On the classification and representations of positive definite ternary quadratic forms

Yifan Luo and Haigang Zhou

School of Mathematical Sciences, Tongji University, Shanghai, China, 200092

1 Introduction

This article is a report of author's talk at the conference "Research on automorphic forms" , which was held at RIMS, Kyoto university between 22rd to 26th, January, 2024. The author's talk was based on our preprint [14], where authors study the classification and representations of positive definite ternary quadratic forms of level $4N$.

Let f be a ternary quadratic form with integer coefficients, given by the equation

$$f(x,y,z) = ax^2 + by^2 + cz^2 + ryz + sxz + txy.$$

Unless otherwise stated, we assume that f is primitive and positive definite. We will also denote f by (a,b,c,r,s,t). Recall that the matrix associated to f is

$$M = M_f = \begin{pmatrix} 2a & t & s \\ t & 2b & r \\ s & r & 2c \end{pmatrix}.$$

Define the discriminant of f to be

$$d = d_f = \frac{\det(M_f)}{2} = 4abc + rst - ar^2 - bs^2 - ct^2.$$

The level of f is the smallest positive integer N such that NM_f^{-1} is even, that is, has integral entries, and even integers on the main diagonal. Let the number of ways to represent integer n by ternary quadratic form f be $R_f(n) := \sharp\{X \in \mathbb{Z}^3 \mid f(X) = \frac{1}{2}X^t M_f X = n\}$.

Classifications and representations are two main topics in the theory of quadratic forms. Regarding the classifications of ternary quadratic forms, two significant equivalent relations exist, namely equivalence class and semi-equivalence class. The set of semi-equivalence class is called genus. Instead of relying on the more familiar Hasse invariants, the classification theory is based on a set of invariants introduced by M. Eichler [5]. G. Shimura [15] classified quadratic forms over an algebraic number field by reformulating Eichler's approach in a more natural style. Extensive tables of positive definite ternary quadratic forms, categorized by discriminant, have been compiled. The tables compiled by H. Brandt and O. Intrau [3] comprehensively document all reduced ternary forms with discriminant $d < 1000$. J. Lehman [11] grouped positive definite ternary quadratic forms differently by level. He gave some correspondences between classes of ternary quadratic forms having the same level with different discriminants and provided a practical method for finding representatives of all classes of ternary forms with a given level. However, the method for determining whether two ternary quadratic forms are semi-equivalent is still cumbersome. In this paper, for squarefree interger N, we group primitive positive definite ternary quadratic forms of level $4N$ more explicitly, which only depends on level, discriminant, and the place where f is anisotropic, as follows.

In the following, we always assume that N is a product of s distinct odd primes. Let N^{odd} (respectively, N^{even}, or N_r) denote one of divisors of N with odd number of (respectively, even number of, or exactly r many) prime factors. We use $G_{4N,d,N^{\text{odd}}}$ (resp. $G_{4N,d,2N^{\text{even}}}$) for the genus of primitive positive definite ternary quadratic forms of level $4N$, discriminant d, and these forms are anisotropic only in the p-adic field where $p \mid N^{\text{odd}}$ (resp.$p \mid 2N^{\text{even}}$).

Theorem 1.1. *Let N be a product of s distinct odd primes. Then for all primitive positive definite ternary quadratic forms of level $4N$, there are 2^{2s+1} genera. These are $G_{4N,N^2/N_r,N^{odd}}$, $G_{4N,4N^2/N_r,N^{odd}}$, $G_{4N,4N^2/N_r,2N^{even}}$ and $G_{4N,16N^2/N_r,N^{odd}}$, where N^{odd} (respectively, N^{even} or N_r) runs over all divisors of N that contain an odd number of (respectively, an even number of, or exactly r many) prime factors.*

The representation problem is the questions if an integer n is represented by an integral quadratic form in s variables and in how many ways n is represented by such an integral quadratic form. The literature on quadratic forms is extensive and highly developed. Here we only focus on the case of ternary quadratic forms. Legendre gave the necessary and sufficient condition of when n is represented by a sum of three squares. Gauss went further, giving an explicit formula of the number of ways to represented n by sum of three squares, which is involved with the class number of binary quadratic forms. However, there are only a few ternary quadratic forms which have representation formulas like that of sum of three squares. Siegel provided a significant quantitative outcome in this regard by presenting $r_{\text{gen}(Q)}(n)$, a weighted average of representations through forms in the genus of a quadratic form $Q(x)$, as an infinite product of local factors. There are many literature on representation of ternary quadratic forms based on Siegel-Weil formula and modular forms. At the end of his book [9], B. Jones gave a specific version formula for positive-definite ternary quadratic forms. This was a significant advancement in making the theory more concrete and applicable to specific cases. However, it still requires the computation of local densities. Recently, X. Guo, Y. Peng, L. Gao and H. Qin [8][6] gave some explicit formulas for the average number of representations over the genus of ternary quadratic form of type $f = x^2 + py^2 + qz^2$, where p and q are odd primes. B. Kane, D. Kim and S. Varadharajan [10] computed explicitly the Siegel–Weil average $r_{\text{gen}(Q)}(n)$ of a genus for ternary quadratic forms corresponding to stable lattices.

Orders of quaternion algebras have a close relation with ternary quadratic forms. For the detail of the relation between ternary forms and orders of quaternion algebras, we refer to the book of J. Voight [16]. In 1987, B. Gross [7] showed that, for definite quaternion algebra ramified at prime p, the weighted sum of theta series corresponding to maximal orders is Eisenstein series of weight $3/2$ whose Fourier coefficients are modified Hurwitz class number. In another words, the modified Hurwitz class number equals a weighted sum of the number of elements of trace 0 and norm n in an maximal order $\mathcal{O}_\mu$, where $\mathcal{O}_\mu$ ranges over the right orders of a set of representatives for left ideal classes of a maximal order $\mathcal{O}$ in a quaternion algebra of discriminant p. In 2019, H. Boylan, N. Skoruppa and the second author [2] recovered the formula obtained by Gross, in the more general case of squarefree N, using the theory of Jacobi forms. More precisely, they showed that the Jacobi Eisenstein series whose Fourier coefficients are modified Hurwitz class number $H^{(N)}(4n - r^2)$ agrees with a weighted sum of theta series corresponding to a set of representatives for the conjugacy classes of $\mathcal{O}$. Recently, Y. Li, N. Skoruppa and the second author [13] extended it to more general cases. More specifically, for all Eichler orders with a same squarefree level in a definite quaternion algebra over the field of rational numbers, they proved that a weighted sum of Jacobi theta series associated with these orders is a Jacobi Eisenstein series which has Fourier coefficients $H^{(N_1,N_2)}(4n - r^2)$.

