LOGICAL DEPTH AND DEEP Π^0_1 CLASSES

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Abstract. Deep Π^0_1 classes, which were introduced in a previous paper of the authors as a generalization of the work of Levin, were so named because they exhib properties similar to those of the notion of logical depth, due to Bennett. In this paper we show that the two notions are in fact much more than just similar, but very closely connected indeed.

1. Introduction

Bennettintroduced the notion of logical depth in [Ben95] as a measure of complexity, formulated in terms the amount of computation time required to reproduce a given object. Whereas the Kolmogorov complexity of a string $\sigma \in 2^{<\omega}$ measures the length of the shortest input given to a fixed universal machine that reproduces σ as its output, logical depth measures the number of steps it takes to recover σ from this shortest input. Bennett further defined a sequence $X \in 2^\omega$ to be

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strongly deep if for every computable function t the logical depth of almost all of the initial segments $X{\upharpoonright}n$ of X is greater than $t(n)$.

Bennett established several fundamental facts about strongly deep sequences, namely that the halting set K is strongly deep, that no computable sequence and no Martin-Löf random sequence is strongly deep, and that strong depth is closed upwards under truth-table reducibility (a result he referred to as the *slow growth law*). Bennett further introduced the notion of weak depth, where a sequence is weakly deep if it is not truth-table reducible to a random sequence.

An analogue of deep sequences for Π^0_1 classes, i.e., effectively closed subsets of 2^ω, was developed by the present authors in [BP16]. The authors isolated the notion of a deep Π^0_1 class as a generalized of work of Levin [Lev13], who implicitly showed that the Π^0_1 class of consistent completions of Peano arithmetic is deep. The basic idea, made precise in the next section, is that a Π^0_1 class $\mathcal{P}$ is deep if the probability of computing some length n initial segment of some member of $\mathcal{P}$ via some Turing functional equipped with a random oracle rapidly approaches zero as n grows without bound. In [BP16], the authors proved a number of results about deep Π^0_1 classes, including an analogue of the slow growth law for deep Π^0_1 classes in the Medvedev degrees, as well as identifying a number of examples of deep Π^0_1 classes based on properties studied in computability theory and algorithmic information theory.

The aim of this study is to show that the relationship between strongly deep sequences and deep Π^0_1 classes is no mere analogy. In particular, we prove that every member of a deep Π^0_1 class is strongly deep, from which it follows that we gain a significant number of newly identified examples of strongly deep sequences based on results from [BP16]. We further show that a strongly deep sequence need not be a member of a deep Π^0_1 class. Next, as every deep Π^0_1 class is negligible, in the sense that the probability of computing a member of such a class with a Turing functional equipped with a random oracle is zero, in light of the fact that all members of deep Π^0_1 classes are strongly deep, it is reasonable to ask whether the collection of strongly deep sequences is negligible. We answer this question in the affirmative, while further showing that the collection of sequences that are deep with respect to any fixed time bound is not negligible.

One takeaway we aim to emphasize in this study is the importance of the slow growth law for the study of depth, akin to the role of randomness preservation in the study of algorithmic randomness. According to the latter, every sequence that is truth-table reducible to a sequence that is random with respect to a computable measure is itself random with respect to a computable measure, which is precisely the dual of the slow growth law for deep sequences. We anticipate that the slow growth law will continue to be a useful tool in the study of notions of depth.

2. Background

2.1. **Turing functionals.** Recall that a *Turing functional* $\Phi :\subseteq 2^\omega \to 2^\omega$ can be defined in terms of a c.e. set S_Φ of pairs of strings (σ, τ) such that if $(\sigma, \tau), (\sigma', \tau') \in S_\Phi$ and $\sigma \preceq \sigma'$, then $\tau \preceq \tau'$ or $\tau' \preceq \tau$. For each $\sigma \in 2^{<\omega}$, we define Φ^σ to be the maximal string in $\{\tau : (\exists \sigma' \preceq \sigma)(\sigma', \tau) \in S_\Phi\}$ in the order given by $\preceq$. To obtain a map defined on 2^ω from the set S_Φ, for each $X \in 2^\omega$, we let Φ^X be the maximal $y \in 2^{<\omega} \cup 2^\omega$ in the order given by $\preceq$ such that $\Phi^{X\upharpoonright n}$ is a prefix of y for all n.

