Complex algebraic geometry around free rigid body dynamics

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1 Introduction

This paper gives a brief overview on complex algebro-geometric aspects of the integrable system of the free rigid body dynamics on the basis of the author's previous papers [4, 14, 15, 19] with Jean-Pierre Françoise and Isao Naruki.

In the theory of completely integrable systems, complex algebraic geometry plays important roles to explain the solvability of the problems, as found in textbooks of the topics, e.g. [1, 2]. Given the Lax equation which describes the integrable system, the spectral curve and the eigenvector mapping are particularly useful tools along the complex algebro-geometric studies. In suitable situations, the flow of the Lax equation is mapped to a linear flow over the Jacobian variety of the spectral curve through the eigenvector mapping.

The rotational motions of the heavy tops, i.e. rigid bodies under the constant gravity, are a class of typical problems in analytical mechanics. From the viewpoint of symplectic and Poisson geometries, the dynamical systems of heavy tops are formulated as Hamiltonian systems on the cotangent bundle $T^*SO(3)$ of the three-dimensional rotation group, equipped with the canonical symplectic structure, and, after the so-called Lie-Poisson reduction procedure, also as those on the dual space $(\mathfrak{so}(3) \ltimes \mathbb{R}^3)^*$ to the Lie algebra of the special Euclidean group, equipped with the Lie-Poisson structure. By the work of Ziglin [22], it is known that, among heavy tops, there are only four different cases where the dynamical system is completely integrable in the sense of Liouville; Euler top, Lagrange top, Kowalevsky top, and Goryachev-Chapligyn top.

Each of these systems allows four constants of motion given by polynomial functions on $\mathbb{R}^3 \times \mathbb{R}^3$ and the flow of the integrable systems are contained in the common level sets of the four constants of motion. For the simplicity, one can think of the

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common level sets from the viewpoint of complex algebraic/analytic geometry. Further, the four integrable systems are also described by Lax equations, their spectral curves, which are complex algebraic curves, and the eigenvector mappings. As one can let the values of the constants of motion vary, the complex algebraic varieties obtained as the complexification of the common level sets for the constants of motion form a complex fibration over certain appropriate base spaces. Similarly, the family of the spectral curves also gives rise to a complex fibrations.

Despite various previous works on integrable heavy tops from the viewpoint of complex algebraic geometry, the singularities of the above complex algebro-geometric objects have not been studied intensively. In other words, the global geometry, particularly described by the geometries around the singular fibres, of the fibrations have not been clarified well.

In view of the above perspective over the complex algebraic geometry of integrable heavy tops, the author of the present paper has been studied the complex algebro-geometric aspects of the Euler top in the collaborations with I. Naruki and J.-P. Françoise [4, 14, 15, 19]. As a result of [4, 15, 19], the singular fibres and the monodromy of the two elliptic fibrations arising from the family of complexified integral curves of the Euler top and from the family of the spectral curves of the Lax (Manakov) equation are classified. Then, the two elliptic fibrations are related by a 4 : 1 meromorphic mapping. The monodromy of the elliptic fibration is concretely determined and also connected to the Birkhoff normal forms of the Euler top around the equilibrium points.

In [14], the relation between the eigenvector mapping of the Lax (Manakov) equation of the Euler top and complex algebraic surfaces are studied and a class of the Kummer surfaces are found to factor the eigenvector mapping.

The paper is organized as follows:
In Section 2, one give a brief overview on the generalities of completely integrable systems in the sense of Liouville. Section 3 deals with the equation of motion for the heavy tops and the four integrable cases are mentioned. In Section 4, one finds a more detailed descriptions of the Euler top, focusing on the Lax (Manakov) equation, its spectral curve, and the Birkhoff normal forms. A brief review is also given about the Birkhoff normal forms around isolated equilibrium points of Hamiltonian systems over symplectic manifolds. Then, the general integral formulas of the derivative of the inverse of the Birkhoff normal forms are applied to the Euler top. On the basis of [4, 15, 19], Section 5 is about the complex algebraic geometry of the elliptic fibration associated to the family of the complexified integral curves of Hamilton's equation (the Euler equation) of the Euler top. It also concerns with the elliptic fibrations consisting of the spectral curves. The classifications of the singular fibres of the two fibrations and the 4 : 1 meromorphic mapping between the two elliptic fibrations are given. Section 6 briefly deals with the eigenvector mapping for the Lax (Manakov) equation of the Euler top and the relation to the Kummer surfaces is mentioned. In Section 7, a few related studies are mentioned.

2 A quick review on generalities of completely integrable systems

We here briefly review the notion of completely integrable systems. We start with the basic definition of Hamiltonian systems on symplectic manifolds. The relations to Poisson manifolds and to the Lax equations are also mentioned from a general point of view.

Hamiltonian systems on symplectic manifolds Let (V, ω) be a symplectic manifold with $\mathrm{dim} V = 2n$, where ω is a non-degenerate closed two-form, namely $\underbrace{\omega \wedge \cdots \wedge \omega}_{n} \neq 0$, $\mathsf{d}\omega = 0$. For a smooth function $H \in \mathcal{C}^\infty(V)$, the Hamiltonian vector field Ξ_H for the Hamiltonian H with respect to ω is defined through $\iota_{\Xi_H}\omega = -\mathsf{d}H$. The Poisson bracket $\{\cdot,\cdot\} : \mathcal{C}^\infty(V) \times \mathcal{C}^\infty(V) \to \mathcal{C}^\infty(V)$ is defined through $\{F_0, F_1\} = -\Xi_{F_0}(F_1) = -\omega(\Xi_{F_0}, \Xi_{F_1})$, where $F_0, F_1 \in \mathcal{C}^\infty(V)$.

