An Algebraic Counterpart of Kripke Semantics Based on Completely Prime Elements[1]

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It is verified in [KF19, KF22] that we can employ the algebraic structure of completely distributive algebraic lattices to model the basic inference system $\mathbf{IPC_2}$ of second-order intuitionistic propositional logic. Actually, we have presented in [KF22] a framework of semantics based on this algebraic structure by analogy with the interpretation of neighbourhood semantics. This is mainly due to making use of the general theory of correspondence, so-called Stone duality, established between sober spaces and spatial lattices.

On the other hand, the structure of Kripke models is well known as a framework of semantics, with respect to which the system $\mathbf{IPC_2}$ is ensured to be complete. We refer its detailed discussion to [SU06]. Furthermore we know that Kripke semantics is considered to be a specific sort of neighbourhood semantics especially based on Alexandrov topology. These backgrounds allow us to have a possibility to present a version of interpretation of algebraic semantics in which we can interpret every proposition as well as the definition of the forcing relation in Kripke semantics. We demonstrate it in this short note, which is actually accomplished by focusing completely prime elements.

Here we give a brief review of the definition of our presented algebraic models, for which we mainly follow the notations and the terminologies in [KF22]. Let $\langle L, \sqsubseteq \rangle$ be a complete lattice. Then it is said to be algebraic if every element x of L is identical with the directed join of all compact elements below x. An element a of L is said to be completely prime if it satisfies

$$a \sqsubseteq \bigsqcup X \Longrightarrow \exists x \in X \quad a \sqsubseteq x$$

for every subset X of L. We use letters a, b, $c, \ldots$ to designate completely prime elements of L, and define $\mathsf{C}L$ to be the set of completely prime elements of L. We also define $\mathsf{C}L(x) = \{a \in \mathsf{C}L \mid a \sqsubseteq x\}$ for every $x \in L$. If a complete lattice $\langle L, \sqsubseteq \rangle$ is algebraic and completely distributive, then we have the equality

$$x = \bigsqcup \mathsf{C}L(x)$$

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for every $x \in L$, as is proved in [Win83, Corollary 8] and [Win09, Corollary 5]. This property is called the prime algebraicity of completely distributive algebraic lattice.

Let $\langle L, \sqsubseteq \rangle$ be a completely distributive algebraic lattice. Then we are supposed to have a mapping d which associate with every $a \in \mathsf{C}L$ a domain $d(a) \subseteq L$ and which have a nested structure that for every $a \in \mathsf{C}L$ there exists an element $x \in L$ satisfying $a \sqsubseteq x$ and

$$\forall b \in \mathsf{C}L(x) \;\; d(a) \subseteq d(b).$$

Then we call the triple $\mathscr{A} = \langle L, \sqsubseteq, d \rangle$ an algebraic model of $\mathbf{IPC_2}$. We define an environment ξ on $\mathscr{A}$ as a mapping which maps every propositional variable to an element of L. Then, for every proposition A and environment ξ, we define the interpretation $[\![A]\!]_\xi \in L$ by induction on the structure of A, as follows:

$$
\begin{aligned}
&[\![\bot]\!]_\xi = \bot, \\
&[\![p]\!]_\xi = \xi(p), \\
&[\![A \land B]\!]_\xi = [\![A]\!]_\xi \sqcap [\![B]\!]_\xi, \\
&[\![A \lor B]\!]_\xi = [\![A]\!]_\xi \sqcup [\![B]\!]_\xi, \\
&[\![A \to B]\!]_\xi = \bigsqcup\{x \in L \mid [\![A]\!]_\xi \sqcap x \sqsubseteq [\![B]\!]_\xi\}, \\
&[\![\forall p.A]\!]_\xi = \bigsqcup\{x \in L \mid \forall a \in \mathsf{C}L(x) \;\; \forall y \in d(a) \;\; a \sqsubseteq [\![A]\!]_{\xi(p:y)}\}, \\
&[\![\exists p.A]\!]_\xi = \bigsqcup\{x \in L \mid \forall a \in \mathsf{C}L(x) \;\; \exists y \in d(a) \;\; a \sqsubseteq [\![A]\!]_{\xi(p:y)}\}.
\end{aligned}
$$

