Epistemic possibility in
Artemov and Protopopescu's intuitionistic epistemic logic

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1 Introduction

Epistemic logics formalize knowledge and related concepts. Artemov and Protopopescu [1] defined an epistemic logic IEL to formalize intuitionistic knowledge. The key point of this logic is that intuitionistic truth implies intuitionistic knowledge. In this paper, we characterize epistemic possibility in IEL. We also extend IEL's Brouwer–Heyting–Kolmogorov interpretation to epistemic possibility.

We contrast IEL to the epistemic logic S5. Both S5 and IEL have a modality K for knowledge such that, for all formula φ,

$$K\varphi \text{ holds iff it is known that } \varphi.$$

Furthermore, both logics satisfy positive introspection $K\varphi \to KK\varphi$ and negative introspection $\neg K\varphi \to K\neg K\varphi$. The key difference between them is that, in S5, knowledge means classical knowledge and, in IEL, knowledge means intuitionistic knowledge. Classical knowledge satisfies the reflection principle: $K\varphi$ implies φ. Intuitionistic knowledge on the other hand satisfies co-reflection: $\varphi \to K\varphi$. While intuitionistic knowledge does not satisfy the reflection principle, it satisfies a weaker version of it: $K\varphi \to \neg\neg\varphi$. We will see that co-reflection is incompatible with classical knowledge.

We extend IEL with a modality $\hat{K}$ for epistemic possibility. To do so, we use the semantics of diamonds in constructive modal logics [5]. We will show that, for all formula φ, $\hat{K}P$ is equivalent to $\neg\neg P$. This implies that φ is epistemic possible iff it one can show that it is impossible to prove the negation of φ.

IEL has also been extended with distributed knowledge by Su, Murai and Sano [6] and to first-order logic by Su and Sano [7]. Note that there are other intuitionistic approaches to epistemic logic in the literature. For a brief survey, see Section 6 of [1]. As far as the author is aware, epistemic possibility has not been studied in other intuitionistic approaches to epistemic logic.

We now describe the structure of this paper. In Section 2, we review the classical epistemic logic S5. In Section 3, we review IEL, its BHK interpretation. In Section 4, we define and characterize epistemic possibility in IEL.

2 The Classical Epistemic Logic S5

In this section we review the classical epistemic logic S5. This is only one of many epistemic logics in the literature; we review it because it is one of the most studied ones and because it shares many of its axioms with IEL. Also note that, as defined, this logic can only treat one epistemic agent. For more on epistemic logics in general, see Fagin *et al.*[3]. For other axiom systems for one agent epistemic logic, see Lenzen [4].

Fix a set of propositional symbols Prop. The formulas of S5 are defined by the following grammar:

$$\varphi := P \mid \neg\varphi \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \varphi \to \varphi \mid K\varphi \mid \hat{K}\varphi,$$

where $P \in \text{Prop}$. Here, $K\varphi$ is read as "it is known that φ" and $\hat{K}\varphi$ as "it is epistemically possible that φ". Note that some of the symbols can be defined in terms of others. In particular, $K\varphi$ will be equivalent to

$\neg\hat{K}\neg\varphi$ and $\hat{K}\varphi$ will be equivalent to $\neg K\neg\varphi$. We use this maximalist definition because the interdefinability of K and $\hat{K}$ does not hold in the intuitionistic setting.

The formulas of S5 will be interpreted over Kripke models whose accessibility relation is an equivalence relation. Formally, an S5 Kripke model M is a triple $\langle W, R, V\rangle$ where:

- W is a set of possible worlds;
- $R \subseteq W \times W$ is an equivalence relation over W; and
- $V : \text{Prop} \to \mathcal{P}(W)$ is a valuation function.