Proposition 2.1 *The Poisson bracket $\{\cdot,\cdot\}$ satisfies the following properties:*

- $\{F_0, F_1\} = -\{F_1, F_0\}$,
- *Leibniz rule;* $\{F_0 F_1, F_2\} = \{F_0, F_2\} F_1 + \{F_1, F_2\} F_0$,
- *Jacobi identity;* $\{\{F_0, F_1\}, F_2\} + \{\{F_2, F_0\}, F_1\} + \{\{F_1, F_2\}, F_0\} = 0$,

for arbitrary functions $F_0, F_1, F_2 \in \mathcal{C}^\infty(V)$. □

Remark 2.2 *In the above arguments, the category of the manifolds, as well the functions and the differential forms, can be of the class $\mathcal{C}^\omega$ or complex analytic instead of the class $\mathcal{C}^\infty$. This is also true in many of the subsequent arguments except for Liouville-Arnol'd Theorem and the Birkhoff normal forms, which can be extended to the $\mathcal{C}^\omega$ category in a suitable manner, but may be more subtle in the complex analytic category.* □

Liouville complete integrability On a $2n$-dimensional symplectic manifold (V, ω), a Hamiltonian system (V, ω, H) is called *completely integrable* in the Liouville sense if there exist n functionally independent functions $F_1, \ldots, F_{n-1}, F_n \equiv H$ on V satisfying $\{F_i, F_j\} = 0$ for any $i, j = 1, \ldots, n$.

The following theorem shows the importance of the notion in a geometric sense. We set $f = (F_1, \ldots, F_n) : V \to \mathbb{R}^n$.

Theorem 2.3 (Liouville-Arnol'd) *Around a regular point $x \in V$ of f, for which $f(x)$ is a regular value of f and $f^{-1}(f(x))$ is connected compact manifold, there exist a neighbourhood $U (\subset \mathbb{R}^n)$ of $f(x)$ and a diffeomorphism $\Psi : f^{-1}(U) \to U \times \mathbb{T}^n$ such that $\Psi^*\left(\sum_{i=1}^n \mathsf{d}I_i \wedge \mathsf{d}\theta_i\right) = \omega|_{f^{-1}(U)}$, where $(I_1, \ldots, I_n; \theta_1, \ldots, \theta_n)$ are coordinates on $U \times \mathbb{T}^n$, $\theta_i + 2\pi = \theta_i$. Further, the flow of the Hamiltonian vector field Ξ_H on $U \times \mathbb{T}^n \cong f^{-1}(U)$ is linear with respect to $(\theta_1, \ldots, \theta_n)$.* □

Hamiltonian systems on Poisson manifolds The notion of the Hamiltonian system can be generalized to Poisson manifolds from symplectic manifolds.

Let $(N, \{\cdot,\cdot\})$ be a Poisson manifold. Namely, there is given a Poisson bracket $\{\cdot,\cdot\} : \mathcal{C}^\infty(N) \times \mathcal{C}^\infty(N) \to \mathcal{C}^\infty(N)$ which is an $\mathbb{R}$-linear mapping satisfying the following conditions:

- $\{F_0, F_1\} = -\{F_1, F_0\}$,
- Leibniz rule; $\{F_0F_1, F_2\} = \{F_0, F_2\} F_1 + \{F_1, F_2\} F_0$,
- Jacobi identity; $\{\{F_0, F_1\}, F_2\} + \{\{F_2, F_0\}, F_1\} + \{\{F_1, F_2\}, F_0\} = 0$,

for arbitrary $F_0, F_1, F_2 \in \mathcal{C}^\infty(N)$.

For the Hamiltonian $H \in \mathcal{C}^\infty(N)$, the associated Hamiltonian vector field Ξ_H with respect to $\{\cdot,\cdot\}$ is defined through $\Xi_H(F_0) = -\{H, F_0\}$, where $F_0 \in \mathcal{C}^\infty(N)$ is arbitrary. A function $F_0 \in \mathcal{C}^\infty(N)$ is called a Casimir function if $\Xi_{F_0} \equiv 0$, i.e. if $\{F_0, F_1\} = 0$ for any $F_1 \in \mathcal{C}^\infty(N)$.

Symplectic stratification A Poisson manifold $(N, \{\cdot,\cdot\})$ can be stratified into the disjoint union of symplectic manifolds. Any two points of the Poisson manifold N are defined to be equivalent if they are (piecewise) connected by the trajectories of Hamiltonian vector fields. This equivalent relation gives rise to the stratification of N into the disjoint union of smooth strata (manifolds) $\mathcal{S}$ equipped with the Poisson bracket $\{\cdot,\cdot\}_\mathcal{S}$ given by $\{F_0|_\mathcal{S}, F_1|_\mathcal{S}\}_\mathcal{S} := \{F_0, F_1\}|_\mathcal{S}$, where $F_0, F_1 \in \mathcal{C}^\infty(N)$. Since the Poisson bracket $\{\cdot,\cdot\}|_\mathcal{S}$ on $\mathcal{S}$ is non-degenerate at each point, it defines a symplectic form $\omega_\mathcal{S}$ on $\mathcal{S}$. This stratification is called the *symplectic stratification* of the Poisson manifold $(N, \{\cdot,\cdot\})$.

The integral curve of the Hamiltonian system on the Poisson manifold $(N, \{\cdot,\cdot\})$ for a Hamiltonian H through a point in $\mathcal{S}$ coincides with the one for the Hamiltonian system on the symplectic manifold $(\mathcal{S}, \omega_\mathcal{S})$ for the Hamiltonian $H|_\mathcal{S}$.

