Tightness and related properties in subtheories of Peano Arithmetic

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Abstract

It was shown by Visser that Peano Arithmetic has the property that any two bi-interpretable extensions of it (in the same language) are equivalent. Enayat proposed to refer to this property of a theory as *tightness*. He proved that PA and other theories such as ZF enjoy even stronger variants of this property, and he asked whether PA has a proper subtheory that is tight.

We report on work in progress with Piotr Gruza and Mateusz Łełyk that shows, among other things, that there are tight proper subtheories of PA and that tightness can be separated from one of its stronger variants called neatness.

Introduction. In this note we report on an ongoing project [GKŁ] concerned with the behaviour of subtheories of first-order arithmetic with respect to some properties of axiomatic theories defined in terms of interpretability.

We assume that the reader has at least an intuitive understanding of what an interpretation of a theory V in a theory U is. We will write $g : U \rhd V$ to indicate that g is such an interpretation. Since we are concerned with theories of arithmetic, which always have a pairing function, we may assume that all our interpretations are *one-dimensional*, i.e. an element of the universe of the interpreted theory V is always interpreted as an element of the universe of the interpreting theory U, not as a k-tuple for some $k \geq 2$. On the other hand, we allow interpretations to be *relative*, in the sense that the universe of V-objects can be a definable proper subset of the universe of U-objects. In principle, we do not require *absolute identity*, i.e. identity between V-objects can be interpreted as an equivalence relation on U-objects other than identity, but in practice all the interpretations that matter for us below will have absolute identity.

We also assume some basic familiarity with Peano Arithmetic (PA) and its subtheories, especially with the theories $\mathrm{I}\Sigma_n$ and $\mathrm{B}\Sigma_n$. All relevant background material on this topic can be found in [Kay91].

Two theories U and V are *bi-interpretable* if there exist interpretations $g\colon U \rhd V$ and $h\colon V \rhd U$ such that $h \circ g$ is U-provably isomorphic to the identity interpretation of U in U and $g \circ h$ is V-provably isomorphic to the identity interpretation of V in V (composition of interpretations and the identity interpretation are defined in a natural way). In [Vis06], Visser proved the following

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result about PA:

$$\text{If } U, V \text{ are bi-interpretable theories extending PA formulated in the language of PA, then } U \equiv V. \quad (1)$$

In fact, [Vis06] contains a stronger result:

$$\text{If } U, V \text{ are theories extending PA formulated in the language of PA, and } U \text{ is a retract of } V\text{, then } U \vDash V. \quad (2)$$

Here U is a *retract* of V in the sense of interpretability if there exist interpretations $g\colon U \rhd V$ and $h\colon V \rhd U$ witnessing "one half" of the definition of bi-interpretability: $h \circ g$ is U-provably isomorphic to the identity interpretation on U.

Enayat [Ena16] proposed to say that a theory T is *tight* if it has a property analogous to (1), that is any two bi-interpretable extensions of T formulated in the language of T are equivalent; and to say that T is *neat* if it has a property analogous to (2). Note that both tightness and neatness are preserved upwards w.r.t. logical strength for theories in the same language. There are also stronger, "semantical" versions of tightness and neatness, called *semantical tightness* and *solidity* respectively [FH21, Ena16], in which the antecedents are stated in terms of interpretations between models of T and the consequents are stated in terms of definable isomorphism rather than implication or elementary equivalence. However, in this text we focus on the syntactically defined properties.

Prior work. In [Ena16], Enayat showed that not only PA, but also other important schematically axiomatized theories such as ZF and second-order arithmetic are neat, and *a fortiori* tight. However, it has turned out that natural proper fragments of those theories tend not to be tight. More specifically, the following theories (and some others not mentioned here) are known not to be tight:

- ZFC^- (ZFC without Power Set and with Collection instead of Replacement), as well as Zermelo set theory Z [FH21].
- the fragment Π^1_n-CA of second-order arithmetic, for any $n \in \mathbb{N}$ [FW23].
- $\mathrm{I}\Sigma_n$, for any $n \in \mathbb{N}$. In fact, the Π_n fragment of true arithmetic and any Π_n-axiomatized subtheory of ZFC, for any $n \in \mathbb{N}$ [EŁ24].