2.2. **Semimeasures.** A *discrete semimeasure* is a function $m : 2^{<\omega} \to [0,1]$ satisfying $\sum_{\sigma\in 2^{<\omega}} m(\sigma) \leq 1$. Similarly, *a continuous semimeasure* is a function $P : 2^{<\omega} \to [0,1]$ satisfying (i) $P(\varnothing) \leq 1$ and (ii) $P(\sigma) \geq P(\sigma 0) + P(\sigma 1)$ for all $\sigma \in 2^{<\omega}$. Given a continuous semimeasure P and some $S \subseteq 2^{<\omega}$, we set $P(S) = \sum_{\sigma\in S} P(\sigma)$.

A discrete semimeasure m is computable if its output values $m(\sigma)$ are computable uniformly in the input $\sigma \in 2^{<\omega}$ (and similarly for continuous semimeasures). Here we will also consider *lower semicomputable* semimeasures (both discrete and continuous), where a function $f : 2^{<\omega} \to [0,1]$ is lower semicomputable if each value $f(\sigma)$ is the limit of a computable, nondecreasing sequence of rationals, uniformly in $\sigma \in 2^{<\omega}$.

An important development due to Levin [LZ70] was the identification of *universal* semimeasures: for discrete semimeasures, m is universal if for every lower semicomputable measure m_0 there is some constant c such that $m_0(\sigma) \leq c \cdot m(\sigma)$. Similarly, a continuous semimeasure M is universal if for every lower semicomputable measure P there is some constant c such that $P(\sigma) \leq c \cdot M(\sigma)$. Hereafter, $\mathbf{m}$ and $\mathbf{M}$ will denote fixed universal discrete and continuous semimeasures, respectively.

2.3. **Initial segment complexity.** Recall that the prefix-free Kolmogorov complexity of a string $\tau \in 2^{<\omega}$ is defined by setting $K(\tau) = \min\{|\sigma| : U(\sigma)\downarrow = \tau\}$, where U is a fixed universal prefix-free machine (i.e., recall that a machine M is prefix-free if for $\sigma, \rho \in 2^{<\omega}$, if $M(\sigma)\downarrow$ and $\sigma \prec \rho$, then $M(\rho)\uparrow$). Moreover, we can define time-bounded versions of Kolmogorov complexity. A function $t : \omega \to \omega$ is called a *time bound* if t is total and non-decreasing. Then for a fixed computable time bound t, the t-time-bounded complexity of $\tau \in 2^{<\omega}$ is defined by setting $K^t(\tau) = \min\{|\sigma| : U(\sigma)\downarrow = \tau \text{ in } \leq t(|\tau|) \text{ steps}\}$.

Note that by Levin's coding theorem, $K(\sigma) = -\log \mathbf{m}(\sigma) + O(1)$ for all $\sigma \in 2^{<\omega}$. A similar relationship holds for computable discrete semimeasures and time-bounded Kolmogorov complexity. First, we define a time-bounded version of $\mathbf{m}$ as follows. As $\mathbf{m}$ is lower semi-computable, for each $s \in \omega$, we have an approximation $\mathbf{m}_s$ of $\mathbf{m}$ (i.e., for each $\sigma \in 2^{<\omega}$, $\mathbf{m}_s(\sigma)$ is the s-th rational number in computable sequence that converges to $\mathbf{m}(\sigma)$). Then given a computable time bound t, we set $\mathbf{m}^t(\sigma) = \mathbf{m}_{t(|\sigma|)}$, which is clearly a computable semimeasure. We will make use of the following lemma from [BDM23] (where for functions f, g, $f \leq^\times g$ means that there is some c such that $f(n) \leq c \cdot g(n)$ for all $n \in \omega$).

Lemma 1 ([BDM23])**.**

(i) *For every computable discrete semimeasure m, there is some computable time bound t such that $m \leq^\times \mathbf{m}^t$.*

(ii) *For every computable time bound t, 2^{-K^t} is a computable discrete semimeasure.*

(iii) *For every computable time bound t, there is some computable time bound t' such that $2^{-K^t} \leq^\times \mathbf{m}^{t'}$).*

In addition, we need the following theorem (see, e.g. [JLL94, Theorem 4.3(2)]).

Theorem 2. *For every computable time bound t, there is a computable time bound t' such that $\mathbf{m}^t \leq^\times 2^{-K^{t'}}$.*

Note that by combining Lemma 1(iii) and Theorem 2, we obtain a resource-bounded analogue of Levin's coding theorem.

In the case of continuous semimeasures, we directly define $KA(\sigma) := -\log \mathbf{M}(\sigma)$ to be the *a priori complexity* of $\sigma \in 2^{<\omega}$. Just as we defined $\mathbf{m}^t$ for any computable time bound t, we can similarly define $\mathbf{M}^t$, which is a computable continuous semimeasure. Moreover, we can establish the analogue of Lemma 1(i): For every computable continuous semimeasure P, there is some computable time bound t such that $P \leq^{\times} \mathbf{M}^t$. We will also define $KA^t := \mathbf{M}^t$ for any given computable time bound t.