We also define $[\![\Gamma]\!]_\xi = \bigsqcap_{A \in \Gamma} [\![A]\!]_\xi$ for every set Γ of propositions. We say that an algebraic model $\mathscr{A}$ is full if for every proposition A, $a \in \mathsf{C}L$ and environment ξ such that $\xi(\mathrm{FV}(A)) \subseteq d(a)$, we can find $x \in d(a)$ and $y \in L$ satisfying $a \sqsubseteq y$ and $x \sqcap y = [\![A]\!]_\xi \sqcap y$. Then a judgement $\Gamma \vdash A$ is said to be valid with respect to a full algebraic model $\mathscr{A}$ if

$$a \sqsubseteq [\![\Gamma]\!]_\xi \Longrightarrow a \sqsubseteq [\![A]\!]_\xi$$

holds in the model $\mathscr{A}$, and we write $\Gamma \models A$ if $\Gamma \vdash A$ is valid with respect to every full algebraic model. With respect to this notion of validity, the formal system $\mathbf{IPC_2}$ is ensured to be complete.

Theorem 1. $\Gamma \vdash A$ is derivable in $\mathbf{IPC_2}$ if and only if $\Gamma \models A$.

Against this framework of algebraic models, we can find a possibility of taking an alternative interpretation in accordance with the definition of the forcing relation in Kripke semantics more faithfully. To be more specifically, we are allowed to adopt the following version of interpretation:

$$
\begin{aligned}
&[\![\bot]\!]^{\mathsf{C}}_\xi = \bigsqcup \emptyset, \\
&[\![p]\!]^{\mathsf{C}}_\xi = \bigsqcup\{a \in \mathsf{C}L \mid a \sqsubseteq \xi(p)\}, \\
&[\![A \land B]\!]^{\mathsf{C}}_\xi = \bigsqcup\{a \in \mathsf{C}L \mid a \sqsubseteq [\![A]\!]^{\mathsf{C}}_\xi \text{ and } a \sqsubseteq [\![B]\!]^{\mathsf{C}}_\xi\}, \\
&[\![A \lor B]\!]^{\mathsf{C}}_\xi = \bigsqcup\{a \in \mathsf{C}L \mid a \sqsubseteq [\![A]\!]^{\mathsf{C}}_\xi \text{ or } a \sqsubseteq [\![B]\!]^{\mathsf{C}}_\xi\}, \\
&[\![A \to B]\!]^{\mathsf{C}}_\xi = \bigsqcup\{a \in \mathsf{C}L \mid \forall b \in \mathsf{C}L(a) \; (\, b \sqsubseteq [\![A]\!]^{\mathsf{C}}_\xi \text{ implies } b \sqsubseteq [\![B]\!]^{\mathsf{C}}_\xi \,)\},
\end{aligned}
$$

$$[\![\forall p.A]\!]^{\mathsf{C}}_{\xi} = \bigsqcup\{a \in \mathsf{C}L \mid \forall b \in \mathsf{C}L(a)\ \ \forall x \in d(b)\ \ b \sqsubseteq [\![A]\!]^{\mathsf{C}}_{\xi(p:x)}\},$$
$$[\![\exists p.A]\!]^{\mathsf{C}}_{\xi} = \bigsqcup\{a \in \mathsf{C}L \mid \exists x \in d(a)\ \ a \sqsubseteq [\![A]\!]^{\mathsf{C}}_{\xi(p:x)}\}.$$

For example, we have

$$\begin{aligned} a \sqsubseteq [\![A \to B]\!]^{\mathsf{C}}_{\xi} &\iff \exists c \in \mathsf{C}L\ \left[\begin{array}{l} a \sqsubseteq c \text{ and} \\ \forall b \in \mathsf{C}L(c)\ \ (\, b \sqsubseteq [\![A]\!]^{\mathsf{C}}_{\xi} \text{ implies } b \sqsubseteq [\![B]\!]^{\mathsf{C}}_{\xi}\,) \end{array}\right] \\ &\iff \forall b \in \mathsf{C}L(a)\ \ (\, b \sqsubseteq [\![A]\!]^{\mathsf{C}}_{\xi} \text{ implies } b \sqsubseteq [\![B]\!]^{\mathsf{C}}_{\xi}\,) \end{aligned}$$

under this version of interpretation because of the prime algebraicity. As this example demonstrates, for every proposition A, $a \in \mathsf{C}L$ and environment ξ we can consider the order relation $a \sqsubseteq [\![A]\!]^{\mathsf{C}}_{\xi}$ to be the condition that the completely prime element a forces the proposition A. In addition to this, we can show that the denotation of every proposition is invariant, as is shown in the lemma below. So the completeness of $\mathbf{IPC_2}$ is still ensured with respect to this interpretation.