Lax equation Completely integrable systems are typically described by Lax equations. We consider the following $m\times m$-matrix-valued ordinary differential equation,

$$\frac{\mathsf{d}M_\lambda}{\mathsf{d}t} = [M_\lambda, N_\lambda], \tag{2.1}$$

called Lax equation with a (complex) parameter λ, which does not depend on the time t.

As is easily checked, we have $\frac{\mathsf{d}}{\mathsf{d}t}\left(\mathrm{Tr}\,(M_\lambda)^\ell\right) = 0$ along the integral curve of the Lax equation for all $\ell \in \mathbb{N}$. In other words, each of the eigenvalues of M_λ is a constant of motion and we are led to think of the set of all the eigenvalues for each parameter λ, which is realized as an affine algebraic curve

$$C : \det(M_\lambda - \mu \mathsf{E}_m) = 0,$$

in $\mathbb{C}^2$, with (λ, μ) being the affine coordinates. This curve C is called the *spectral curve* associated to the Lax equation (2.1).

The eigenspaces of M_λ are generically a complex line in $\mathbb{C}^3$ parametrized by $(\lambda, \mu) \in C$ and hence give rise to a line bundle on the spectral curve C. Thus, we have a holomorphic mapping of the common level surface $\mathcal{T}$ to the Picard variety of C

$$\varphi_C : \mathcal{T} \to \mathrm{Pic}(C).$$

An (often incorrect) folklore One might expect that the real part of (a connected component of) $\mathcal{T}$ is isomorphic to (a connected component of) $\mathrm{Pic}(C)$ and the flow of the integrable system is linearized on $\mathrm{Pic}(C)$. However, this expectation is not true in many cases and the situation depends on the feature of each completely integrable system.

3 Heavy rigid bodies and integrable cases

A heavy rigid body stands for a rigid body under gravity force around a fix point and its rotational motion can be described by the Euler equation

$$\begin{cases} \dfrac{\mathsf{d}\Gamma}{\mathsf{d}t} &= \Gamma \times \nabla_M H, \\ \dfrac{\mathsf{d}M}{\mathsf{d}t} &= M \times \nabla_\Gamma H + \Gamma \times \nabla_M H, \end{cases} \tag{3.1}$$

where $(M, \Gamma) \in \mathbb{R}^3 \times \mathbb{R}^3$ and $\nabla H = (\nabla_M H, \nabla_\Gamma H)$ is the gradient vector field of the Hamiltonian $H(M, \Gamma) := \dfrac{1}{2}\langle M, \mathsf{I}^{-1}(M)\rangle + \langle L, \Gamma\rangle$ with respect to the standard inner product on $\mathbb{R}^3 \times \mathbb{R}^3$, given as the direct sum of the standard inner product $\langle \cdot, \cdot \rangle$ on $\mathbb{R}^3$. Here, $L = (L_1, L_2, L_3) \in \mathbb{R}^3$ is a constant vector given by the gravity and the 3×3 positive-definite symmetric matrix I is the inertia tensor of the rigid body, which can be assumed to be a diagonal matrix as $\mathsf{I} = \mathrm{diag}(I_1, I_2, I_3)$ without loss of generality. It is known that the Euler equation (3.1) is Hamilton's equation with respect to the Poisson structure on $\mathbb{R}^3 \times \mathbb{R}^3$:

$$\{F_0, F_1\}(M, \Gamma) := \langle M, \nabla_M F_0 \times \nabla_M F_1\rangle + \langle \Gamma, \nabla_M F_0 \times \nabla_\Gamma F_1 - \nabla_M F_1 \times \nabla_\Gamma F_0\rangle$$

where F_0, F_1 are smooth functions on $\mathbb{R}^3 \times \mathbb{R}^3$ which is nothing but the Lie-Poisson bracket on the dual space to the semi-direct product Lie algebra $\mathfrak{so}(3) \ltimes \mathbb{R}^3 \cong \mathbb{R}^3 \times \mathbb{R}^3$. As the Poisson bracket allows two functionally independent Casimir functions $C_1(M, \Gamma) = \langle \Gamma, \Gamma \rangle$ and $C_2(M, \Gamma) = \langle M, \Gamma\rangle$, namely $\{C_1, \cdot\} = 0$ and $\{C_2, \cdot\} = 0$, we have three functionally independent constants of motion C_1, C_2, and H which Poisson commute with each other. The Hamiltonian system (3.1) can be restricted to a generic four-dimensional symplectic leaf of the Poisson space $(\mathbb{R}^3 \times \mathbb{R}^3, \{\cdot, \cdot\})$, which is described as the intersection of the level sets of C_1 and C_2. These common level sets are known to be diffeomorphic to T^*S^2 with the canonical symplectic form if $C_2 = 0$ or with a modified (magnetic) symplectic forms if $C_2 \neq 0$. The restricted system on a four-dimensional symplectic leaf is completely integrable in the sense of Liouville if it admits an additional constant of motion which is independent of C_1, C_2, H. It is known that this is the case only in the following four cases by the result [22] of Ziglin:

- Euler top: $L = 0$.
- Lagrange top: $\mathsf{I} = \mathrm{diag}(I_1, I_1, I_3),\ L = (0, 0, L_3)$.
- Kowalevsky top: $\mathsf{I} = \mathrm{diag}(2I_3, 2I_3, I_3),\ L = (L_1, 0, 0)$.
- Goryachev-Chapligyn top: $\mathsf{I} = \mathrm{diag}(4I_3, 4I_3, I_3),\ L = (L_1, 0, 0)$. (Integrable only on $C_1 =$ constant and $C_2 = 0$.)

