ON THE SUPERSINGULAR LOCUS OF SHIMURA VARIETIES FOR QUATERNIONIC UNITARY GROUPS (ANNOUNCEMENT)

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Abstract. In this article, we announce mainly the results of [24]. We study the supersingular locus of a Shimura variety attached to a unitary similitude group of a skew-Hermitian form over a totally indefinite quaternion algebra over a totally real number field, that is, a PEL-Shimura variety of type C. It is unexpected that the corresponding principally polarized O_B-varieties may not exist, which is different from the case of Siegel modular varieties. We give necessary and sufficient conditions for the existence of such objects. Under such a condition we show that the superspecial locus in the fiber at p of the associated Shimura variety is non-empty. We also give an explicit formula for the number of irreducible components of the supersingular locus when p is odd and unramified in the quaternion algebra. Using the methods of this article, we obtain explicit results on bad reduction of Shimura curves.

1. The supersingular locus of Siegel modular varieties

Let p be a rational prime number and $N \geq 3$ a positive integer with $(p, N) = 1$. We write $\overline{\mathbb{F}}_p$ for an algebraic closure of the field $\mathbb{F}_p$ of order p. Let $\mathbf{A}_{g,N}$ denote the moduli scheme over $\mathbb{Z}_{(p)}$ of principally polarized abelian varieties of dimension $g \geq 1$ with a level N-structure, and let $\mathcal{A}_{g,N} := \mathbf{A}_{g,N} \otimes \overline{\mathbb{F}}_p$ denote the geometric special fiber. We recall that an abelian variety A over $\overline{\mathbb{F}}_p$ is said to be *superspecial* (resp. *supersingular*) if it is isomorphic (resp. isogenous) to a product of supersingular elliptic curves over $\overline{\mathbb{F}}_p$. Let $\mathcal{A}^{\rm sp}_{g,N} \subset \mathcal{A}^{\rm ss}_{g,N} \subset \mathcal{A}_{g,N}$ be the superspecial (resp. supersingular) locus of $\mathcal{A}_{g,N}$, that is, the subspace parameterizing the superspecial (resp. supersingular) abelian varieties in $\mathcal{A}_{g,N}$. Then $\mathcal{A}^{\rm sp}_{g,N}$ is the unique 0-dimensional Ekedahl-Oort stratum, and $\mathcal{A}^{\rm ss}_{g,N}$ is the unique closed Newton stratum of $\mathcal{A}_{g,N}$ [26]. An explicit formula for the cardinality of $\mathcal{A}^{\rm sp}_{g,N}$ was given by Ekedahl [7], using Hashimoto-Ibukiyama's mass formula [10, Proposition 9]. In [17], Li and Oort investigated the geometry of the supersingular locus, and in particular they derived a formula relating the number of irreducible components to the class number of a genus of quaternion Hermitian lattices. An explicit formula for the class number was given in [29].

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Theorem 1 ([7, 10, 17, 29]). *We write $\zeta(s)$ for the Riemann zeta function and GSp_{2g} for the symplectic similitude group of degree $2g$. Further we put*

$$C(g,N) := |\mathrm{GSp}_{2g}(\mathbb{Z}/N\mathbb{Z})| \cdot \frac{(-1)^{g(g+1)}/2}{2^g} \cdot \prod_{i=1}^{g} \zeta(1-2i).$$

(1) *The cardinality of the superspecial locus $\mathcal{A}^{\mathrm{sp}}_{g,N}$ is equal to*

$$C(g,N) \cdot \prod_{i=1}^{g} (p^i + (-1)^i).$$

(2) *The supersingular locus $\mathcal{A}^{\mathrm{ss}}_{g,N}$ is equi-dimensional of dimension $\lfloor g^2/4 \rfloor$ and the number of its irreducible components is equal to $C(g,N)\cdot\lambda_p$ where λ_p is given by*

$$\lambda_p = \begin{cases} \prod_{i=1}^{g}(p^i + (-1)^i) & \text{if } g \text{ is odd;} \\ \prod_{i=1}^{c}(p^{4i-2}-1) & \text{if } g = 2c \text{ is even.} \end{cases}$$

2. Simura varieties for quaternionic unitary groups

The aim of this announcement is to study the supersingular locus of a PEL Shimura variety of type C and, in particular, generalize Theorem 1.

Let F be a totally real field of degree d with ring of integers O_F, and O_B a maximal O_F-order in a totally indefinite quaternion algebra B over F which is stable under a positive involution $*$ of B. Let $b \mapsto \bar{b}$ denote the canonical involution of B. There is an element $\gamma \in B^\times$ such that

$$\gamma + \bar{\gamma} = 0, \quad \text{and} \quad b^* = \gamma \bar{b} \gamma^{-1} \tag{2.1}$$

for all $b \in B$.