For any pair of relatively prime positive squarefree integers (N_1, N_2) and any negative discriminant $-D$, Hurwitz class numbers $H(D)$ can be modified as follows:

$$H^{(N_1,N_2)}(D) = H(D/f_{N_1,N_2}^2) \prod_{p|N_1} \left(1 - \left(\frac{-D/f_{N_1,N_2}^2}{p}\right)\right) \prod_{p|N_2} \frac{2pf_p - p - 1 - \left(\frac{-D/f_{N_1,N_2}^2}{p}\right)(2f_p - p - 1)}{p-1}, \tag{1.1}$$

where f_{N_1,N_2} is the largest positive integer containing only prime factors of N_1N_2 whose square divides D such that $-D/f_{N_1,N_2}^2$ is still a negative discriminant. The products run through all primes p dividing N_1 and N_2, respectively. We use f_p for the exact p-power dividing f_{N_1,N_2}. In particular, when $f_p = 1$, the above fraction containing f_p becomes $1 + \left(\frac{-D/f_{N_1,N_2}^2}{p}\right)$. Here $(\frac{\cdot}{p})$ is the Kronecker symbol, and for integer m, Kronecker symbol

$$\left(\frac{m}{2}\right) = \begin{cases} 1 & \text{if} \quad m \equiv \pm 1 \pmod 8, \\ -1 & \text{if} \quad m \equiv \pm 3 \pmod 8, \\ 0 & \text{otherwise}. \end{cases}$$

Set

$$H^{(N_1,N_2)}(0) = -\frac{1}{12} \prod_{p|N_1} (1-p) \prod_{p|N_2} (1+p),$$

and $H^{(N_1,N_2)}(D) = 0$ for every positive integer $D \equiv 1, 2 \pmod 4$.

Consider the bijections between Eichler orders and ternary quadratic forms and associate the results of Y. Li et al.[13] and their modified Hurwitz class number $H^{(N_1,N_2)}(D)$, we derive more explicit formula of for the average number of representations over the genus of definite positive ternary quadratic form of level $4N$.

Theorem 1.2. *For any squarefree poisitive integer N and any divisors N^{odd} and N^{even} of N with an odd respectively even number of prime factors, and for any nonnegative integer n, one has*

$$\sum_{f \in G_{4N,N^2/N_r,N^{odd}}} \frac{R_f(n)}{|\operatorname{Aut}(f)|} = 2^{-s-1} H^{(N^{odd},N/N^{odd})}(4N_r n),$$

$$\sum_{f \in G_{4N,4N^2/N_r,N^{odd}}} \frac{R_f(n)}{|\operatorname{Aut}(f)|} = 2^{-s-2} H^{(N^{odd},2N/N^{odd})}(4N_r n),$$

$$\sum_{f \in G_{4N,4N^2/N_r,2N^{even}}} \frac{R_f(n)}{|\operatorname{Aut}(f)|} = 2^{-s-2} H^{(2N^{even},N/N^{even})}(4N_r n),$$

and

$$\sum_{f \in G_{4N,16N^2/N_r,N^{odd}}} \frac{R_f(n)}{|\operatorname{Aut}(f)|} = 2^{-s-1} H^{(N^{odd},N/N^{odd})}(N_r n).$$

Here as above $R_f(n)$ denotes the number of representations of n by the form f, we use $\operatorname{Aut}(f)$ *for the number of automorphs of f. The sums are over a complete set of equivalent classes in the given genus classes, respectively.*

Remark 1.3. The conditions hold for $N^{\text{even}} = 1$ and $N_r = 1$. When $N = 1$, all positive definite ternary quadratic forms of level 4 are only in one genus which has one class, and the sum of three squares is their representative element

$$r_3(n) = 12H^{(2,1)}(4n).$$

If the class number of the genus of a positive definite ternary quadratic forms f of level $4N$ is one, we can give the explicit formula of $R_f(n)$. In the table 2 in section 3, we list 73 formulas of $R_f(n)$.

Li et al. [13] also gave a simple formula for the type number of Eichler orders with squarefree level by modified Hurwitz class number. Based on the type number formula, we can give the formula of the class number of ternary quadratic forms of level $4N$.

Theorem 1.4. *Let $|C(4N)|$ denote the number of classes of primitive positive definite ternary quadratic forms of level $4N$. Then we have*

$$|C(4N)| = 2^s \left(\frac{N}{6} + \frac{5}{4} - \frac{1}{4}\left(\frac{-4}{N}\right) - \frac{1}{6}\left(\frac{-3}{N}\right) + \frac{1}{2}\left(1 - \left(\frac{N}{3}\right)^2\right) + \frac{1}{4}\sum_{\substack{d|N \\ d\neq 1}} (3H(4d) + H(8d)) \right).$$

In the table 3 in section 3, we give a list of $|C(4N)|$, the number of classes of primitive positive definite ternary quadratic forms of level $4N$ for $N < 1000$.

2 Some applications

In this section, we will give three applications of our main results. Construct a basis of Eisenstein space of modular forms of weight 3/2, and give new proofs of Berkovich and Jagy's genus identity and Du's identity.

2.1 Modular forms of weight 3/2

Let $\mathscr{E}(4N, \frac{3}{2}, \chi_l)$ be the Eisenstein series space of modular forms of weight 3/2, level $4N$ and character χ_l, which is the orthogonal complement of the subspace of cusp forms of weight 3/2, level $4N$ and the character χ_l with respect to Petersson inner product. Pei [17] gave an explicit basis of the Eisenstein space $\mathscr{E}(4N, \frac{3}{2}, \chi_l)$.