For the details about these completely integrable cases of heavy tops, including the precise form of the fourth constant K of motion, see e.g. [2]. Particularly, the intersection of the level hypersurfaces of C_1, C_2, H, K includes the Lagrangian torus of the integrable system.

It should be pointed out that the constants of motion of these completely integrable systems are polynomials in the variables $(M, \Gamma) = (M_1, M_2, M_3, \Gamma_1, \Gamma_2, \Gamma_3)$ and hence it is natural to complexify the whole settings from the viewpoint of complex algebraic/analytic geometry. We regard $(M, \Gamma) = (M_1, M_2, M_3, \Gamma_1, \Gamma_2, \Gamma_3)$ as affine coordinates in $\mathbb{C}^3 \times \mathbb{C}^3$, the constants J_1, J_2, J_3, L_1, L_2, L_3 as complex numbers. The inner product $\langle \cdot, \cdot, \rangle$ and the Poisson bracket $\{\cdot, \cdot\}$ are respectively considered as complex quadratic forms on $\mathbb{C}^3$ and on the space of all the holomorphic functions on $\mathbb{C}^3 \times \mathbb{C}^3$.

In this direction, the notion of algebraic complete integrability has been introduced as a generalization of Liouville-Arnol'd Theorem (Theorem 2.3) into the complex algebraic category, with the mapping f in Theorem 2.3 being a complex analytic fibration by Abelian varieties. See [1] as well as [20] for the details on the algebraic complete integrability. The notion of algebraic complete integrability is a central idea in many studies on complex algebraic geometry around integrable systems. Nevertheless, the (global) geometric properties of the complex analytic/algebraic fibrations, as well as the singularities/singular fibres, remain unclear in many cases. In what follows, we focus on the complex analytic/algebraic geometry around the Euler top along the lines of the author's researches in [4, 14, 15, 19].

4 Integrable system of Euler top

We consider the Euler top, or the free rigid body, which physically describes the rotational motion of a rigid body under no external force.

The system is essentially described by the Euler equation

$$\frac{\mathsf{d}}{\mathsf{d}t} M = M \times \left(\mathsf{I}^{-1}(M)\right),$$

where $M = (M_1, M_2, M_3)^{\mathrm{T}} \in \mathbb{R}^3$ is the angular momentum.

It is known that the Euler equation has two first integrals, i. e. the Hamiltonian $H(M) = \frac{1}{2}\langle M, \mathsf{I}^{-1}(M)\rangle$ and the squared norm of the total angular momentum $K(M) = \frac{1}{2}\langle M, M\rangle$. The restriction to the level surface $K = k$ (const.) ($\cong S^2_{\sqrt{2k}}$) is a completely integrable Hamiltonian system for the Hamiltonian $H|_{K=k}$ with respect to the symplectic form $\omega = \frac{\mathsf{d}M_1 \wedge \mathsf{d}M_2}{3M_3} = \frac{\mathsf{d}M_2 \wedge \mathsf{d}M_3}{3M_1} = \frac{\mathsf{d}M_3 \wedge \mathsf{d}M_1}{3M_2}$. Further, the integral curve of the Euler equation coincides with the intersection of the two quadric level surfaces

$$\begin{cases} H(M) = h, \\ K(M) = k, \end{cases}$$

up to the connected components. The intersection is in general the real part of a complex smooth elliptic curve.

4.1 Lax (Manakov) equation and the spectral curve

Using the Lie-algebra isomorphism $(\mathbb{R}^3, \times) \ni M \overset{\cong}{\mapsto} \widehat{M} \in \mathfrak{so}(3)$, the Euler equation can be written in terms of matrices as

$$\frac{\mathsf{d}}{\mathsf{d}t}\widehat{M} = \left[\widehat{M}, \mathcal{J}^{-1}(\widehat{M})\right], \qquad \widehat{M} \in \mathfrak{so}(3) \cong \left(\mathbb{R}^3, \times\right),$$

where the linear operator $\mathcal{J} : \mathfrak{so}(3) \ni U \mapsto \mathsf{J}U + U\mathsf{J} \in \mathfrak{so}(3)$ is related to I by the matrix $\mathsf{J} = \mathrm{diag}(J_1, J_2, J_3)$, $J_2 + J_3 = I_1$, $J_3 + J_1 = I_2$, $J_1 + J_2 = I_3$. This means that $\widehat{\mathsf{I}^{-1}(M)} = \mathcal{J}^{-1}\widehat{M}$ for all $M \in \mathbb{R}^3$. It is a well-known result by Manakov [11] that the Euler equation is equivalent to the following Lax (Manakov) equation with a (time-independent) complex parameter $\lambda \in \mathbb{C}$:

$$\frac{\mathsf{d}}{\mathsf{d}t}\left(\widehat{M} + \lambda\mathsf{J}^2\right) = \left[\widehat{M} + \lambda\mathsf{J}^2, \mathcal{J}^{-1}(\widehat{M}) + \lambda\mathsf{J}\right]. \tag{4.1}$$

The eigenvalues of the matrices $\widehat{M} + \lambda\mathsf{J}^2$, which are parametrized by $\lambda \in \mathbb{C}$, are constants of motion for the Lax equation, and hence for the Euler equation, since the two equation are equivalent to each other. We think about the spectral curve

$$\left\{(\lambda, \mu) \in \mathbb{C}^2 \mid \det\left(\widehat{M} + \lambda\mathsf{J}^2 - \mu\mathsf{E}_3\right) = 0\right\},$$

where E_3 is identity matrix. This algebraic curve is known to be a smooth elliptic curve if $I_1, I_2, I_3, \frac{k}{h}$ are distinct

4.2 A brief review on Birkhoff normal forms

We quickly review the Birkhoff normal forms of Hamiltonian around an isolated equilibrium point of a Hamiltonian system on a symplectic manifold. The Birkhoff normal forms can be regarded as the generalization of Morse Lemma (cf. [13]) respecting the symplectic structure. However, the normal form has much larger varieties than in the case of Morse Lemma.