Lemma 2. We have $[\![A]\!]_{\xi} = [\![A]\!]^{\mathsf{C}}_{\xi}$ for every proposition A.

Proof. It is verified easily by induction on the structure of A. Here we give a proof for several non-trivial cases:

Case 1: Suppose $A \equiv B \to C$. Then we have

$$\begin{aligned} [\![B \to C]\!]^{\mathsf{C}}_{\xi} &= \bigsqcup\{a \in \mathsf{C}L \mid [\![B]\!]^{\mathsf{C}}_{\xi} \sqcap a \sqsubseteq [\![C]\!]^{\mathsf{C}}_{\xi}\} \\ &= \bigsqcup\{a \in \mathsf{C}L \mid [\![B]\!]_{\xi} \sqcap a \sqsubseteq [\![C]\!]_{\xi}\} \end{aligned}$$

by induction hypothesis, from which $[\![B \to C]\!]^{\mathsf{C}}_{\xi} \sqsubseteq [\![B \to C]\!]_{\xi}$ follows immediately. To see the converse, suppose that $x \in L$ and $[\![B]\!]_{\xi} \sqcap x \sqsubseteq [\![C]\!]_{\xi}$. Then, for every $a \in \mathsf{C}L(x)$ we have $[\![B]\!]_{\xi} \sqcap a \sqsubseteq [\![B]\!]_{\xi} \sqcap x \sqsubseteq [\![C]\!]_{\xi}$, which implies $a \sqsubseteq [\![B \to C]\!]^{\mathsf{C}}_{\xi}$. So we obtain $x = \bigsqcup \mathsf{C}L(x) \sqsubseteq [\![B \to C]\!]^{\mathsf{C}}_{\xi}$.

Case 2: Suppose $A \equiv \forall p.B$. Then we have

$$[\![\forall p.A]\!]^{\mathsf{C}}_{\xi} = \bigsqcup\{a \in \mathsf{C}L \mid \forall b \in \mathsf{C}L(a)\ \ \forall x \in d(b)\ \ b \sqsubseteq [\![A]\!]_{\xi(p:x)}\}$$

by induction hypothesis, from which $[\![\forall p.B]\!]^{\mathsf{C}}_{\xi} \sqsubseteq [\![\forall p.B]\!]_{\xi}$ follows immediately. To see the converse, suppose that $x \in L$ and $b \sqsubseteq [\![A]\!]_{\xi(p:y)}$ for every $b \in \mathsf{C}L(x)$ and $y \in d(b)$. Suppose also that $a \in \mathsf{C}L(x)$. Then we have $b \sqsubseteq [\![A]\!]_{\xi(p:y)}$ for every $b \in \mathsf{C}L(a)$ and $y \in d(b)$ since $b \in \mathsf{C}L(a) \subseteq \mathsf{C}L(x)$. This implies $a \sqsubseteq [\![\forall p.A]\!]^{\mathsf{C}}_{\xi}$. So we obtain $x = \bigsqcup \mathsf{C}L(x) \sqsubseteq [\![\forall p.A]\!]^{\mathsf{C}}_{\xi}$.

Case 3: Suppose $A \equiv \exists p.B$. Then we first obtain

$$[\![\exists p.A]\!]_{\xi} = \bigsqcup\{a \in \mathsf{C}L \mid \forall b \in \mathsf{C}L(a)\ \ \exists x \in d(b)\ \ b \sqsubseteq [\![A]\!]^{\mathsf{C}}_{\xi(p:x)}\}$$

by analogy with the proof of the preceding case. This entails $[\![\exists p.A]\!]_{\xi} \sqsubseteq [\![\exists p.A]\!]^{\mathsf{C}}_{\xi}$ since $a \in \mathsf{C}L(a)$ for every $a \in \mathsf{C}L$. To see the converse, suppose we can find $x \in d(a)$ such that $a \sqsubseteq [\![A]\!]^{\mathsf{C}}_{\xi(p:x)}$. Then, for every $b \in \mathsf{C}L(a)$ we obtain $x \in d(a) \subseteq d(b)$ and $b \sqsubseteq a \sqsubseteq [\![A]\!]^{\mathsf{C}}_{\xi(p:x)}$. So $[\![\exists p.A]\!]^{\mathsf{C}}_{\xi} \sqsubseteq [\![\exists p.A]\!]_{\xi}$ results. □

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