Denote the generating functions of average representation number of ternary quadratic forms over each genus as following:

$$\theta_{G_{4N,N^2,N^{\text{odd}}}}(z) := 2^{s+1} \sum_{n=0}^{\infty} \left(\sum_{f\in G_{4N,N^2,N^{\text{odd}}}} \frac{R_f(n)}{|\operatorname{Aut}(f)|} \right) q^n = \sum_{n=0}^{\infty} H^{(N^{\text{odd}}, N/N^{\text{odd}})}(4n) q^n, \tag{2.1}$$

$$\theta_{G_{4N,4N^2,N^{\text{odd}}}}(z) := 2^{s+2} \sum_{n=0}^{\infty} \left(\sum_{f\in G_{4N,4N^2,N^{\text{odd}}}} \frac{R_f(n)}{|\operatorname{Aut}(f)|} \right) q^n = \sum_{n=0}^{\infty} H^{(N^{\text{odd}}, 2N/N^{\text{odd}})}(4n) q^n, \tag{2.2}$$

and

$$\theta_{G_{4N,4N^2,2N^{\text{even}}}}(z) := 2^{s+2} \sum_{n=0}^{\infty} \left(\sum_{f\in G_{4N,4N^2,2N^{\text{even}}}} \frac{R_f(n)}{|\operatorname{Aut}(f)|} \right) q^n = \sum_{n=0}^{\infty} H^{(2N^{\text{even}}, N/N^{\text{even}})}(4n) q^n. \tag{2.3}$$

It is well-known that these functions are in the space of Eisenstein series of weight 3/2, $\mathscr{E}(4N, \frac{3}{2}, \mathrm{id})$.

Theorem 2.1. *Let I denote the set of all positive divisor of $2N$ except 1, $d \in I$. Set*

$$\theta_{d,2N/d}(z) := \sum_{n=0}^{\infty} H^{(d,2N/d)}(4n) q^n. \tag{2.4}$$

Let l be the divisor of N, and χ_l denote the primitive characters such that $\chi_l(n) = \left(\frac{l}{n}\right)$ for $(n, 4l) = 1$. Then the set $\{\theta_{d,2N/d}(lz)\}_I$ is a basis of space of Eisenstein series of weight 3/2 with character χ_l, $\mathscr{E}(4N, \frac{3}{2}, \chi_l)$.

2.2 Berkovich and Jagy's genus identity

Berkovich and Jagy [1] established the following interesting identity connecting the weighted sum of the representation numbers and the sum of three squares $r_3(n)$:

$$r_3(p^2n) - pr_3(n) = 48 \sum_{f\in TG_{1,p}} \frac{R_f(n)}{|\operatorname{Aut}(f)|} - 96 \sum_{f\in TG_{2,p}} \frac{R_f(n)}{|\operatorname{Aut}(f)|}, \tag{2.5}$$

where a sum over forms in a genus should be understood to be the finite sum resulting from taking a single representative from each equivalence class of forms.

We now give a new proof of the identity (2.5). By the level and discriminant of ternary quadratic forms, it becomes evident that $TG_{1,p}$ (resp. $TG_{2,p}$) coincides $G_{4p,p^2,p}$ (resp. $G_{4p,16p^2,p}$). In terms of the results of Theorem 1.2, we have

$$\sum_{f\in TG_{1,p}} \frac{R_f(n)}{|\operatorname{Aut}(f)|} = \sum_{f\in G_{4p,p^2,p}} \frac{R_f(n)}{|\operatorname{Aut}(f)|} = \frac{1}{4}H^{(p,1)}(4n); \tag{2.6}$$

and

$$\sum_{f\in TG_{2,p}} \frac{R_f(n)}{|\operatorname{Aut}(f)|} = \sum_{f\in G_{4p,16p^2,p}} \frac{R_f(n)}{|\operatorname{Aut}(f)|} = \frac{1}{4}H^{(p,1)}(n). \tag{2.7}$$

For a squarefree integer M and an odd prime p with $(M,p)=1$ and a negative discriminant $-D$, one [2] has

$$H^{(M,1)}(p^2D) - pH^{(M,1)}(D) = H^{(pM,1)}(D). \tag{2.8}$$

So we have

$$H^{(2,1)}(4p^2n) - pH^{(2,1)}(4n) = H^{(2p,1)}(4n) \tag{2.9}$$

and

$$H^{(p,1)}(16n) - 2H^{(p,1)}(4n) = H^{(2p,1)}(4n). \tag{2.10}$$

Furthermore, from the above equation we can get

$$H^{(p,1)}(4n) - 2H^{(p,1)}(n) = H^{(2p,1)}(4n). \tag{2.11}$$

Combining the eqatuions (2.6), (2.7), (2.9), (2.11) and $r_3(n) = 12H^{(2,1)}(4n)$, we obtain the identity (2.5).

In fact, combining Theorem 1.2 and the equality of the modified Hurwitz class number $H^{N_1,N_2}(D)$, we can derive more identities similar to (2.5).

2.3 Du's equality

In 2016, Du [4] gave an interesting equality as follows. For squarefree D with odd number of prime factors, let $B(D)$ be the unique quaternion algebra over $\mathbb{Q}$ of discriminant D, and

$$V(D) = \{x \in B(D) | \operatorname{tr}(x) = 0\}.$$

For a positive interger N prime to D, let $L_D(N) = \mathcal{O}(N) \cap V(D)$, where $\mathcal{O}(N)$ is an Eichler order in B of conductor N, and

$$r_{D,N} = r_{\operatorname{gen}(L)}(m) = \frac{\sum_{L_1\in \operatorname{gen}(L)} \frac{r_{L_1}(m)}{|\operatorname{Aut}(L_1)|}}{\sum_{L_1\in \operatorname{gen}(L)} \frac{1}{|\operatorname{Aut}(L_1)|}}.$$

Now let D be a square-free positive integer with even number of prime factors, $p \neq q$ be two different primes not dividing D, and N be a positive integer prime to Dpq. Then

$$-\frac{2}{q-1} r_{Dp,N}(m) + \frac{q+1}{q-1} r_{Dp,Nq}(m) = -\frac{2}{p-1} r_{Dq,N}(m) + \frac{p+1}{p-1} r_{Dq,Np}(m) \tag{2.12}$$

for every positive integer m.