Let us sketch how to prove the non-tightness of $\mathrm{I}\Sigma_n$. Let U be true arithmetic, i.e. $U = \mathrm{Th}(\mathbb{N})$. Let V be the theory of the model $\mathbb{K}$ defined as follows: take an arithmetically definable completion of $\mathrm{PA} + \neg\,\mathrm{Con}(\mathrm{PA})$, let $\mathbb{H}$ be its Henkin model (which is definable in $\mathbb{N}$, by the *arithmetized completeness theorem*, cf. [Kay91, Chapter 13.2]), and let $\mathbb{K}$ consist of the Σ_{n+1}-definable elements of $\mathbb{H}$ (namely, those $a \in H$ such that $\{a\}$ is Σ_{n+1}-definable without parameters in $\mathbb{H}$). See Figure 1.

It is known that $\mathbb{K}$ is a Σ_{n+1}-elementary substructure of $\mathbb{H}$: thus, it satisfies $\mathrm{I}\Sigma_n$, which is a Π_{n+2}-axiomatized theory. We also know that $\mathbb{K} \not\equiv \mathbb{N}$, because $\mathbb{K}$ satisfies $\neg\,\mathrm{Con}(\mathrm{PA})$.

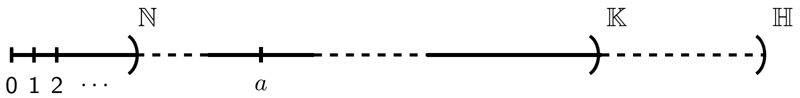

Figure 1: Construction of the model $\mathbb{K}$ witnessing the non-tightness of $I\Sigma_n$. The solid horizontal lines represent $\mathbb{K}$, which is a pointwise Σ_{n+1}-definable substructure of the Henkin structure $\mathbb{H}$. The dashed horizontal lines represent the rest of $\mathbb{H}$.

However, the theories of $\mathbb{N}$ and $\mathbb{K}$, i.e. U and V, are bi-interpretable. On the level of models, $\mathbb{N}$ can interpret $\mathbb{K}$ by formalizing the construction of $\mathbb{H}$ and $\mathbb{K}$ described above (here it matters that $\mathbb{H}$ was definable in $\mathbb{N}$), while $\mathbb{K}$ can interpret $\mathbb{N}$ as a particular definable cut: a number a is *not* standard if every element has a Σ_{n+1} definition with Gödel number below a. Moreover, one can check that following one interpretation and then the other, in whichever order, leads to a model definably isomorphic to the one we started in. The isomorphism between $\mathbb{N}$ and its copy in $\mathbb{K}$ is given by mapping $m \in \mathbb{N}$ to the m-th smallest element of $\mathbb{K}$. We let the reader figure out the details of the isomorphism between $\mathbb{K}$ and its copy defined inside the standard cut of $\mathbb{K}$.

Putting it all together, $\mathrm{Th}(\mathbb{N})$ and $\mathrm{Th}(\mathbb{K})$ are bi-interpretable theories extending $I\Sigma_n$ which are not equivalent, so $I\Sigma_n$ is not tight.

Main theorem. The non-tightness results mentioned above give additional force to the observation, made already by Enayat in [Ena16], that proofs of tightness-like properties for a theory like PA and ZF use the 'full power' of the theory. Enayat asked whether any of those theories has a neat proper subtheory[1]. Another natural question is whether tightness can be separated from its apparently stronger cousins: for example, is there a theory that is tight but not neat?

We are now able to answer both of these questions in the affirmative for the case of PA. Namely, in [GKŁ] we prove among other things the following.

Theorem 1. *For every $n \in \mathbb{N}$ there exists:*

(a) a neat c.e. subtheory T_n of PA *that contains* $I\Sigma_n$ *but not* $B\Sigma_{n+1}$*,*

(b) a tight but not neat c.e. subtheory S_{n+1} of PA *that contains* $B\Sigma_{n+1}$ *but not* $I\Sigma_{n+1}$*.*

Below, we sketch the proof of (b) for $n = 0$. To keep things reasonably straightforward, we focus on a simplified version where we do not pay attention to making S_1 c.e. and contained in PA. At the end of the sketch, we briefly explain how to get rid of these simplifications. Afterwards, we comment on some ideas that go into the proof of (a).

A tight but not neat theory. In the proof of non-tightness of $I\Sigma_n$, we used a pointwise Σ_{n+1}-definable model $\mathbb{K}$, which is a typical model of $I\Sigma_n$ but not

[1] In [Ena16], the question is stated for the semantic variant of neatness, solidity. Our answer discussed below works in that setting as well. Also, to rule out trivial examples, one should probably require the proper subtheory to have at least some minimal amount of strength.