Let (V, ω) be a $2n$-dimensional symplectic manifold. We consider the Hamiltonian system (V, ω, H) for the Hamiltonian H and assume that $x_0 \in V$ is an isolated elliptic equilibrium, i.e. $X_H(x_0) = 0$ and $X_H(x) = 0$ for $x \neq x_0$ in a sufficiently small neighbourhood of x_0.

We assume that x_0 is an elliptic equilibrium point, i.e. the linearization matrix of the Hamiltonian vector field X_H has only purely imaginary eigenvalues. If there are suitable coordinates $(p_1, \cdots, p_n; q_1, \cdots, q_n)$ around x_0 such that $\omega = \sum_{i=1}^{n} \mathsf{d}p_i \wedge \mathsf{d}q_i$, in which case $(p_1, \cdots, p_n; q_1, \cdots, q_n)$ are called Darboux coordinates, and if the Hamiltonian H is written by a (formal) power series $\mathcal{H}$ in n variables as

$$H = \mathcal{H}\left(\frac{p_1^2 + q_1^2}{2}, \cdots, \frac{p_n^2 + q_n^2}{2}\right),$$

then the function $\mathcal{H}$ is called Birkhoff normal form of H around x_0.

In general, it is known that the eigenvalues of the linearization matrix for a Hamiltonian vector field around an isolated equilibrium point consists of the following tuples of complex numbers:
pairs of real numbers $\pm\alpha$ $(\alpha \in \mathbb{R})$, pairs of purely imaginary numbers $\pm\beta\sqrt{-1}$ $(\beta \in \mathbb{R})$; quadruple of complex numbers $\pm\alpha \pm \beta\sqrt{-1}$ $(\alpha, \beta \in \mathbb{R})$, and 0.

According to the types of the eigenvalues, the Birkhoff normal form $\mathcal{H}$ is formal convergent power series in different types of quadratic functions in the Darboux coordinates.

Note that the Birkhoff normal forms can be considered also for the Hamiltonian systems in the complex holomorphic category. In this case, there is no difference among the non-zero eigenvalues of the linearization matrix of the Hamiltonian vector field.

Remark 4.1 *The convergence of local canonical transforms which put the Hamiltonian into the Birkhoff normal form indicates the complete integrability of the Hamiltonian system around the equilibrium point. There is an algorithms to obtain the Birkhoff normal form in the framework of formal power series. See e.g. [12]. In general, however, Birkhoff normal forms are know to be divergent by an earlier result by C. L. Siegel [18].*

For a completely integrable system, the existence of canonical transforms for the Birkhoff normal forms are discussed by several authors. J. Vey [21] has proved the convergence for analytic systems under the so-called non-degeneracy condition of the equilibrium point.[1] *H. Ito [8] improved the result, replacing the non-degeneracy condition by the non-resonance condition. H. Eliasson [3] proves the convergence of such a canonical transformation in the $\mathcal{C}^\infty$ category under the non-degeneracy condition. N. T. Zung [23] gave a proof by using a geometric method around torus actions.*

Birkhoff normal forms are applied in many different ways. See e.g. [12]. Concerning the generalized vesions of free rigid body dynamics, the Birkhoff normal forms are applied in the stability analysis. See e.g. [9, 10, 16, 17], as well as the references therein.

□

[1] This non-degeneracy is different from the non-degenerate critical point in the sense of Morse Lemma. An isolated equilibrium point x_0 of a completely integrable Hamiltonian system (V, ω, H) on a symplectic manifold, with constants of motions $F_1, \ldots, F_{n-1}, F_n(= H)$, is called non-degenerate if the linearization matrices of the Hamiltonian vector fields $X_{F_1}, \ldots, X_{F_n}$ form a Cartan subalgebra in the Lie algebra of the infinitesimal symplectic transformations $\mathfrak{sp}(T_{x_0}V, \omega_{x_0})$.

To return to the case of the Euler top, we can assume that all the systems below are real analytic. We focus on the situation of two-dimensional symplectic manifold. Under this assumption, the Birkhoff normal form is a convergent series. Let (V, ω, H) be an analytic Hamiltonian system on two-dimensional symplectic manifold.

Proposition 4.2 *If $x_0 \in V$ is an elliptic equilibrium of (V, ω, H), for which the linearization matrix has only purely imaginary eigenvalues, then there is a coordinate system (p, q) around x_0 and an invertible convergent power series $\mathcal{H}$ such that $\omega = \mathsf{d}p \wedge \mathsf{d}q$ and*

$$H = \mathcal{H}\left(\frac{p^2 + q^2}{2}\right).$$

□

Similarly, we have the following proposition.

Proposition 4.3 *If $x_0 \in V$ is a hyperbolic equilibrium of (V, ω, H), then there is a coordinate system (P, Q) around x_0 and an invertible convergent power series $\mathcal{H}$ such that $\omega = \mathsf{d}P \wedge \mathsf{d}Q$ and*

$$H = \mathcal{H}(PQ).$$

□

Although the Birkhoff normal form is by definition a local object around the equilibrium point, we can think over its relation to the global aspects of the Hamiltonian system, particularly from complex analytical point of view. In this direction, the following formulae by period integrals are useful.

We take a one-form η around the isolated equilibrium x_0 which satisfies $\eta \wedge \mathsf{d}H = \omega$. Then, we can describe the relation between the Birkhoff normal form around an elliptic equilibrium and a period integral.