Now we give a new proof of the equality (2.12). In terms of the results of Theorem 1.2,

$$r_{Dp,Nq}(m) = \frac{H^{(Dp,Nq)}(4m)}{H^{(Dp,Nq)}(0)} = \frac{2H^{(Dp,N)}(4m) - H^{(Dpq,N)}(4m)}{(1+q)H^{(Dp,N)}(0)}.$$

In terms of the definition (1.1), we have

$$H^{(D,Nq)}(4m) = 2H^{(D,N)}(4m) - H^{(Dq,N)}(4m).$$

The left hand of the equality (2.12)

$$-\frac{2}{q-1} r_{Dp,N}(m) + \frac{q+1}{q-1} r_{Dp,Nq}(m) = \frac{-H^{(Dpq,N)}(4m)}{(q-1)H^{(Dp,N)}(0)} = \frac{H^{(Dpq,N)}(4m)}{H^{(Dpq,N)}(0)}.$$

Similarly, the right hand of the equality (2.12)

$$-\frac{2}{p-1} r_{Dq,N}(m) + \frac{p+1}{p-1} r_{Dq,Np}(m) = \frac{H^{(Dpq,N)}(4m)}{H^{(Dpq,N)}(0)}.$$

3 Examples and tables

In this section we will give some examples of representation of ternary quadratic forms. If the class number of a genus is one, we can give an exact formula of representation number of n by the ternary quadratic forms. For squarefree integers N, there are 73 genera of ternary quadratic forms of level $4N$ with one class. Let $\mathcal{O} \subset Q_N$ be an Eichler order. Its type number equals 1 if its level (N, F) is one of the following: (2,1), (3,1), (5,1), (7,1), (13,1), (30,1), (42,1), (70,1), (78,1), (2,3), (2,5), (2,7), (2,11), (2,15), (2,23), (3,2), (3,5), (3,11), (5,2), (7,3)[12, p.94], and we get 73 genera with one class. We give the explicit formulas for the representation number of ternary quadratic forms as follows.

Table 1: Genera with one class

Genus	N_f	d_f	$R_f(n)$
$G_{4,4,2}$	4	4	$R_{(1,1,1,0,0,0)}(n) = 12H^{(2,1)}(4n)$
$G_{12,9,3}$	$4\cdot 3$	3^2	$R_{(1,1,3,0,0,-1)}(n) = 6H^{(3,1)}(4n)$
$G_{12,3,3}$	$4\cdot 3$	3	$R_{(1,1,1,0,0,-1)}(n) = 6H^{(3,1)}(12n)$
$G_{12,144,3}$	$4\cdot 3$	$16\cdot 3^2$	$R_{(3,4,4,-4,0,0)}(n) = 6H^{(3,1)}(n)$
$G_{12,48,3}$	$4\cdot 3$	$16\cdot 3$	$R_{(1,4,4,-4,0,0)}(n) = 6H^{(3,1)}(3n)$
$G_{20,25,5}$	$4\cdot 5$	5^2	$R_{(2,2,2,-1,-1,-1)}(n) = 3H^{(5,1)}(4n)$
$G_{20,5,5}$	$4\cdot 5$	5	$R_{(1,1,2,1,1,1)}(n) = 3H^{(5,1)}(20n)$
$G_{20,400,5}$	$4\cdot 5$	$16\cdot 5^2$	$R_{(3,7,7,-6,-2,-2)}(n) = 3H^{(5,1)}(n)$
$G_{20,80,5}$	$4\cdot 5$	$16\cdot 5$	$R_{(3,3,3,2,2,2)}(n) = 3H^{(5,1)}(5n)$
$G_{28,49,7}$	$4\cdot 7$	7^2	$R_{(1,2,7,0,0,-1)}(n) = 2H^{(7,1)}(4n)$