Lastly, we define monotone complexity in terms of monotone machines, where a monotone machine $M : 2^{<\omega} \to 2^{<\omega}$ satisfies the property that for $\sigma, \tau \in \mathrm{dom}(M)$, if $\sigma \preceq \tau$, then either $M(\sigma) \preceq M(\tau)$ or $M(\tau) \preceq M(\sigma)$. Given a universal monotone machine U, we set $Km(\tau) = \min\{|\tau| \colon U(\sigma){\downarrow} \succeq \tau\}$. Given a computable time bound, we can also define Km^t in the obvious way.

2.4. **Randomness and depth notions.** Given a computable measure μ on 2^ω (i.e., a measure on 2^ω where the values $\mu([\![\sigma]\!])$ are computable uniformly in $\sigma \in 2^{<\omega}$), recall that a *μ-Martin-Löf test* is a uniformly Σ^0_1 sequence $(U_i)_{i\in\omega}$ such that $\mu(U_i) \leq 2^{-i}$. Recall further that a sequence $X \in 2^\omega$ passes the test $(U_i)_{i\in\omega}$ if $X \notin \bigcap_{i\in\omega} U_i$ and X is *μ-Martin-Löf random* if it passes all μ-Martin-Löf tests. In the case that μ is the Lebesgue measure on 2^ω (which we denote by λ), we will refer to λ-Martin-Löf random sequences simply as Martin-Löf random sequences.

Next, $X \in 2^\omega$ is *strongly deep* if for every computable time bound t, we have $K^t(X{\upharpoonright}n) - K(X{\upharpoonright}n) \to \infty$. A slightly stronger notion is given by order-depth, where $X \in 2^\omega$ is *order-deep* if there is a computable order function $g : \omega \to \omega$ such that $K^t(X{\upharpoonright}n) - K(X{\upharpoonright}n) \geq g(n)$ for almost every $n \in \omega$. Here we use the term 'order function', or simply 'order' to mean a non-decreasing and unbounded function. When h

is such a function, $h^{-1}(k)$ denotes the smallest n such that $h(n) \geq k$. Note that h^{-1} is computable when h is.

In the rest of the paper we will sometimes use an equivalent characterization of order-depth, given by the following lemma, whose proof we will omit here.

Lemma 3. *For $X \in 2^\omega$, the following are equivalent.*

(i) *X is order-deep*

(ii) *For some computable increasing function h, for any computable time bound t and almost all n, $K^t(X{\upharpoonright}h(n)) - K(X{\upharpoonright}h(n)) \geq^+ n$.*

(iii) *For some computable increasing function h, for any computable time bound t and almost all n, $\frac{\mathbf{m}(X{\upharpoonright}h(n))}{\mathbf{m}^t(X{\upharpoonright}h(n))} \geq^\times 2^n$.*

One of the key properties of strong depth is the slow growth law, given in terms of truth-table reductions. Recall that a *tt*-functional is a Turing functional that is total on all oracles; equivalently, there is a computable function f such that for all $X \in 2^\omega$, $|\Phi^{X{\upharpoonright}f(n)}| \geq n$.

Theorem 4 (Slow Growth Law [Ben95]). *For $X, Y \in 2^\omega$, if X is strongly deep and $X \leq_{tt} Y$, then Y is strongly deep.*

The slow growth law also holds for order-depth.

Bennett proved that no computable sequence and no Martin-Löf random sequence is strongly deep. Hereafter, we will refer to sequences that are not strongly deep as being *shallow*. Bennett further showed that the halting set $\emptyset' = \{e : \phi_e(e)\downarrow\}$ (where $(\phi_e)_{e\in\omega}$ is a standard enumeration of the partial computable functions) is strongly deep.

Bennett defined a weaker notion of depth: a sequence $X \in 2^\omega$ is *weakly deep* if X is not *tt*-reducible to a Martin-Löf random. By the slow growth law and the fact that no Martin-Löf random sequences are strongly deep, it follows that every strongly deep sequence is weakly deep; as shown by Bennett [Ben95], the converse does not hold. Note that it is a folklore result that a sequence is not truth-table reducible to a Martin-Löf random sequence if and only if it is not random with respect to a computable measure, thereby providing an alternative characterization of weak depth.