The valuation function assign to each propositional symbol P the set of worlds $V(P)$ where P holds. We extend V to the set of all formulas as follows:

- $V(\neg\varphi) := W \setminus V(\varphi)$;
- $V(\varphi \wedge \psi) := V(\varphi) \cap V(\psi)$;
- $V(\varphi \vee \psi) := V(\varphi) \cup V(\psi)$;
- $V(\varphi \to \psi) := (W \setminus V(\varphi)) \cup V(\psi)$;
- $V(K\varphi) := \{w \in W \mid \text{for all } v \in W,\ wRv \text{ implies } v \in V(\varphi)\}$; and
- $V(\hat{K}\varphi) := \{w \in W \mid \text{there is } v \in W \text{ such that } wRv \text{ and } v \in V(\varphi)\}$.

We say a formula φ holds over a model $M = \langle W, R, V\rangle$ iff $V(\varphi) = W$.

We now define the logic S5. The axioms of S5 are:

- the classical tautologies;
- $K := K(\varphi \to \psi) \to (K\varphi \to K\varphi)$;
- $T := K\varphi \to \varphi$;
- $4 := K\varphi \to KK\varphi$; and
- $5 := \neg K\varphi \to K\neg K\varphi$.

S5 is also closed under the following inference rules:

- *modus ponens* := if $\varphi \to \psi \in$ IEL and $\varphi \in$ IEL, then $\psi \in$ IEL;
- necessitation := if $\varphi \in$ IEL, then $K\varphi \in$ IEL.

This axiomatization is complete for S5 models:

Theorem 1. *Let φ be a formula, then* S5 *proves φ iff φ holds over all* S5 *models.*

Proof. This holds by a canonical model argument. For details, see any standard reference such as [2]. □

The principle of reflection T will not be an axiom of IEL; instead a weaker form of reflection will appear there. The intuitionistic epistemic logic IEL satisfies the co-reflection principle $coT := \varphi \to K\varphi$. Co-reflection implies positive introspection 4 and negative introspection 5. If we add coT to S5, the resulting modal logic is equivalent propositional logic without fixed-point operators. This happens because T and coT together imply $\varphi \leftrightarrow K\varphi$ for all formula φ. Therefore co-reflection is not compatible with S5.

3 The Intuitionistic Epistemic Logic IEL

We now review the axiomatization, the BHK interpretation, and the semantics of IEL. For a more through exposition, see Artemov and Protopopescu's original paper [1].

Fix a set of propositional symbols Prop. The formulas of IEL are defined by the following grammar:

$$\varphi := P \mid \bot \mid \neg\varphi \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \varphi \to \varphi \mid K\varphi,$$

where $P \in$ Prop. Here, $K\varphi$ is read as "it is intuitionistically known that φ". Note that the set of propositional connectors cannot be simplified in IEL.

The axioms of IEL are:

- the intuitionistic tautologies;
- $K := K(\varphi \to \psi) \to (K\varphi \to K\varphi)$;
- $coT := \varphi \to K\varphi$; and
- $T' := K\varphi \to \neg\neg\varphi$.

IEL is also closed under *modus ponens*: if $\varphi \to \psi \in$ IEL and $\varphi \in$ IEL, then $\psi \in$ IEL.

Before we provide the BHK interpretation and semantics for IEL, we show that positive introspection and negative introspection hold in IEL, and that IEL is closed under necessitation.

Proposition 2. *Let φ be a modal formula. Then:*

- IEL *proves $K\varphi \to KK\varphi$;*
- IEL *proves $\neg K\varphi \to K\neg K\varphi$; and*
- *if* IEL *proves φ, then* IEL *proves $\Box\varphi$.*

Proof. Both $K\varphi \to KK\varphi$ and $\neg K\varphi \to K\neg K\varphi$ are instances of the co-reflection axiom *coT*. If IEL proves φ, then IEL proves $\Box\varphi$ by co-reflection and *modus ponens.* □

The BHK interpretation for propositional connectors is defined as usual:

- $\bot$ has no proof;
- a proof of $\varphi \wedge \psi$ consists in a proof of φ and a proof of ψ;
- a proof of $\varphi \vee \psi$ consists in giving either a proof of φ or a proof of ψ;
- a proof of $\varphi \to \psi$ consists in a construction which given a proof of φ returns a proof of ψ;
- a proof of $\neg\varphi$ consists in a construction which given a proof of φ returns a proof of $\bot$.