Theorem 4.4 *By Φ, we denote the inverse of the Birkhoff normal form $\mathcal{H}$ around an elliptic equilibrium point, where $H = 0$. Then, we have*

$$\Phi'(h) = -\frac{1}{2\pi} \int_{H=h} \eta.$$

□

The integral path $H = h$ stands for the integral curve of the Hamiltonian system.

For the hyperbolic equilibrium, we have a similar relations as follows:

Theorem 4.5 *We denote by Φ the inverse of the Birkhoff normal form $\mathcal{H}$ around a hyperbolic equilibrium point, where $H = 0$. Then, we have*

$$\Phi'(h) = \frac{1}{2\pi\sqrt{-1}} \int_{\gamma} \eta.$$

□

The integral path γ is taken from the homotopy class of the closed arc $P = \sqrt{\epsilon} e^{\sqrt{-1}\theta}, Q = \sqrt{\epsilon} e^{-\sqrt{-1}\theta}, \theta : 0 \to 2\pi$ in the complexified integral curve, where (P, Q) are the Darboux coordinates such that $H = \mathcal{H}(PQ)$ and $\epsilon = \Phi(h) = \mathcal{H}^{-1}(h)$.

4.3 Birkhoff normal forms for the Euler top

We calculate the derivative of the inverse for Birkhoff normal forms around the six equilibria for the Euler top restricted to the sphere $K = k$.

We assume that $I_1 < I_2 < I_3$. Then, the equilibria on p_1- and p_3-axes are elliptic, while those on p_2-axis are hyperbolic.

The derivative of inverse for Birkhoff normal form around the elliptic equilibria on p_1-axis can be written as

$$-\frac{1}{3\pi}\sqrt{\frac{2}{k}}\frac{1}{\sqrt{(d-c)(a-b)}}\mathcal{K}\left(\frac{(d-a)(b-c)}{(d-c)(b-a)}\right) =: S(a,b,c,d),$$

where $a = \dfrac{1}{I_1}, b = \dfrac{1}{I_2}, c = \dfrac{1}{I_3}, d = \dfrac{h}{k}$. Here, $\mathcal{K}(s) := \int_0^1 \frac{\mathrm{d}x}{\sqrt{(1-x^2)(1-sx^2)}}$ is the complete elliptic integral of the first kind.

Similarly, the derivative of inverse for Birkhoff normal form around the elliptic equilibria on p_3-axis is expressed as

$$-\frac{1}{3\pi}\sqrt{\frac{2}{k}}\frac{1}{\sqrt{(d-a)(c-b)}}\mathcal{K}\left(\frac{(d-c)(b-a)}{(d-a)(b-c)}\right) = S(c,b,a,d).$$

Now, around the hyperbolic equilibria on p_2-axis, the derivative of inverse for Birkhoff normal form is in the form as follows:

$$\frac{\sqrt{-1}}{3\pi}\sqrt{\frac{2}{k}}\frac{1}{\sqrt{(d-c)(b-a)}}\mathcal{K}\left(\frac{(d-b)(a-c)}{(d-c)(a-b)}\right) = -\sqrt{-1}S(b,a,c,d).$$

In what follows, we consider the analytic continuation of these functions $S(a,b,c,d)$, $S(c,b,a,d)$, $S(b,a,c,d)$ with respect to the complex parameters (a,b,c,d).

The complete elliptic integral $\mathcal{K}$ satisfies the following special Gauß hypergeometric differential equation:

$$s(1-s)f''(s) + (1-2s)f'(s) - \frac{1}{4}f(s) = 0.$$

By the connection formula of the Gauß hypergeometric differential equation, we have the following formula.

Proposition 4.6 *The analytic continuations of* $S(a,b,c,d)$, $S(b,a,c,d)$, $S(c,b,a,d)$ *satisfy*

$$S(a,b,c,d) + S(b,a,c,d) = S(c,b,a,d).$$

□

More details around the discussion in this subsection can be found in [4, 19].

5 Two elliptic fibrations arising from Euler top

From now on, we complexify the settings for the simplicity. The integral curve of Euler equation can be compactified in $\mathbb{CP}^3 : (x : y : z : w)$ by the projective curve

$$\begin{cases} x^2 + y^2 + z^2 + w^2 = 0, \\ ax^2 + by^2 + cz^2 + dw^2 = 0. \end{cases} \tag{5.1}$$

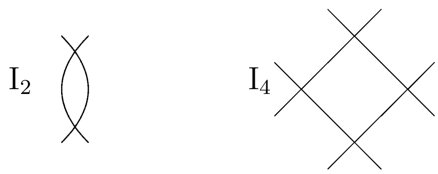
This curve is known to be a smooth elliptic curve if a, b, c, d are distinct. See [2] or [15] for the details about this fact.

The family of curves (5.1) parameterized by a, b, c, d gives rise to an elliptic fibration $\pi_F : F \to \mathbb{CP}^3 : (a : b : c : d)$ as follows:
The total space F is a smooth four-fold defined by (5.1) in $\mathbb{CP}^3 \times \mathbb{CP}^3$, where $((x : y : z : w), (a : b : c : d))$ denotes the pair of homogeneous coordinates. Then, restricting the projection $\mathbb{C}^3 \times \mathbb{C}^3 \ni ((x : y : z : w), (a : b : c : d)) \mapsto (a : b : c : d)$ to the manifold F, we have an elliptic fibration $\pi_F : F \to \mathbb{CP}^3$.