Definition 2 ([14, 15]). An *integral PEL datum of type C* is a septuple $\mathscr{D} = (B, *, O_B, V, \psi, \Lambda, h_0)$ where

(i) $(B, *, O_B)$ is as above;

(ii) (V, ψ) is a $\mathbb{Q}$-valued skew-Hermitian $(B,*)$-module, that is, V is a finite free left B-module and $\psi : V \times V \to \mathbb{Q}$ is a non-degenerate $\mathbb{Q}$-bilinear pairing such that

$$\psi(y,x) = -\psi(x,y) \quad \text{and} \quad \psi(ax,y) = \psi(x, a^* y) \tag{2.2}$$

for all $a \in B$ and $x, y \in V$;

(iii) Λ is an O_B-lattice in V;

(iv) $h_0 : \mathbb{C} \to \mathrm{End}_{B\otimes_{\mathbb{Q}}\mathbb{R}}(V_{\mathbb{R}})$ is an $\mathbb{R}$-algebra homomorphism such that

$$\psi(h_0(i)x, h_0(i)y) = \psi(x,y) \quad \text{for all} \quad x, y \in V_{\mathbb{R}},$$

and that the symmetric form $(x,y) := \psi(h_0(i)x, y)$ is positive definite on $V_{\mathbb{R}}$.

A datum $\mathscr{D}$ is said to be *principal* if Λ is self-dual with respect to ψ in the sense that

$$\Lambda = \Lambda^{\vee,\psi} := \{x \in V \mid \psi(x, \Lambda) \subset \mathbb{Z}\}.$$

Note that for each positive integer m there exists a unique $\mathbb{Q}$-valued skew-Hermitian $(B, *)$-module (V, ψ) of rank m up to isomorphism.

Let $\mathscr{D} = (B, *, O_B, V, \psi, \Lambda, h_0)$ be an integral PEL datum of type C. The map h_0 endows $V_\mathbb{R}$ with a complex structure, and hence it gives a decomposition $V_\mathbb{C} = V^{-1,0} \oplus V^{0,-1}$ of complex subspaces. Here, $V^{-1,0}$ (resp. $V^{0,-1}$) denotes the subspace where $h_0(z)$ acts by z (resp. $\bar{z}$). Let $\mathrm{char}_F(b) \in O_F[T]$ be the reduced characteristic polynomial of $b \in O_B$, and let $\mathrm{char}(b) := \mathrm{Nr}_{F/\mathbb{Q}}\mathrm{char}_F(b) \in \mathbb{Z}[T]$ be the reduced characteristic polynomial from B to $\mathbb{Q}$. As in [32, Section 2.3], the characteristic polynomial of $b \in O_B$ on $V^{-1,0}$ is given by

$$\mathrm{char}(b \mid V^{-1,0}) = \mathrm{char}(b)^m \in \mathbb{Z}[T]. \tag{2.3}$$

For a commutative $\mathbb{Q}$-algebra R, we write $V_R := V \otimes_\mathbb{Q} R$. Further we write $\mathrm{End}_{B\otimes_\mathbb{Q} R}(V_R)$ for the ring of $B \otimes_\mathbb{Q} R$-linear endomorphisms of V_R. We define a $\mathbb{Q}$-group $\mathbf{G} = \mathrm{GU}_\mathbb{Q}(V, \psi)$ by

$$\mathbf{G}(R) = \{g \in \mathrm{End}_{B\otimes_\mathbb{Q} R}(V_R) \mid \exists\, c(g) \in R^\times \\ \text{s.t. } \psi(gx, gy) = c(g)\psi(x, y),\ \forall\, x, y \in V_R\}.$$

The group $\mathbf{G}$ is connected and reductive. Further it satisfies the Hasse principle, that is, the local-to-global map $H^1(\mathbb{Q}, \mathbf{G}) \to \prod_{v\leq\infty} H^1(\mathbb{Q}_v, \mathbf{G})$ is injective, where $\mathbb{Q}_v$ denotes the completion of $\mathbb{Q}$ at a place v ([14, Section 7]).

We define a homomorphism $h : \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\,\mathbb{G}_{m,\mathbb{C}} \to \mathbf{G}_\mathbb{R}$ by restricting h_0 to $\mathbb{C}^\times$. Composing $h_\mathbb{C}$ with the map $\mathbb{C}^\times \to \mathbb{C}^\times \times \mathbb{C}^\times$ where $z \mapsto (z, 1)$ then gives $\mu_h : \mathbb{C}^\times \to \mathbf{G}(\mathbb{C})$. Moreover, there is an isomorphism $\mathrm{End}_{B\otimes_\mathbb{Q}\mathbb{C}}(V_\mathbb{C}) \simeq \mathrm{Mat}_{2m}(\mathbb{C})^d$, inducing an embedding of $\mathbf{G}(\mathbb{C})$ into $\mathrm{GL}_{2m}(\mathbb{C})^d$. Up to conjugation in $\mathbf{G}(\mathbb{C})$, the cocharacter μ_h is expressed as

$$\mu_h(z) = ((\mathrm{diag}(z^m, 1^m), \ldots, (\mathrm{diag}(z^m, 1^m)) \in \mathbf{G}(\mathbb{C}) \subset \mathrm{GL}_{2m}(\mathbb{C})^d. \tag{2.4}$$

Let X be the $\mathbf{G}(\mathbb{R})$-conjugacy class of h. Then the pair $(\mathbf{G}, X)$ is a Shimura datum [5, (2.1.1)]. The reflex field of $(\mathbf{G}, X)$ is $\mathbb{Q}$ [22, Section 7].