See Troelstra and van Dalen [8] for a discussion of the BHK interpretation for intuitionistic propositional logic. Artemov and Protopopescu [1] proposed the following interpretation for $K\varphi$:

- a proof of $K\varphi$ is conclusive evidence of verification that φ has a proof.

Examples of what conclusive evidence of verification may mean can be found in Section 2 of [1].

As in S5, the semantics of IEL is based on Kripke models. In IEL, we use bi-relational models; with one relation corresponding to the relation of Kripke models for intuitionistic propositional logic and one corresponding to relations in Kripke models for classical modal logics. Formally, IEL models are tuples $M = \langle W, \preceq, R, V\rangle$ where:

- W is a set of possible worlds;

- $\preceq$ and R are a binary relations over W; and
- $V : \text{Prop} \to \mathcal{P}(W)$ is a valuation function.

We require that IEL model M satisfies the following properties:

- $\preceq$ is reflexive and transitive;
- if $w \preceq v$ and $w \in V(P)$, then $v \in V(P)$;
- if wRv, then $w \preceq v$;
- $w \preceq v$ implies that, for all u, if vRu then wRu;
- for all w there is v such that wRv.

Again, the valuation function assign to each propositional symbol P the set of worlds $V(P)$ where P holds. The relation $\preceq$ is the intuitionistic relation and R the modal relation. One can think of advancing through the relation $\preceq$ as increasing of information; this justifies that it is a preorder and that propositions do not become false. R being a subset of $\preceq$ implies that any epistemic alternative can be attained by an increase of information. The fourth condition implies that any epistemic alternative available after an increase of information were already available before the increase. At last, there is always an epistemic alternative available.

We extend V to the set of all formulas as follows:

- $V(\bot) := \emptyset$;
- $V(\neg\varphi) := \{w \in W \mid \text{for all } v, \text{ if } w \preceq v \text{ then } v \notin V(\varphi)\}$;
- $V(\varphi \wedge \psi) := V(\varphi) \cap V(\psi)$;
- $V(\varphi \vee \psi) := V(\varphi) \cup V(\psi)$;
- $V(\varphi \to \psi) := \{w \in W \mid \text{for all } v, \text{ if } w \preceq v \text{ and } v \in V(\varphi) \text{ then } v \in V(\psi)\}$; and
- $V(K\varphi) := \{w \in W \mid \text{for all } v \in W,\ wRv \text{ implies } v \in V(\varphi)\}$.

We say a formula φ holds over a model $M = \langle W, \preceq, R, V\rangle$ iff $V(\varphi) = W$. We say a formula φ holds over a world w in a model $M = \langle W, \preceq, R, V\rangle$ iff $w \in V(\varphi)$. Note that semantics of K in IEL coincides with the semantics of K in S5, while semantics of the propositional connectives coincide with their semantics in intuitionistic propositional logic.

By structural induction of formulas, the monotonicity of V also extends to all modal formulas:

Proposition 3. *Let φ be a formula and M be an* IEL *model. If $w \in V(\phi)$ and $w \preceq v$, then $v \in V(\varphi)$.*

This axiomatization is complete for IEL models:

Theorem 4. *Let φ be a formula, then* IEL *proves φ iff φ holds over all* IEL *models.*

Proof. The proof of completeness uses canonical models for intuitionistic logics. See Section 4 of Artemov and Protopopescu [1] for details. ☐

4 Epistemic possibility in IEL

We are now ready to define a modality for epistemic possibility in IEL. After the definition, we show that:

Theorem 5. *For all* IEL *model* $M = \langle W, \preceq, R, V \rangle$*, world* w*, and formula* φ*,*

$$w \in V(\hat{K}P) \text{ iff } w \in V(\neg\neg P).$$

Proof. By Lemma 8 and Lemma 9. □

From the BHK interpretation of negation, we get that:

φ is epistemic possible iff it one can show that it is impossible to prove the negation of φ.