The singular locus of the fibration $\pi_F : F \to \mathbb{CP}^3$ is given by the divisor $D : a = b, a = c, a = d, b = c, b = d, c = d$. Over the singular locus, the singular fibres are classified as follows:

- If only two of a, b, c, d coincide, the fibre is of type I_2 (see Figure 1) in Kodaira's list of singular fibres of elliptic surfaces.
- If two of a, b, c, d coincide and the other two also coincide, the fibre is of type I_4 (see Figure 1).
- If three of a, b, c, d are equal without further coincidence, the fibre is a smooth rational curve with multiplicity two.
- If $a = b = c = d$, the fibre is a quadric surface.

Figure 1: Singular fibres of types I_2 and I_4



The family of spectral curves is also parameterized by the parameters $(a : b : c : d) \in \mathbb{CP}^3$. It induces another elliptic fibration over $\mathbb{CP}^3$. This elliptic fibration is bimeromorphic to the elliptic fibration in Weierstraß normal form defined as follows: We take the holomorphic line bundle $G = \mathcal{O}_{\mathbb{CP}^3}(1)$ over $\mathbb{CP}^3$ with the first Chern number 1. Then, we consider the projectification $\mathbb{P}\left(G^{\otimes 2} \oplus G^{\otimes 2} \oplus \mathcal{O}_{\mathbb{CP}^3}\right)$ of the rank three vector bundle $G^{\otimes 2} \oplus G^{\otimes 2} \oplus \mathcal{O}_{\mathbb{CP}^3}$ over $\mathbb{CP}^3$, where the structure sheaf $\mathcal{O}_{\mathbb{CP}^3}$ is identified with the trivial line bundle. In the total space of the $\mathbb{CP}^2$-bundle $\mathbb{P}\left(G^{\otimes 2} \oplus G^{\otimes 2} \oplus \mathcal{O}_{\mathbb{CP}^3}\right)$, we consider the hypersurface W defined by

$$\mathsf{y}^2\mathsf{z} = 4\left(\mathsf{x} - \mathsf{e}_1\right)\left(\mathsf{x} - \mathsf{e}_2\right)\left(\mathsf{x} - \mathsf{e}_3\right),$$

where $(\mathsf{x} : \mathsf{y} : \mathsf{z})$ denotes the homogeneous fibre coordinates of the $\mathbb{CP}^2$-bundle. Here, the parameters $\mathsf{e}_1, \mathsf{e}_2, \mathsf{e}_3$ are the sections of the line bundle $G^{\otimes 2}$ given by the homogeneous quadratic polynomials

$$\begin{aligned} \mathsf{e}_1 &= (a-b)(c-d) + (a-c)(b-d), \\ \mathsf{e}_2 &= -(a-b)(c-d) + (a-d)(b-c), \\ \mathsf{e}_3 &= -(a-c)(b-d) - (a-d)(b-c). \end{aligned}$$

Restricting the canonical projection of the $\mathbb{CP}^2$-bundle to W, we naturally obtain an elliptic fibration $\pi_W : W \to \mathbb{CP}^3$ in Weierstraß normal form. This elliptic fibration is flat and yet it allows singularity on the total space. Modifying both the total and the base spaces, we have a smooth flat elliptic fibration, denoted by $\pi_{\widehat{W}} : \widehat{W} \to \widehat{B}$, whose singular fibres are in Kodaira's list of singular fibres of elliptic surfaces.

More precisely, the modification of the base space $\mathbb{CP}^3$ is carried out as follows:

1. We blow up $\mathbb{CP}^3$ with the centre at $(1:1:1:1)$: $B \to \mathbb{CP}^3$.

2. We consider the natural projection of B to the exceptional set $E \cong \mathbb{CP}^2$: $\tau_B : B \to E$.

3. We blow up E at the four intersection points with the loci $a = b = c$, $b = c = d$, $c = d = a$, $d = a = b$: $\widehat{E} \to E$.

4. We blow up B along the four curves corresponding to these four points: $\widehat{B} \to B$.

The singular fibres of $\pi_{\widehat{W}} : \widehat{W} \to \widehat{B}$ are given as follows:

- The fibres over the generic points on the proper transforms of the six planes $a = b$, $a = c$, $a = d$, $b = c$, $b = d$, $c = d$ are of type I_2.

- The fibres over the points on the proper transforms of the three lines $a = b, c = d$; $a = c, b = d$; $a = d, b = c$ are of type I_4.

- The fibres over the generic points on the exceptional divisors C_1, C_2, C_3, and C_4 are of type I_0^*.

- The fibres over the points on the intersection of the exceptional divisors C_1 (respectively C_2, C_3, and C_4) and the proper transforms of the three lines $a = b$, $b = c$, $c = a$, (respectively $b = c$, $c = d$, $d = b$; $c = d$, $d = a$, $a = c$; $d = a$, $a = b$, $b = d$) are of type I_1^* in Kodaira's notation.

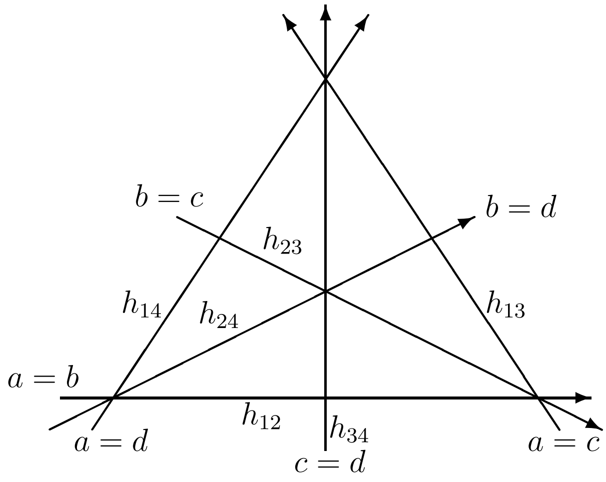
The two fibrations F and $\widehat{W}$ on $\mathbb{CP}^3$ are related as follows:

Theorem 5.1 ([15]) *There is a* $4:1$ *meromorphic mapping from* $\pi_F : F \to \mathbb{CP}^3$ *onto* $\pi_{\widehat{W}} : \widehat{W} \to \widehat{B}$, *which respects the fibration and which induces a* $4:1$ *isogeny on each regular fibre.* □

To close this section, we mention the monodromy of the elliptic fibration π_F.