2.5. **Deep Π^0_1 classes and negligiblity.** As noted in the introduction, the authors in [BP16] introduced the notion of a deep Π^0_1 class as the abstraction of a phenomenon first isolated by Levin in [Lev13] Given a Π^0_1 class $\mathcal{P}$, recall that there is a canonical co-c.e. tree $T \subseteq 2^{<\omega}$ such that $\mathcal{P} = [T]$, i.e., $\mathcal{P}$ is the collection of all infinite paths through T; more specifically, this tree T is the set of all initial segments of members of $\mathcal{P}$. For $n \in \omega$, let T_n be the set of all strings in T of length n. We say that a Π^0_1 class $\mathcal{P}$ is *deep* if there is some order g such that $\mathbf{M}(T_n) \leq 2^{-g(n)}$. Equivalently, $\mathcal{P}$ is deep is there is some order h such that $\mathbf{M}(T_{h(n)}) \leq 2^{-n}$.

An analogue of the slow growth law holds for deep Π^0_1 classes in a suitable degree structure, namely the strong degrees (also referred to as the Medvedev degrees). Given Π^0_1 classes $\mathcal{P}$ and $\mathcal{Q}$, we say that $\mathcal{P}$ is strongly reducible to $\mathcal{Q}$, written $\mathcal{P} \leq_s \mathcal{Q}$, if there is some Turing functional Φ such that for every $Y \in \mathcal{Q}$, there is some $X \in \mathcal{P}$ such that $X = \Phi(Y)$; equivalently, we have $\Phi(\mathcal{Q}) = \mathcal{P}$. As noted in [BP16], we can assume here that Φ is a *tt*-functional, a fact that will be useful in this study. Then we have:

Theorem 5 (Slow Growth Law for Π^0_1 classes, [BP16])**.** *For Π^0_1 classes $\mathcal{P}, \mathcal{Q} \subseteq 2^\omega$, if $\mathcal{P}$ is deep and $\mathcal{P} \leq_s \mathcal{Q}$, then $\mathcal{Q}$ is deep.*

Depth for Π^0_1 classes implies a property that holds more broadly for subsets of 2^ω, namely the property of being negligible. First, observe that a lower semicomputable semimeasure P can be trimmed back to a measure $\overline{P} \leq P$ (see [BHPS14] details). In particular, we can trim back the universal lower semicomputable semimeasure $\mathbf{M}$ to get a measure $\overline{\mathbf{M}}$. One key result concerning $\overline{\mathbf{M}}$ is that for a measurable set $\mathcal{A} \subseteq 2^\omega$, $\overline{\mathbf{M}}(\mathcal{A}) = 0$ if and only if $\lambda(\{X : (\exists Y \in \mathcal{A})\ Y \leq_T X\}) = 0$; that is, from the point of view of Lebesgue measure, only relatively few sequences can compute of member of $\mathcal{A}$. Following Levin (see for example [Lev84]), we call such sets $\mathcal{A}$ *negligible*. As we can equivalently consider the collection of random sequences that compute a member of $\mathcal{A}$, we can recast negligibility in terms of probabilistic computation: a collection $\mathcal{A}$ is negligible if the probability of probabilistically computing a member of $\mathcal{A}$ is zero. Note that every deep Π^0_1 class is thus negligible; in fact, we can interpret the property of depth for a Π^0_1 class $\mathcal{P}$ as the property that the probability of computing the first n bits of a member of $\mathcal{P}$ converges to 0 effectively in $n \in \omega$. As shown in [BP16], not every negligible Π^0_1 class is deep.

3. Members of deep Π^0_1 classes

When deep Π^0_1 classes were defined in [BP16], the authors referred to the notion as a type of depth in analogy with Bennett's original notion of logical depth (as, for instance, an analogue of the slow growth law for deep Π^0_1 classes was established in [BP16]). We now show that the connection between these two depth notions is much closer than merely satisfying an analogy, as we prove that the members of deep Π^0_1 classes are strongly deep; in fact, we prove the stronger result that all such members are order-deep.

Theorem 6. *Every member of a deep Π^0_1 class $\mathcal{P}$ is order-deep.*

Proof. Let T be the canonical co-c.e. tree associated to the Π^0_1 class $\mathcal{P}$. Let h be a computable order such that

$$\sum_{\sigma \in T_{h(n)}} \mathbf{M}(\sigma) \leq 2^{-2n}.$$

for all n.