In this section, we consider formulas defined by the grammar:

$$\varphi := P \mid \bot \mid \neg\varphi \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \varphi \to \varphi \mid K\varphi \mid \hat{K}\varphi,$$

We extend valuations to formulas with $\hat{K}$ as follows:

Definition 1. Let $M = \{W, \preceq, R, V\}$ be an IEL model. Then $w \in V(\hat{K}\varphi)$ iff

for all v, if $w \preceq v$ then there is u such that vRu and $u \in V(\varphi)$.

This semantics are the same as the semantics of diamonds in constructive modal logics [5]. It permits extended valuation function V to be monotone with respect to $\preceq$:

Proposition 6. *Let* $M = \{W, \preceq, R, V\}$ *be an* IEL *model and* φ *be a formula. If* $w \in V(\hat{K}\varphi)$ *and* $w \preceq v$*, then* $v \in V(\hat{K}\varphi)$*.*

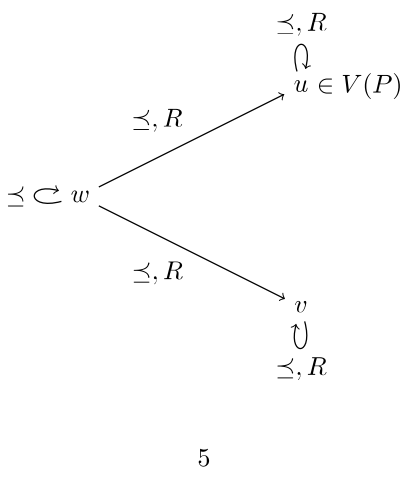
Proof. Suppose that the model M and the worlds w and v are as in the hypothesis of the theorem. By the hypothesis, for all u, if $w \preceq u$ then there is u' such that uRu' and $u' \in V(\varphi)$. By the transitivity of $\preceq$, if $v \preceq u$, then $w \preceq u$. Therefore, for all u, if $v \preceq u$ then there is u' such that uRu' and $u' \in V(\varphi)$. That is, $v \in V(\hat{\varphi})$. □

Not that while the semantics of the knowledge modality K in IEL is the same as in S5; the same cannot not happen to the semantics of $\hat{K}$. Indeed, if we used the classical semantics for $\hat{K}$, the monotonicity of the valuation function V would fail:

Proposition 7. *There is an* IEL *model* $M = \{W, \preceq, R, V\}$ *with worlds* $w, v \in W$ *satisfying:*

- $w \preceq v$*;*
- *there is* u *such that* wRu *and* $u \in V(P)$*;*
- *there is no* u *such that* vRu *and* $u \in V(P)$*.*

Proof. The following model witnesses the proposition:



□

We are now ready to characterize $\hat{K}\varphi$ in IEL as $\neg\neg\varphi$. For readability, we divide the proof in two lemmas.

Lemma 8. *For all* IEL *model* $M = \langle W, \preceq, R, V\rangle$*, world* w*, and formula* φ*, if* $w \in V(\hat{K}\varphi)$ *then* $w \in V(\neg\neg\varphi)$.

Proof. We have $w \in V(\neg\neg\varphi)$ iff

$$\text{for all } v \succeq w, \text{ there is } u \text{ such that } v \preceq u \text{ and } u \in V(\varphi).$$

We have $w \in V(\neg\neg P)$ implies $w \in V(\hat{K}P)$, since vRu implies $v \preceq u$. □

Lemma 9. *For all* IEL *model* $M = \langle W, \preceq, R, V\rangle$*, world* w*, and formula* φ*, if* $w \in V(\neg\neg\varphi)$ *then* $w \in V(\hat{K}\varphi)$.

Proof. We prove this by contradiction. Suppose that $\neg\neg P$ holds at w and $\hat{K}P$ fails at w. Then there is v such that $w \preceq v$ and, for all v', vRv' implies $v' \notin V(P)$. By the definition of IEL models, there is u' such that uRu' Since uRu' implies $u \preceq w'$ and $\preceq$ is monotone with respect to the valuation, $u' \in V(P)$. As $v \preceq u$ and uRu', we have vRu'. Therefore $v \preceq u'$ and $u' \notin V(P)$. Now, since $\neg\neg P$ holds at w, there is u such that $v \preceq u$ and $u \in V(P)$. This is a contradiction. □

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