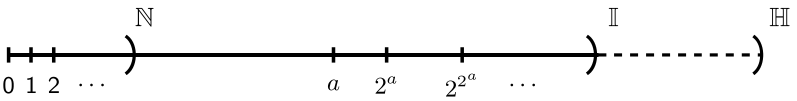

Figure 2: Construction of the model $\mathbb{I}$ of the tight but not neat theory S_1. The solid horizontal line represents $\mathbb{I}$, which is an initial segment of the Henkin structure $\mathbb{H}$. The dashed horizontal line represents the rest of $\mathbb{H}$.

$\mathrm{B}\Sigma_{n+1}$. We now take advantage of the fact that a suitably chosen proper cut in a model of a sufficiently strong theory satisfies $\mathrm{B}\Sigma_{n+1}$ but not $\mathrm{I}\Sigma_{n+1}$.

Consider the model $\mathbb{I}$ defined as follows: once more, begin with an arithmetically definable completion of $\mathrm{PA} + \neg\,\mathrm{Con(PA)}$, and let $\mathbb{H}$ be its Henkin model. Now, however, instead of taking exactly the Σ_{n+1}-definable elements of $\mathbb{H}$ (which will never form an initial segment of $\mathbb{H}$), let a be the smallest inconsistency proof for PA in $\mathbb{H}$, and let $\mathbb{I}$ be the initial segment of $\mathbb{H}$ generated by the standard iterations of the exponential function on a, that is by $2^a, 2^{2^a}, \ldots$ See Figure 2.

As mentioned, we know on general grounds that $\mathbb{I}$ satisfies $\mathrm{B}\Sigma_1$, and it satisfies the axiom exp (totality of the exponential function) by our choice of $\mathbb{I}$. We also know that it does not satisfy $\mathrm{I}\Sigma_1$: the standard cut, which is a proper cut of $\mathbb{I}$, is Σ_1-definable in $\mathbb{I}$, because it consists of exactly those ℓ for which there exists a number that is the result of iterating the exponential function ℓ times starting with a.

There are interpretations between $\mathbb{N}$ and $\mathbb{I}$ very similar to those between $\mathbb{N}$ and $\mathbb{K}$: the interpretation of $\mathbb{I}$ in $\mathbb{N}$ is based on formalizing the construction of $\mathbb{H}$ and $\mathbb{I}$, and we have just seen that $\mathbb{N}$ can be interpreted in $\mathbb{I}$ as a definable cut. Once again, mapping $m \in \mathbb{N}$ to the m-th smallest element of $\mathbb{I}$ gives an $\mathbb{N}$-definable isomorphism between $\mathbb{N}$ and its copy in $\mathbb{I}$.

As might be expected, it is also true that $\mathbb{I}$ is isomorphic to its copy defined inside the standard cut of $\mathbb{I}$. However, perhaps contrary to what might be expected, the isomorphism cannot be defined in $\mathbb{I}$. In fact, there is no definable injection from $\mathbb{I}$ into a proper initial segment of $\mathbb{I}$, because of the following result.

Theorem. *[Kay95] For each n, the theory $\mathrm{B}\Sigma_n + \exp + \neg\mathrm{I}\Sigma_n$ proves the* cardinality scheme*: "for each number ℓ, there is no definable injective multifunction with the universe as domain and with range bounded by ℓ".*

We can now define (our simplified version of) S_1 as the following theory:

$$\{\mathrm{I}\Sigma_1 \to \varphi : \varphi \in \mathrm{Th}(\mathbb{N})\} \cup \{\neg\,\mathrm{I}\Sigma_1 \to \psi : \psi \in \mathrm{Th}(\mathbb{I})\}.$$

Note that this makes sense, because $\mathrm{I}\Sigma_n$ is finitely axiomatizable for each $n \geq 1$. Clearly, S_1 is a subtheory of $\mathrm{Th}(\mathbb{N})$ containing $\mathrm{B}\Sigma_1$ but not $\mathrm{I}\Sigma_1$. We have to show that S_1 is tight, but also that it is not neat.

We begin with the latter. As our pair of theories witnessing the failure of neatness, we take $U = \mathrm{Th}(\mathbb{N})$ and $V = \mathrm{Th}(\mathbb{I})$. Both of these theories are extensions of S_1. Also, $\mathrm{Th}(\mathbb{N})$ is a retract of $\mathrm{Th}(\mathbb{I})$: we have already mentioned that the isomorphism between $\mathbb{N}$ and its copy in $\mathbb{I}$ is definable in $\mathbb{N}$; clearly then, $\mathrm{Th}(\mathbb{N})$ knows that this is an isomorphism. However, $\mathrm{Th}(\mathbb{N}) \not\vdash \mathrm{Th}(\mathbb{I})$; in fact, $\mathrm{Th}(\mathbb{N})$ and $\mathrm{Th}(\mathbb{I})$ are distinct consistent complete theories, so they are mutually contradictory. Thus, S_1 is not neat.