$G_{28,7,7}$	$4 \cdot 7$	7	$R_{(1,1,2,0,-1,0)}(n) = 2H^{(7,1)}(28n)$
$G_{28,784,7}$	$4 \cdot 7$	$16 \cdot 7^2$	$R_{(4,7,8,0,-4,0)}(n) = 2H^{(7,1)}(n)$
$G_{28,112,7}$	$4 \cdot 7$	$16 \cdot 7$	$R_{(1,4,8,-4,0,0)}(n) = 2H^{(7,1)}(7n)$
$G_{52,169,13}$	$4 \cdot 13$	13^2	$R_{(2,5,5,-3,-1,-1)}(n) = H^{(13,1)}(4n)$
$G_{52,13,13}$	$4 \cdot 13$	13	$R_{(1,2,2,-1,0,-1)}(n) = H^{(13,1)}(52n)$
$G_{52,2704,13}$	$4 \cdot 13$	$16 \cdot 13^2$	$R_{(7,8,15,8,2,4)}(n) = H^{(13,1)}(n)$
$G_{52,208,13}$	$4 \cdot 13$	$16 \cdot 13$	$R_{(3,3,7,2,2,2)}(n) = H^{(13,1)}(13n)$
$G_{60,900,30}$	$4 \cdot 3 \cdot 5$	$4 \cdot 3^2 \cdot 5^2$	$2R_{(3,10,10,-10,0,0)}(n) = 3H^{(30,1)}(4n)$
$G_{60,300,30}$	$4 \cdot 3 \cdot 5$	$4 \cdot 3 \cdot 5^2$	$2R_{(1,10,10,-10,0,0)}(n) = 3H^{(30,1)}(12n)$
$G_{60,180,30}$	$4 \cdot 3 \cdot 5$	$4 \cdot 3^2 \cdot 5$	$2R_{(2,2,15,0,0,-2)}(n) = 3H^{(30,1)}(20n)$
$G_{60,60,30}$	$4 \cdot 3 \cdot 5$	$4 \cdot 3 \cdot 5$	$2R_{(2,2,5,0,0,-2)}(n) = 3H^{(30,1)}(60n)$
$G_{84,1764,42}$	$4 \cdot 3 \cdot 7$	$4 \cdot 3^2 \cdot 7^2$	$R_{(1,21,21,0,0,0)}(n) = H^{(42,1)}(4n)$
$G_{84,588,42}$	$4 \cdot 3 \cdot 7$	$4 \cdot 3 \cdot 7^2$	$R_{(3,7,7,0,0,0)}(n) = H^{(42,1)}(12n)$
$G_{84,252,42}$	$4 \cdot 3 \cdot 7$	$4 \cdot 3^2 \cdot 7$	$R_{(3,3,7,0,0,0)}(n) = H^{(42,1)}(28n)$
$G_{84,84,42}$	$4 \cdot 3 \cdot 7$	$4 \cdot 3 \cdot 7$	$R_{(1,1,21,0,0,0)}(n) = H^{(42,1)}(84n)$
$G_{140,4900,70}$	$4 \cdot 5 \cdot 7$	$4 \cdot 5^2 \cdot 7^2$	$2R_{(2,18,35,0,0,-2)}(n) = H^{(70,1)}(4n)$
$G_{140,980,70}$	$4 \cdot 5 \cdot 7$	$4 \cdot 5 \cdot 7^2$	$2R_{(6,6,7,0,0,-2)}(n) = H^{(70,1)}(20n)$
$G_{140,700,70}$	$4 \cdot 5 \cdot 7$	$4 \cdot 5^2 \cdot 7$	$2R_{(5,6,6,-2,0,0)}(n) = H^{(70,1)}(28n)$
$G_{140,140,70}$	$4 \cdot 5 \cdot 7$	$4 \cdot 5 \cdot 7$	$2R_{(1,2,18,-2,0,0)}(n) = H^{(70,1)}(140n)$
$G_{156,6084,78}$	$4 \cdot 3 \cdot 13$	$4 \cdot 3^2 \cdot 13^2$	$2R_{(6,13,21,0,-6,0)}(n) = H^{(78,1)}(4n)$
$G_{156,2028,78}$	$4 \cdot 3 \cdot 13$	$4 \cdot 3 \cdot 13^2$	$2R_{(2,7,39,0,0,-2)}(n) = H^{(78,1)}(12n)$
$G_{156,468,78}$	$4 \cdot 3 \cdot 13$	$4 \cdot 3^2 \cdot 13$	$2R_{(1,6,21,-6,0,0)}(n) = H^{(78,1)}(52n)$
$G_{156,156,78}$	$4 \cdot 3 \cdot 13$	$4 \cdot 3 \cdot 13$	$2R_{(2,33,7,0,-2,0)}(n) = H^{(78,1)}(156n)$
$G_{12,36,2}$	$4 \cdot 3$	$4 \cdot 3^2$	$R_{(2,2,3,0,0,-2)}(n) = 3H^{(2,3)}(4n)$
$G_{12,12,2}$	$4 \cdot 3$	$4 \cdot 3$	$R_{(1,2,2,-2,0,0)}(n) = 3H^{(2,3)}(12n)$
$G_{20,100,2}$	$4 \cdot 5$	$4 \cdot 5^2$	$R_{(1,5,5,0,0,0)}(n) = 2H^{(2,5)}(4n)$
$G_{20,20,2}$	$4 \cdot 5$	$4 \cdot 5$	$R_{(1,1,5,0,0,0)}(n) = 2H^{(2,5)}(20n)$
$G_{28,196,2}$	$4 \cdot 7$	$4 \cdot 7^2$	$2R_{(3,5,5,-4,-2,-2)}(n) = 3H^{(2,7)}(4n)$
$G_{28,28,2}$	$4 \cdot 7$	$4 \cdot 7$	$2R_{(2,2,3,2,2,2)}(n) = 3H^{(2,7)}(28n)$
$G_{44,484,2}$	$4 \cdot 11$	$4 \cdot 11^2$	$R_{(2,6,11,0,0,-2)}(n) = H^{(2,11)}(4n)$
$G_{44,44,2}$	$4 \cdot 11$	$4 \cdot 11$	$R_{(1,2,6,-2,0,0)}(n) = H^{(2,11)}(44n)$
$G_{60,900,2}$	$4 \cdot 3 \cdot 5$	$4 \cdot 3^2 \cdot 5^2$	$2R_{(5,6,9,-6,0,0)}(n) = H^{(2,15)}(4n)$
$G_{60,300,2}$	$4 \cdot 3 \cdot 5$	$4 \cdot 3 \cdot 5^2$	$2R_{(2,3,15,0,0,-2)}(n) = H^{(2,15)}(12n)$
$G_{60,180,2}$	$4 \cdot 3 \cdot 5$	$4 \cdot 3^2 \cdot 5$	$2R_{(1,6,9,-6,0,0)}(n) = H^{(2,15)}(20n)$
$G_{60,60,2}$	$4 \cdot 3 \cdot 5$	$4 \cdot 3 \cdot 5$	$2R_{(2,3,3,0,0,-2)}(n) = H^{(2,15)}(60n)$
$G_{92,2116,2}$	$4 \cdot 23$	$4 \cdot 23^2$	$2R_{(5,10,14,10,2,4)}(n) = H^{(2,23)}(4n)$
$G_{92,92,2}$	$4 \cdot 23$	$4 \cdot 23$	$2R_{(2,3,5,-2,0,-2)}(n) = H^{(2,23)}(92n)$
$G_{12,36,3}$	$4 \cdot 3$	$4 \cdot 3^2$	$R_{(1,3,3,0,0,0)}(n) = 2H^{(3,2)}(4n)$
$G_{12,12,3}$	$4 \cdot 3$	$4 \cdot 3$	$R_{(1,1,3,0,0,0)}(n) = 2H^{(3,2)}(12n)$
$G_{60,225,3}$	$4 \cdot 3 \cdot 5$	$3^2 \cdot 5^2$	$R_{(1,4,15,0,0,-1)}(n) = H^{(3,5)}(4n)$
$G_{60,75,3}$	$4 \cdot 3 \cdot 5$	$3 \cdot 5^2$	$R_{(2,2,5,0,0,-1)}(n) = H^{(3,5)}(12n)$
$G_{60,45,3}$	$4 \cdot 3 \cdot 5$	$3^2 \cdot 5$	$R_{(2,2,3,0,0,-1)}(n) = H^{(3,5)}(20n)$
$G_{60,15,3}$	$4 \cdot 3 \cdot 5$	$3 \cdot 5$	$R_{(1,1,4,0,-1,0)}(n) = H^{(3,5)}(60n)$
$G_{60,3600,3}$	$4 \cdot 3 \cdot 5$	$16 \cdot 3^2 \cdot 5^2$	$R_{(4,15,16,0,-4,0)}(n) = H^{(3,5)}(n)$
$G_{60,1200,3}$	$4 \cdot 3 \cdot 5$	$16 \cdot 3 \cdot 5^2$	$R_{(5,8,8,-4,0,0)}(n) = H^{(3,5)}(3n)$
$G_{60,720,3}$	$4 \cdot 3 \cdot 5$	$16 \cdot 3^2 \cdot 5$	$R_{(3,8,8,-4,0,0)}(n) = H^{(3,5)}(5n)$