The locus $D : a = b, a = c, a = d, b = c, b = d, c = d$ form the A_3 plane arrangement in $\mathbb{CP}^3 : (a : b : c : d)$ as in Figure 2. Around each component of D, we associate a generator of the fundamental group $\pi_1\left(\mathbb{CP}^3 \setminus \mathrm{Supp}(D)\right)$ for the complement of the support $\mathrm{Supp}(D)$ in $\mathbb{CP}^3$ as is indicated in Figure 2. The six

Figure 2: Singular locus D



generators $h_{12}, h_{13}, h_{14}, h_{23}, h_{24}, h_{34}$ obeys the relations

$$\begin{aligned} h_{12}h_{23}h_{13} &= h_{23}h_{13}h_{12} = h_{13}h_{12}h_{23}, \\ h_{23}h_{34}h_{24} &= h_{34}h_{24}h_{23} = h_{24}h_{23}h_{34}, \\ h_{12}h_{24}h_{14} &= h_{24}h_{14}h_{12} = h_{14}h_{12}h_{24}, \\ h_{34}h_{14}h_{13} &= h_{14}h_{13}h_{34} = h_{13}h_{34}h_{14}, \\ h_{12}h_{34} &= h_{34}h_{12}, \\ h_{13}h_{23}^{-1}h_{24}h_{23} &= h_{23}^{-1}h_{24}h_{23}h_{13}, \\ h_{23}h_{14} &= h_{14}h_{23}, \\ 1 &= h_{13}h_{12}h_{23}h_{34}h_{24}h_{14}. \end{aligned}$$

Now, we regard the derivative of the inverse for Birkhoff normal forms described by $S(c, b, a, d)$, $S(a, b, c, d)$ as a basis of the first cohomology group of regular fibres of π_F, which are complex one-dimensional tori.

Theorem 5.2 ([4, 19]) *With respect to this basis, the monodromy of π_F is given by the correspondence*

$$h_{14}, h_{23} \mapsto \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, h_{13}, h_{24} \mapsto \begin{bmatrix} -1 & 2 \\ -2 & 3 \end{bmatrix}, h_{12}, h_{34} \mapsto \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}.$$

□

The same result holds in the case of π_W.

In [19], the monodromy of the derivatives for the Birkhoff normal forms are also studied in the case of the simple pendulum.

6 Eigenvector mapping of Euler top

In the previous sections, we have considered the spectral curve for the Euler top. This algebraic curve reflects the totality of the eigenvalues of the matrix $\widehat{M} + \lambda \mathsf{J}^2$

appearing in the Lax (Manakov) equation (4.1). For each point (λ, μ) in the spectral curve $\det\left(\widehat{M} + \lambda \mathsf{J}^2 - \mu \mathsf{E}_3\right) = 0$, we can also consider the eigenvector v determined by

$$\left(M + \lambda \mathsf{J}^2 - \mu \mathsf{E}_3\right) v = 0.$$

As the eigenvector v is determined up to multiples in $\mathbb{C}^*$, it induces a rational mapping

$$g : C \times C' - \cdots \to \mathbb{CP}^2$$

of the product of the integral and the spectral curves. The following result shows a relation between this eigenvector mapping and a Kummer surface.

Theorem 6.1 (**[14]**) *For generic parameters a, b, c, d, there exists a Kummer surface*[2] *Σ between $C \times C'$ and $\mathbb{CP}^2$ such that the eigenvector mapping g factors as the composition of two $2:1$ rational mappings:*

$$\begin{array}{ccc} C \times C' & \overset{g}{- \cdots \to} & \mathbb{CP}^2 \\ & \searrow \quad \nearrow & \\ & \Sigma & \end{array}$$

□

In fact, the Kummer surface Σ is the desingularization of the double covering of $\mathbb{CP}^2$ branched along a singular sextic curve consisting of a smooth quadric curve and a smooth quartic curve which are tangent at four points. The details about the branch locus on Σ, as well as the geometry of elliptic fibrations of Σ, are found in [14].

Further, we can observe the following degeneration of Kummer surfaces.

Theorem 6.2 *When two of the parameters a, b, c coincide, the Kummer surface Σ is degenerated to a sum of two rational surfaces.* □

7 Further perspectives

In the above sections, complex algebro-geometric aspects of the Euler top are explained in view of the elliptic fibrations and Kummer surfaces.

A similar problem of an elliptic fibration associated to Lagrange top has been studied recently in [7].

Kummer surfaces appear in other integrable systems of rigid bodies. As an example, there is a Kummer surface appearing in the Clebsch top with the Weber's condition, which is a completely integrable system formulated as a Hamiltonian system on the Poisson space $(\mathbb{R}^3 \times \mathbb{R}^3, \{\cdot, \cdot\})$ as in Section 2. See [5,6] and the references therein for more details about the geometries around this integrable systems.

[2] The quotient of an Abelian surface $\mathbb{C}^2/\Lambda$, where $\Lambda \cong \mathbb{Z}^4$ is a lattice in $\mathbb{C}^2$, with respect to the involution given by the multiplication of -1 has 16 A_1-singularities and the desingularization of this singular surface, which is a K3 surface, is called a Kummer surface.

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