s > t$, $\gcd(s,t) = 1$, $s \not\equiv t \pmod 2$ and $3 \nmid (s-t)$. When $s < (1+\sqrt{3})t$, for a nonnegative integer p with $p \le \lfloor t/(s-t) \rfloor$, we have*

$$g_p(s^2-t^2, 2st+t^2, s^2+st+t^2)$$

$$= (s-t-1)t(2s+t) + \big(p(s-t)+s-1\big)(s^2+st+t^2) - (s+t)(s-t)\,.$$

When $s > (1+\sqrt{3})t$, for a nonnegative integer p with $p \le \lfloor (s-t)/t \rfloor$, we have

$$\begin{aligned} & g_p(s^2-t^2, 2st+t^2, s^2+st+t^2) \\ & = (s+2t-1)(s+t)(s-t) + \big((p+1)t-1\big)(s^2+st+t^2) - t(2s+t)\,. \end{aligned}$$

6 Final comments

Diophantine equations of the type $x^2+y^2=z^r$ $(r \ge 2)$ seem to be more popular. Their solutions can be also parameterized:

$$\begin{aligned} x &= \sum_{k=0}^{\lfloor r/2 \rfloor} (-1)^k \binom{r}{2k} s^{r-2k} t^{2k}\,, \\ y &= \sum_{k=0}^{\lfloor (r-1)/2 \rfloor} (-1)^k \binom{r}{2k+1} s^{r-2k-1} t^{2k+1}\,, \\ z &= s^2+t^2\,, \end{aligned}$$

where s and t are of opposite parity with $\gcd(s,t)=1$.

However, the situation becomes much more complicated, and many detailed discussion is needed. In addition, if the value r is different, the situation of the Apéry set is different, so we cannot discuss about general r.

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