$G_{60,240,3}$	$4\cdot 3\cdot 5$	$16\cdot 3\cdot 5$	$R_{(1,4,16,-4,0,0)}(n)=H^{(3,5)}(15n)$
$G_{132,1089,3}$	$4\cdot 3\cdot 11$	$3^2\cdot 11^2$	$2R_{(6,7,10,7,3,6)}(n)=H^{(3,11)}(4n)$
$G_{132,363,3}$	$4\cdot 3\cdot 11$	$3\cdot 11^2$	$2R_{(2,7,7,3,1,1)}(n)=H^{(3,11)}(12n)$
$G_{132,99,3}$	$4\cdot 3\cdot 11$	$3^2\cdot 11$	$2R_{(2,3,5,-3,-1,0)}(n)=H^{(3,11)}(44n)$
$G_{132,33,3}$	$4\cdot 3\cdot 11$	$3\cdot 11$	$2R_{(1,2,5,1,1,1)}(n)=H^{(3,11)}(132n)$
$G_{132,17424,3}$	$4\cdot 3\cdot 11$	$16\cdot 3^2\cdot 11^2$	$2R_{(7,19,39,-18,-6,-2)}(n)=H^{(3,11)}(n)$
$G_{132,5808,3}$	$4\cdot 3\cdot 11$	$16\cdot 3\cdot 11^2$	$2R_{(8,13,17,2,4,8)}(n)=H^{(3,11)}(3n)$
$G_{132,1584,3}$	$4\cdot 3\cdot 11$	$16\cdot 3^2\cdot 11$	$2R_{(5,5,17,-2,-2,-2)}(n)=H^{(3,11)}(11n)$
$G_{132,528,3}$	$4\cdot 3\cdot 11$	$16\cdot 3\cdot 11$	$2R_{(4,7,7,-6,0,-4)}(n)=H^{(3,11)}(33n)$
$G_{20,100,5}$	$4\cdot 5$	$4\cdot 5^2$	$R_{(2,3,5,0,0,-2)}(n)=2H^{(5,2)}(4n)$
$G_{20,20,5}$	$4\cdot 5$	$4\cdot 5$	$R_{(1,2,3,-2,0,0)}(n)=2H^{(5,2)}(20n)$
$G_{84,441,7}$	$4\cdot 3\cdot 7$	$3^2\cdot 7^2$	$2R_{(2,8,8,-5,-1,-1)}(n)=H^{(7,3)}(4n)$
$G_{84,147,7}$	$4\cdot 3\cdot 7$	$3\cdot 7^2$	$2R_{(3,3,5,-2,-2,-1)}(n)=H^{(7,3)}(12n)$
$G_{84,63,7}$	$4\cdot 3\cdot 7$	$3^2\cdot 7$	$2R_{(2,2,5,2,2,1)}(n)=H^{(7,3)}(28n)$
$G_{84,21,7}$	$4\cdot 3\cdot 7$	$3\cdot 7$	$2R_{(1,2,3,-1,-1,0)}(n)=H^{(7,3)}(84n)$
$G_{84,7056,7}$	$4\cdot 3\cdot 7$	$16\cdot 3^2\cdot 7^2$	$2R_{(8,11,23,2,8,4)}(n)=H^{(7,3)}(n)$
$G_{84,2352,7}$	$4\cdot 3\cdot 7$	$16\cdot 3\cdot 7^2$	$2R_{(5,12,12,-4,-4,-4)}(n)=H^{(7,3)}(3n)$
$G_{84,1008,7}$	$4\cdot 3\cdot 7$	$16\cdot 3^2\cdot 7$	$2R_{(5,8,8,4,4,4)}(n)=H^{(7,3)}(7n)$
$G_{84,336,7}$	$4\cdot 3\cdot 7$	$16\cdot 3\cdot 7$	$2R_{(3,3,11,2,2,2)}(n)=H^{(7,3)}(21n)$

For the class number larger than one, we give one example. Let level $4N=4\cdot 5\cdot 7=140$, there are $2^{2\times 2+1}=32$ genera in the set of all primitive positive definite ternary quadratic forms of level 140. The class number of primitive positive definite ternary quadratic forms of level 140 is $|C(140)|=76$. We give 32 formulas of the weighted sums of representation of ternary quadratic forms on each genus.