Let t be a computable time bound. By virtue of the inequalities $\mathbf{M} \geq^{\times} \mathbf{m} \geq^{\times} \mathbf{m}^t$, the above inequality implies

$$\sum_{\sigma \in T_{h(n)}} \mathbf{m}^t(\sigma) \leq 2^{-2n}.$$

Since $\mathbf{m}^t$ is computable and T is co-c.e., one can effectively compute a sequence s_n such that

$$\sum_{\sigma \in T_{h(n)[s_n]}} \mathbf{m}^t(\sigma) \leq 2^{-2n}.$$

Let now p be the computable discrete semimeasure defined by $p(\tau) = 2^n \cdot \mathbf{m}^t(\tau)$ if τ belongs to $T_{h(n)}[s_n]$ for some $n > 0$, and $p(\tau) = 0$ otherwise. That p is computable is clear from the definition (and the computability of the sequence s_n), and that it is a semimeasure follows from

$$\sum_{\tau} p(\tau) = \sum_{n>0} \sum_{\tau \in T_{h(n)}[s_n]} 2^n \cdot \mathbf{m}^t(\tau) \leq \sum_{n>0} 2^n \cdot 2^{-2n} \leq 1.$$

Now, if X is a member of $\mathcal{P}$, that is, X is a path through T, then for each $n > 0$, we have $X{\upharpoonright}h(n) \in T_{h(n)}$ and thus

$$p(X{\upharpoonright}h(n)) = 2^n \cdot \mathbf{m}^t(X{\upharpoonright}h(n)).$$

As $\mathbf{m} \geq^{\times} p$ (because p is a discrete semi-measure), we have

$$\mathbf{m}(X{\upharpoonright}h(n)) \geq^{\times} 2^n \cdot \mathbf{m}^t(X{\upharpoonright}h(n))$$

By Lemma 3, we can conclude that X is order-deep. □

3.1. **Additional examples of deep sequences.** Theorem 6 also allows us to derive a number of examples of deep sequences. In [BP16] it was shown that the following collections of sequences form deep Π^0_1 classes:

(1) the collection of consistent completions of Peano arithmetic;
(2) the collection of (α, c)-shift-complex sequences for computable $\alpha \in (0, 1)$ and $c \in \omega$, where a sequence $X \in 2^\omega$ is (α, c)-shift-complex if $K(\tau) \geq \alpha|\tau| - c$ for every substring τ of X;
(3) the collection of DNC_q functions with $\sum_{n\in\omega} \frac{1}{q(n)} = \infty$, where $f : \omega \to \omega$ is a DNC_q function if f is total function such that $f(n) \neq \phi_n(n)$ and $f(n) < q(n)$ for all $n \in \omega$;
(4) the collection of codes of infinite sequences of finite sets $(F_0, F_1, \dots)$ of strings of maximal complexity, i.e., there are computable functions $\ell, f, d : \omega \to \omega$ such that for all $n \in \omega$, (i) $|F_n| = f(n)$, (ii) $|\sigma| = \ell(n)$ for $\sigma \in F_n$, and (iii) $K(\sigma) \geq \ell(n) - d(n)$ for $\sigma \in F_n$; and
(5) the collection of codes of K-compression functions with constant c, where $g : 2^{<\omega} \to \omega$ is a K-compression function with constant c if (i) $g(\sigma) \leq K(\sigma) + c$ for all $\sigma \in 2^{<\omega}$ and (ii) $\sum_{\sigma\in 2^{<\omega}} 2^{-g(\sigma)} \leq 1$.

As an immediate consequence of Theorem 6 and the above results from [BP16], we have:

Corollary 7. *Every sequence in the following collections is strongly deep:*

(1) *the collection of consistent completions of Peano arithmetic;*

(2) *the collection of shift-complex sequences (i.e., the sequences that are (α, c)-shift complex for some computable $\alpha \in (0,1)$ and $c \in \omega$);*

(3) *the collection of codes of* DNC_q *functions with* $\sum_{n\in\omega} \frac{1}{q(n)} = \infty$*;*

(4) *the collection of codes of infinite sequences of finite sets of strings of maximal complexity; and*

(5) *the collection of codes of K-compression functions (i.e., K-compression functions with constant c for some $c \in \omega$).*

We can obtain further examples of members of deep Π^0_1 classes using a version of the slow growth law for members of deep Π^0_1 classes.

Lemma 8. *If X is a member of a deep Π^0_1 class and $X \leq_{tt} Y$, then Y is a member of a deep Π^0_1 class.*

Proof. Let $\mathcal{P}$ be a Π^0_1 class contain X, and let Φ be a total Turing functional satisfying $\Phi(Y) = X$. Then by the slow growth law for deep Π^0_1 classes, $\Phi^{-1}(\mathcal{P})$ is a deep Π^0_1 class that contains Y, which must be strongly deep by Theorem 6. ☐

Theorem 9.