On the other hand, if there were two theories U, V witnessing the non-tightness of S_1, then we could assume w.l.o.g. that they are complete. The theory S_1 has only two complete extensions, namely $\mathrm{Th}(\mathbb{N})$ and $\mathrm{Th}(\mathbb{I})$, so w.l.o.g. $U = \mathrm{Th}(\mathbb{N})$ and $V = \mathrm{Th}(\mathbb{I})$. But this would imply in particular that $\mathrm{Th}(\mathbb{I})$ is a retract of $\mathrm{Th}(\mathbb{N})$ – which can be ruled out by a slightly more careful version of the cardinality scheme argument described above! So, S_1 is tight.

To make S_1 computably axiomatized, and a subtheory of PA, one essentially "replaces $\mathrm{Th}(\mathbb{N})$ by PA" in the definitions above. This also requires replacing $\mathrm{Th}(\mathbb{I})$ by a computably axiomatized subtheory that captures enough of the situation for the arguments to work. Then one still has to check that PA is sufficiently strong for the construction of the Henkin model $\mathbb{H}$ to go through. This involves another change: we can no longer insist $\mathbb{H}$ satisfying $\mathrm{PA} + \neg\,\mathrm{Con}(\mathrm{PA})$, because PA does not know that this theory is consistent. We can use $\mathrm{I}\Sigma_1 + \neg\,\mathrm{Con}(\mathrm{I}\Sigma_1)$ instead.

Generalizing the proof so as to obtain S_{n+1} for all $n \geq 0$ involves mostly routine tricks.

Other results. In the construction of the neat theory containing $\mathrm{I}\Sigma_n$ but not $\mathrm{B}\Sigma_{n+1}$ needed for Theorem 1 part (a), we go back to the model $\mathbb{K}$ used in the proof of non-tightness of $\mathrm{I}\Sigma_n$, but with a crucial change. The standard model $\mathbb{N}$ is replaced by its expansion $(\mathbb{N}, \mathrm{Th}(\mathbb{N}))$, where $\mathrm{Th}(\mathbb{N})$ is the interpretation of a new unary relation symbol that we refer to as a *truth predicate*. We want the "new version" of $\mathbb{K}$ to be a structure in the original language of PA, but at the same time we want $\mathrm{Th}(\mathbb{K})$ to be bi-interpretable with $\mathrm{Th}(\mathbb{N})$. To this end, we employ a Σ_1*-flexible formula*, which is a Σ_1 formula that consistently with PA can be equivalent to any other Σ_1 formula. We ensure that in $\mathbb{K}$, the elements of the standard cut that satisfy the flexible formula are exactly the Gödel numbers of sentences true in $\mathbb{N}$.

Our theory is then roughly

$$\{\mathrm{B}\Sigma_{n+1} \to \varphi : \varphi \in \mathrm{Th}(\mathbb{N})\} \cup \{\neg\,\mathrm{B}\Sigma_{n+1} \to \psi : \psi \in \mathrm{Th}(\mathbb{K})\},$$

where of course $\mathbb{K}$ is the new version. To turn this into the theory T_n which is c.e. and (properly) contained in PA, we "replace $\mathrm{Th}((\mathbb{N}, \mathrm{Th}(\mathbb{N})))$ by CT[PA]" – where CT[PA] is a well-known axiomatic theory of truth that extends PA by adding a unary relation symbol Tr and saying that it satisfies the usual inductive conditions for a truth predicate. Naturally, this leads to some further changes downstream.

Part (a) of our theorem can be improved by making T_n "even more distant from PA" in various ways: for example, we can demand that T_n be unable to interpret PA. This requires a more involved construction.

We find it interesting that the proof of Theorem 1 employs a relatively large set of tools from the proof theory and model theory of arithmetic: e.g. arithmetized completeness, the cardinality scheme, pointwise definable models, axiomatic truth theories, flexible formulas. It also seems interesting that our constructions of peculiar, necessarily non–finitely-axiomatizable tight theories rely on a method originally used to prove *non*-tightness of finitely axiomatized theories.

The paper [GKŁ] will also contain other results, for example additional separations between tightness-like properties.

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