Table 2: Representation of ternary quadratic forms of level 140

Genus	d_f	$R_f(n)$
$G_{140,1225,5}$	$5^2\cdot 7^2$	$3R_{(3,3,35,0,0,-1)}(n)+2R_{(3,12,12,-11,-2,-2)}(n)+3R_{(5,7,10,0,-5,0)}(n)=3H^{(5,7)}(4n)$
$G_{140,1225,7}$	$5^2\cdot 7^2$	$R_{(1,9,35,0,0,-1)}(n)+2R_{(4,9,11,9,1,2)}(n)=H^{(7,5)}(4n)$
$G_{140,245,5}$	$5\cdot 7^2$	$3R_{(1,7,9,0,-1,0)}(n)+2R_{(4,4,4,1,1,1)}(n)+3R_{(1,2,35,0,0,-1)}(n)=3H^{(5,7)}(20n)$
$G_{140,245,7}$	$5\cdot 7^2$	$R_{(3,3,7,0,0,-1)}(n)+2R_{(3,5,5,3,2,2)}(n)=H^{(7,5)}(20n)$
$G_{140,175,5}$	$5^2\cdot 7$	$3R_{(1,5,10,-5,0,0)}(n)+2R_{(4,4,4,3,3,3)}(n)+3R_{(1,5,9,0,-1,0)}(n)=3H^{(5,7)}(28n)$
$G_{140,175,7}$	$5^2\cdot 7$	$R_{(3,3,5,0,0,-1)}(n)+2R_{(2,2,12,-1,-1,-1)}(n)=H^{(7,5)}(28n)$
$G_{140,35,5}$	$5\cdot 7$	$3R_{(1,3,3,-1,0,0)}(n)+2R_{(1,1,12,1,1,1)}(n)+3R_{(1,2,5,0,0,-1)}(n)=3H^{(5,7)}(140n)$
$G_{140,35,7}$	$5\cdot 7$	$R_{(1,1,9,0,-1,0)}(n)+2R_{(1,3,4,3,1,1)}(n)=H^{(7,5)}(140n)$
$G_{140,4900,2}$	$4\cdot 5^2\cdot 7^2$	$2R_{(5,14,21,-14,0,0)}(n)+2R_{(7,10,20,-10,0,0)}(n)=H^{(2,35)}(4n)$
$G_{140,4900,5}$	$4\cdot 5^2\cdot 7^2$	$4R_{(3,12,35,0,0,-2)}(n)+2R_{(5,7,35,0,0,0)}(n)+2R_{(7,10,20,-10,0,0)}(n)=H^{(5,14)}(4n)$
$G_{140,4900,7}$	$4\cdot 5^2\cdot 7^2$	$R_{(1,35,35,0,0,0)}(n)+4R_{(4,9,35,0,0,-2)}(n)+4R_{(11,11,15,10,10,8)}(n)=H^{(7,10)}(4n)$
$G_{140,4900,70}$	$4\cdot 5^2\cdot 7^2$	$2R_{(2,18,35,0,0,-2)}(n)=H^{(70,1)}(4n)$
$G_{140,980,2}$	$4\cdot 5\cdot 7^2$	$2R_{(2,7,18,0,-2,0)}(n)+2R_{(1,14,21,-14,0,0)}(n)=H^{(2,35)}(20n)$
$G_{140,980,5}$	$4\cdot 5\cdot 7^2$	$4R_{(4,7,9,0,-2,0)}(n)+2R_{(1,7,35,0,0,0)}(n)+2R_{(2,4,35,0,0,-2)}(n)=H^{(5,14)}(20n)$
$G_{140,980,7}$	$4\cdot 5\cdot 7^2$	$R_{(5,7,7,0,0,0)}(n)+4R_{(3,5,19,-4,-2,-2)}(n)+4R_{(3,7,12,0,-2,0)}(n)=H^{(7,10)}(20n)$
$G_{140,980,70}$	$4\cdot 5\cdot 7^2$	$2R_{(6,6,7,0,0,-2)}(n)=H^{(70,1)}(20n)$
$G_{140,700,2}$	$4\cdot 5^2\cdot 7$	$2R_{(2,5,18,0,-2,0)}(n)+2R_{(2,3,25,0,0,-2)}(n)=H^{(2,35)}(28n)$

$G_{140,700,5}$	$4 \cdot 5^2 \cdot 7$	$4R_{(4,5,9,0,-2,0)}(n) + 2R_{(1,5,35,0,0,0)}(n) + 2R_{(1,10,20,-10,0,0)}(n) = H^{(5,14)}(28n)$
$G_{140,700,7}$	$4 \cdot 5^2 \cdot 7$	$R_{(5,5,7,0,0,0)}(n) + 4R_{(3,5,12,0,-2,0)}(n) + 4R_{(2,8,13,6,2,2)}(n) = H^{(7,10)}(28n)$
$G_{140,700,70}$	$4 \cdot 5^2 \cdot 7$	$2R_{(5,6,6,-2,0,0)}(n) = H^{(70,1)}(28n)$
$G_{140,140,2}$	$4 \cdot 5 \cdot 7$	$2R_{(1,6,6,-2,0,0)}(n) + 2R_{(2,3,7,0,0,-2)}(n) = H^{(2,35)}(140n)$
$G_{140,140,5}$	$4 \cdot 5 \cdot 7$	$4R_{(1,3,12,-2,0,0)}(n) + 2R_{(1,5,7,0,0,0)}(n) + 2R_{(2,4,5,0,0,-2)}(n) = H^{(5,14)}(140n)$
$G_{140,140,7}$	$4 \cdot 5 \cdot 7$	$R_{(1,1,35,0,0,0)}(n) + 4R_{(1,4,9,-2,0,0)}(n) + 4R_{(3,4,4,-2,-2,-2)}(n) = H^{(7,10)}(140n)$
$G_{140,140,70}$	$4 \cdot 5 \cdot 7$	$2R_{(1,2,18,-2,0,0)}(n) = H^{(70,1)}(140n)$
$G_{140,19600,5}$	$16 \cdot 5^2 \cdot 7^2$	$3R_{(7,20,40,20,0,0)}(n) + 2R_{(3,47,47,-46,-2,-2)}(n) + 3R_{(12,12,35,0,0,4)}(n) = 3H^{(5,7)}(n)$
$G_{140,19600,7}$	$16 \cdot 5^2 \cdot 7^2$	$R_{(4,35,36,0,-4,0)}(n) + 2R_{(11,15,39,-10,-6,-10)}(n) = H^{(7,5)}(n)$
$G_{140,3920,5}$	$16 \cdot 5 \cdot 7^2$	$3R_{(4,7,36,0,-4,0)}(n) + 2R_{(11,11,11,-6,-6,-6)}(n) + 3R_{(4,8,35,0,0,-4)}(n) = 3H^{(5,7)}(5n)$
$G_{140,3920,7}$	$16 \cdot 5 \cdot 7^2$	$R_{(7,12,12,-4,0,0)}(n) + 2R_{(3,19,19,10,2,2)}(n) = H^{(7,5)}(5n)$
$G_{140,2800,5}$	$16 \cdot 5^2 \cdot 7$	$3R_{(1,20,40,-20,0,0)}(n) + 2R_{(9,9,9,-2,-2,-2)}(n) + 3R_{(4,5,36,0,-4,0)}(n) = 3H^{(5,7)}(7n)$
$G_{140,2800,7}$	$16 \cdot 5^2 \cdot 7$	$R_{(5,12,12,-4,0,0)}(n) + 2R_{(8,8,13,-4,-4,-4)}(n) = H^{(7,5)}(7n)$
$G_{140,560,5}$	$16 \cdot 5 \cdot 7$	$3R_{(4,5,8,0,-4,0)}(n) + 2R_{(4,4,13,4,4,4)}(n) + 3R_{(1,12,12,-4,0,0)}(n) = 3H^{(5,7)}(35n)$
$G_{140,560,7}$	$16 \cdot 5 \cdot 7$	$R_{(1,4,36,-4,0,0)}(n) + 2R_{(4,5,9,-2,0,-4)}(n) = H^{(7,5)}(35n)$