(i) *The halting set $\emptyset' = \{e \in \omega \colon \phi_e(e)\downarrow\}$ is a member of a deep Π^0_1 class.*

(ii) *For every $X \in 2^\omega$, $X' = \{e \colon \phi^X_e(e)\downarrow\}$ is a member of a deep Π^0_1 class.*

(iii) *Every non-trivial index set is a member of a deep Π^0_1 class.*

Proof. (i) There is a DNC_2 function f such that $f \leq_{tt} \emptyset'$ (see [Nie09, Remark 1.8.30]). Since the collection of DNC_2 functions forms a deep Π^0_1 class, the result follows from Lemma 8.

(ii) Since $\emptyset' \leq_{tt} X'$ for every $X \in 2^\omega$ and $\emptyset'$ is a member of a deep Π^0_1

class by part (i), the result follows from Lemma 8.

(iii). By Rice's theorem, every non-trivial index set C satisfies $\emptyset' \leq_1 C$ or $\overline{\emptyset'} \leq_1 C$, and so the result follows again from part (i) and Lemma 8. □

3.2. **Separating depth notions.** In light of Theorem 6, it is natural to ask whether every order-deep sequence is a member of a deep Π^0_1 class. We show that this does not hold by establishing several propositions of independent interest.

Recall that a sequence X is complex if there is some computable order g such that $K(X{\upharpoonright}n) \geq g(n)$. As shown explicitly in [HP17], one can equivalently define a sequence to be complex in terms of a priori complexity, i.e., X is complex if and only if there is some computable order h such that $KA(X{\upharpoonright}n) \geq h(n)$. We use this second characterization to derive the following:

Proposition 10. *Every member of a deep Π^0_1 class is complex.*

Proof. Let $\mathcal{P}$ be a deep Π^0_1 class with associated co-c.e. tree T. Then there is some computable order $h : \omega \to \omega$ such that $\mathbf{M}(T_n) \leq 2^{-h(n)}$. Given $X \in \mathcal{P}$, since $X{\upharpoonright}n \in T_n$, we have $\mathbf{M}(X{\upharpoonright}n) \leq \mathbf{M}(T_n) \leq 2^{-h(n)}$. Taking the negative logarithm yields $KA(X{\upharpoonright}n) \geq h(n)$, from which the conclusion follows. □

Next, we have:

Proposition 11. *No sequence that is Turing equivalent to an incomplete c.e. set is a member of a deep Π^0_1 class.*

Proof. Suppose that X is a member of a deep Π^0_1 class and is Turing equivalent to some incomplete c.e. set Y. By Proposition 10, X is complex. It follows from work of Kjos-Hanssen, Merkle, and Stephan

[KHMS11] that X has DNC degree. But then Y is an incomplete c.e. set of DNC degree, which contradicts Arslanov's completeness criterion (see, e.g., [Nie09, Theorem 4.1.11]). □

Theorem 12. *There is an order-deep sequence that is not complex (hence is not a member of any deep Π^0_1 class).*

Proof. In [JLL94], Juedes, Lathrop, and Lutz introduced the notion of weakly useful sequence (we do not recall the definition of this notion here and refer the reader to their paper) and showed that (i) every weakly useful sequence is order-deep and (ii) every high degree contains a weakly useful sequence. Our theorem then follows: Let X be high, incomplete, and weakly useful (hence order-deep). By the same reasoning in the proof of Proposition 11, X is not complex. □

We note another consequence of Proposition 11, namely that the leftmost path of every deep Π^0_1 class is Turing complete. Indeed, the leftmost path of a Π^0_1 class has c.e. degree and thus must be Turing complete by Proposition 11.

Having separated order-depth from being a member of a deep Π^0_1 class, we can use a similar line of reasoning to further separate order-depth from Bennett's original notion of depth. We need one auxiliary result.

Theorem 13 (Moser, Stephan [MS17])**.** *Every order-deep sequence is either high or of DNC degree.*

Theorem 14. *There is a strongly deep sequence that is not order-deep.*

Proof. Downey, MacInerney, and Ng [DMN17] constructed a low, deep sequence A of c.e. degree. As A can neither be high nor of DNC degree (as it is incomplete), it follows from Theorem 13 that A is not order-deep. □

We have seen by Theorem 6 that members of deep Π^0_1 classes are order-deep, and by Proposition 10 that they are complex. We end this section by showing that these two properties alone are not enough to characterize members of deep Π^0_1 classes.

Theorem 15. *There exists a sequence X which is complex, order-deep, and not a member of any deep Π^0_1 class.*

We will need the following auxiliary lemma of independent interest. Recall that the join $Y \oplus Z$ of two sequences Y and Z is the sequence obtained by interleaving their bits: $Y \oplus Z = Y(0)Z(0)Y(1)Z(1)\ldots$. Similarly, for two strings σ and τ of the same length we can define $\sigma \oplus \tau$ in the same way.

Lemma 16. *If a sequence Y is not a member of any deep Π^0_1 class, then for almost every Z (in the sense of Lebesgue measure), $Y \oplus Z$ is not a member of any deep Π^0_1 class.*

Proof. We prove this lemma by contrapositive. Suppose that Y is such that for positive measure many sequences Z, $Y \oplus Z$ is a member of a deep Π^0_1 class. If this is so, as there are only countably many deep Π^0_1 classes, this means that there is a fixed deep Π^0_1 class $\mathcal{C}$ such that with probability $> \delta$ over Z (with δ a positive rational), we have that $Y \oplus Z$ belongs to $\mathcal{C}$. Consider the Π^0_1 class $\mathcal{D}$ consisting of sequences A such that for any n, there are at least $\delta \cdot 2^n$ strings τ such that $(A{\upharpoonright}n) \oplus \tau$ is in the canonical tree T of $\mathcal{C}$. By definition, Y belongs to $\mathcal{D}$. We claim that $\mathcal{D}$ is deep, which will prove the lemma. Since $\mathcal{C}$ is deep, there is a computable order h such that $\mathbf{M}(T_{2n}) < 1/h(n)$ for all n.

Let P be the continuous semimeasure defined on strings of even length by $P(\sigma \oplus \tau) = \mathbf{M}(\sigma)\lambda(\tau)$. If S is the canonical co-c.e. tree of $\mathcal{D}$, then by definition of $\mathcal{D}$, we have

$$P(T_{2n}) \geq \mathbf{M}(S_n) \cdot \delta.$$

By universality of $\mathbf{M}$, we also have $\mathcal{P}(T_{2n}) \leq^\times \mathbf{M}(T_{2n}) < 1/h(n)$. Putting these two inequalities together, it follows that

$$\mathbf{M}(S_n) \leq^\times \frac{1}{\delta \cdot h(n)}$$

which shows $\mathcal{D}$ is a deep Π^0_1 class. □

Proof of Theorem 15. Having proven Lemma 16, take now Y a sequence that is order-deep and not a member of any deep Π^0_1 class (whose existence was established in Theorem 12). Pick Z at random and form the sequence $X = Y \oplus Z$. With probability 1 over Z:

- X is complex. Indeed $K(X{\upharpoonright}2n) \geq^+ K(Z{\upharpoonright}n) \geq^+ n$ by the Levin-Schnorr theorem.
- X is not a member of any deep Π^0_1 class by Lemma 16.

Moreover, regardless of the value of Z, X *tt*-computes Y which is order-deep, hence by the slow growth law for order-deep sequences, X is itself order-deep. These three properties tell us that with probability 1 over Z, $X = Y \oplus Z$ is as desired.

□

4. Strong Depth is Negligible

While we know that deep Π^0_1 classes must all be negligible, we have established (Theorems 6 and 12) that the collection of strongly deep sequences forms a strict superclass of the collection of members of all deep Π^0_1 classes. It is therefore natural to ask whether the collection of strongly deep sequences is negligible, which we answer in the affirmative.

Theorem 17. *The class of strongly deep sequences is negligible.*

Proof. For the sake of contradiction, assume there exists a functional Φ such that

$$\lambda\{X \ : \ \Phi^X \text{ is deep }\} > 0.9$$

(where we choose this latter value without loss of generality by the Lebegue Density Theorem). Let p be a computable semimeasure such that $\liminf_n \frac{\mathbf{m}(n)}{p(n)} < \infty$. The existence of such a p follows from the existence of *Solovay functions* (see [BDNM15]), which are functions $f \geq^+ K$ such that $\liminf_n f(n) - K(n) < \infty$. Setting $p(n) = 2^{-f(n)-c}$ with f a Solovay function and c large enough gives us the properties of p we need.

We now define a computable discrete semimeasure as follows. For every $n \in \omega$, effectively find a family of clopen sets $\{C_\sigma \ : \ |\sigma| = n\}$ such that $\Phi^X \succeq \sigma$ for all $X \in C_\sigma$ and $\sum_{|\sigma|=n} \lambda(C_\sigma) > 0.9$. Then, set for all σ of length n:

$$q(\sigma) = \lambda(C_\sigma) \cdot p(n)$$

It is clear that q is computable. Moveover, q is a discrete semimeasure, since

$$\sum_{\sigma\in 2^{<\omega}} q(\sigma) = \sum_{n\in\omega}\sum_{|\sigma|=n} \lambda(C_\sigma)\cdot p(n) = \sum_{n\in\omega} p(n) \sum_{\sigma\in 2^{<\omega}\,:\,|\sigma|=n} \lambda(C_\sigma) \leq \sum_{n\in\omega} p(n) \leq 1.$$

For any Y that is strongly deep, we must have $\frac{\mathbf{m}(Y\restriction n)}{q(Y\restriction n)} \to \infty$. For all (n, d), define

$$B^d_n = \{\sigma \ : \ |\sigma| = n \ \text{ and } \ \mathbf{m}(\sigma) > d \cdot q(\sigma)\}$$

By our hypothesis on the functional Φ, this means that for every constant d, for almost all n, $\lambda(\{X : \Phi^X{\upharpoonright}n \in B_n^d\}) > 0.9$.