Table 3: Class number of primitive positive definite ternary quadratic forms of level $4N$

$4N$	$\lvert C(4N)\rvert$	$4N$	$\lvert C(4N)\rvert$	$4N$	$\lvert C(4N)\rvert$
4	1	340	108	668	98
12	8	348	140	692	86
20	8	356	54	708	200
28	10	364	124	716	96
44	14	372	124	724	86
52	12	380	148	732	224
60	48	388	50	740	212
68	16	404	60	748	212
76	18	412	62	764	110
84	52	420	384	772	82
92	22	428	60	780	600
116	22	436	52	788	88
124	26	444	164	796	106
132	68	452	56	804	224
140	76	460	156	812	236
148	22	476	180	820	216
156	80	492	160	836	244
164	30	508	68	844	96
172	28	516	164	852	248
188	38	524	80	860	276
204	92	532	152	868	232
212	32	548	66	876	244
220	100	556	74	884	236
228	88	564	176	892	114
236	44	572	200	908	116
244	36	580	172	916	106

260	96	596	76	924	680
268	38	604	80	932	102
276	112	620	200	940	252
284	50	628	76	948	264
292	40	636	216	956	132
308	112	644	200	964	110
316	48	652	74	988	264
332	54	660	496	996	268

Acknowledgements

This research was supported partially by the National Natural Science Foundation of China(NSFC), Grant no. 12271405. We would like to express our appreciation to Professor Hiroki Aoki for his invitation to deliver a talk at the conference "Research on automorphic forms" , which was held at RIMS, Kyoto university between 22rd to 26th, January, 2024.

References

[1] Alexander Berkovich and William C. Jagy. On representation of an integer as the sum of three squares and ternary quadratic forms with the discriminants p^2, $16p^2$. *J. Number Theory*, 132(1):258–274, 2012. 5

[2] Hatice Boylan, Nils-Peter Skoruppa, and Haigang Zhou. Counting zeros in quaternion algebras using Jacobi forms. *Trans. Amer. Math. Soc.*, 371(9):6487–6509, 2019. 2, 5

[3] Heinrich Brandt and Oskar Intrau. Tabellen reduzierter positiver ternärer quadratischer Formen. *Abh. Sächs. Akad. Wiss. Math.-Nat. Kl.*, 45(4):261, 1958. 1

[4] Tuoping Du. Ternary quadratic forms and Heegner divisors. *Ramanujan J.*, 39(1):61–82, 2016. 5

[5] Martin Eichler. *Quadratische Formen und orthogonale Gruppen*, volume Band 63 of *Die Grundlehren der mathematischen Wissenschaften*. Springer-Verlag, Berlin-New York, 1974. Zweite Auflage. 1

[6] Lei Gao and Hourong Qin. Ternary quadratic forms and the class numbers of imaginary quadratic fields. *Comm. Algebra*, 47(11):4605–4640, 2019. 2

[7] Benedict H. Gross. Heights and L-series. In *Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Berkeley, Calif., 1986)*, pages 425–433. Amer. Math. Soc., Providence, RI, 1987. 2

[8] Xuejun Guo, Yuzhen Peng, and Hourong Qin. On the representation numbers of ternary quadratic forms and modular forms of weight 3/2. *J. Number Theory*, 140:235–266, 2014. 2

[9] Burton W. Jones. *The Arithmetic Theory of Quadratic Forms*, volume no. 10 of *Carcus Monograph Series*. Mathematical Association of America, Buffalo, NY, 1950. 2

[10] Ben Kane, Daejun Kim, and Srimathi Varadharajan. Explicit class number formulas for Siegel-Weil averages of ternary quadratic forms. *Trans. Amer. Math. Soc.*, 376(3):1625–1652, 2023. 2

[11] J. Larry Lehman. Levels of positive definite ternary quadratic forms. *Math. Comp.*, 58(197):399–417, S17–S22, 1992. 1

[12] Yan-Bin Li. *Jacobi Forms of Squarefree Level and Eichler Orders in Definite Quaternion Algebra*. Tongji University, 2021. PhD Thesis. 6

[13] Yan-Bin Li, Nils-Peter Skoruppa, and Haigang Zhou. Eichler orders and Jacobi forms of squarefree level. *J. Number Theory*, 236:349–387, 2022. 2, 3, 4

[14] Yifan Luo and Haigang Zhou. The classification and representations of positive definite ternary quadratic forms of level $4N$. *prepint, 10.48550/arxiv.2402.17443*, 2024. 1

[15] Goro Shimura. Classification, construction, and similitudes of quadratic forms. *Amer. J. Math.*, 128(6):1521–1552, 2006. 1

[16] John Voight. *Quaternion algebras*, volume 288 of *Graduate Texts in Mathematics*. Springer, Cham, [2021] ©2021. 2

[17] Xueli Wang and Dingyi Pei. *Modular forms with integral and half-integral weights*. Science Press Beijing, Beijing; Springer, Heidelberg, 2012. 4