Now, consider the quantity $\sum_{\sigma \in B_n^d} \lambda(C_\sigma)$. On the one hand,

$$\begin{aligned}
\sum_{\sigma \in B_n^d} \lambda(C_\sigma) &= \sum_{\sigma \in B_n^d} \frac{q(\sigma)}{p(n)} && \text{(by definition of } q\text{)} \\
&\leq \sum_{\sigma \in B_n^d} \frac{\mathbf{m}(\sigma)}{d \cdot p(n)} && \text{(by definition of } B_n^d\text{)} \\
&\leq \frac{1}{d \cdot p(n)} \sum_{|\sigma|=n} \mathbf{m}(\sigma) \\
&\leq \frac{\mathbf{m}(n) \cdot O(1)}{d \cdot p(n)} && \text{(using the identity } \sum_{|\sigma|=n} \mathbf{m}(\sigma) =^{\times} \mathbf{m}(n)\text{)}
\end{aligned}$$

On the other hand, for almost all n:

$$\begin{aligned}
\sum_{\sigma \in B_n^d} \lambda(C_\sigma) &\geq \lambda(\{X : \Phi^X{\upharpoonright}n \in B_n^d\}) - 0.1 && \text{(because } \lambda(\bigcup_{|\sigma|=n} C_\sigma) > 0.9\text{)} \\
&\geq 0.9 - 0.1 \\
&\geq 0.8
\end{aligned}$$

Putting the two together, we have established that for all d, for almost all n, $\frac{\mathbf{m}(n)}{p(n)} > d/O(1)$, i.e., $\lim_n \frac{\mathbf{m}(n)}{p(n)} = \infty$. This contradicts the choice of p. □

Note, by contrast, that the collection of weakly deep sequences is not negligible. Indeed, as shown by Muchnik et al. [MSU98], no 1-generic sequence is Martin-Löf random with respect to a computable measure, and thus every 1-generic is weakly deep. Moreover, as shown by Kautz

[Kau91], every 2-random sequence computes a 1-generic, and hence the collection of 1-generics is not negligible.

As the collection of strongly deep sequences is negligible, it is worth asking whether the collection of sequences that are strongly deep with respect to one fixed computable time bound is negligible. We first introduce some notation. For a computable time bound t and $c \in \omega$, let $D^t_c(n) = \{X \in 2^\omega \colon K^t(X{\restriction}n) - K(X{\restriction}n) \geq c\}$, which is clopen uniformly in n, hence $\bigcap_{n\geq m} D^t_c(n)$ is a Π^0_1 class. In addition, we set $D^t_c = \bigcup_{m\in\omega}\bigcap_{n\geq m} D^t_c(n)$. Then we have:

Theorem 18. *Let t be a computable time bound and $c \in \omega$. Then D^t_c is not negligible and hence does not consist entirely of strongly deep sequences.*

Proof. Suppose on the contrary that D^t_c is negligible for some computable time bound t and $c \in \omega$. Then for each $m \in \omega$, $\bigcap_{n\geq m} D^t_c(n)$ is a negligible Π^0_1 class. As shown in [BP16, Theorem 5.2], no weakly 2-random sequence can compute a member of a negligible Π^0_1 class, hence no weakly 2-random sequence can compute a member of $\bigcap_{n\geq m} D^t_c(n)$ (recall that $X \in 2^\omega$ is weakly 2-random if X is not contained in any Π^0_2 of Lebesgue measure zero). It follows that no weakly 2-random sequence can compute a member of D^t_c. However, there is some weakly 2-random sequence that computes a sequence of high Turing degree [Kau91], and as shown by Juedes, Lathrop, and Lutz [JLL94], every high degree contains a strongly deep sequence. In particular, every high degree contains an element of D^t_c, and thus there is some weakly 2-random sequence that computes a member of D^t_c, a contradiction. Note further that under the assumption that D^t_c consists entirely of strongly deep sequences, then by Theorem 17, it would follow that D^t_c is negligible, which we have shown cannot hold. □

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