Let $\mathbb{A}_f$ denote the finite adele ring of $\mathbb{Q}$. For any compact open subgroup $\mathsf{K} \subset \mathbf{G}(\mathbb{A}_f)$, the Shimura variety associated to $(\mathbf{G}, X)$ of level K is defined by

$$\mathrm{Sh}_\mathsf{K}(\mathbf{G}, X)_\mathbb{C} := \mathbf{G}(\mathbb{Q})\backslash X \times \mathbf{G}(\mathbb{A}_f)/\mathsf{K}.$$

This is a quasi-projective normal complex algebraic variety. Further, it admits the canonical model $\mathrm{Sh}_\mathsf{K}(\mathbf{G}, X)$ defined over the reflex field $\mathbb{Q}$.

3. Moduli spaces

For an abelian scheme A over a base scheme S, let $\mathrm{End}_S(A)$ denote the ring of endomorphisms of A.

Definition 3. Let $(B, *, O_B)$ be as in the previous section.

(1) An *O_B-abelian scheme* over a base scheme S is a pair (A, ι), where A is an abelian scheme over S and ι is a monomorphism of rings $\iota : O_B \to \mathrm{End}_S(A)$.

(2) A *(principally) polarized O_B-abelian scheme* is a triple (A, λ, ι), where (A, ι) is an O_B-abelian scheme and $\lambda : A \to A^t$ is a (principal) polarization such that $\lambda \circ \iota(b^*) = \iota(b)^t \circ \lambda$.

(3) We say that an O_B-abelian scheme (A, ι) over a $\mathbb{Z}_{(p)}$-scheme S satisfies the *determinant condition* if we have the equality of characteristic polynomials of degree $2dm$:

$$\mathrm{char}\,(\iota(b) \mid \mathrm{Lie}(A)) = \mathrm{char}\,(b \mid V^{-1,0}) \in O_S[T] \quad \text{for all} \quad b \in O_B.$$

Note that the determinant condition implies the S-scheme A has relative dimension $2dm$.

In the Hilbert-Siegel case ($B = \mathrm{Mat}_2(F)$, $O_B = \mathrm{Mat}_2(O_F)$, and $* = t$), it is known that there always exists a *principally* polarized O_B-abelian variety (A, λ, ι) over $\mathbb{C}$ (for example, using Morita equivalence, one may take a product of m-copies of a d-dimensional principally polarized O_F-abelian variety). However, for a general triple $(B, *, O_B)$, the existence of such abelian varieties is not always true. Our first result gives a necessary and sufficient condition for the existence of principally polarized O_B-abelian varieties.

Theorem 4. *Let $(B, *, O_B)$ be as above and m be a positive integer. Then the following statements are equivalent:*

- (a) *There exists a complex principally polarized O_B-abelian variety of dimension $2dm$.*
- (b) *For a (unique) $\mathbb{Q}$-valued skew-Hermitian $(B, *)$-module (V, ψ) of rank m, there exists a self-dual O_B-lattice Λ in (V, ψ).*
- (c) *Either m is even, or for any finite place v of F ramified in B one has $\mathrm{ord}_{\Pi_v}(\gamma)$ is odd. Here, Π_v denotes a uniformizer of the completion $B_v = B \otimes_F F_v$ at v, and $\mathrm{ord}_{\Pi_v}(\,\cdot\,)$ denotes the Π_v-adic valuation.*
- (d) *There exists a principally polarized O_B-abelian variety of dimension $2dm$ over an algebraically closed field k of characteristic p which satisfies the determinant condition.*

Under these conditions, a self-dual O_B-lattice Λ as in Statement (b) *is unique up to isomorphism.*

Note that the determinant condition in Statement (d) can not be omitted. Indeed, there is a datum $(B, *, O_B)$ where B is an indefinite quaternion $\mathbb{Q}$-algebra and p divides the discriminant of $B/\mathbb{Q}$ such that

- there exists a principally polarized O_B-abelian surface over k;
- a principally polarized O_B-abelian surface over $\mathbb{C}$ does not exist;

see Theorem 7.

Hereafter we impose the conditions in Theorem 4 on $(B, *, O_B)$ and m. We fix a *principal* integral PEL datum $\mathscr{D} = (B, *, O_B, V, \psi, \Lambda, h_0)$ of type C, a prime number p, and an integer $N \geq 3$ with $p \nmid N$. The lattice Λ gives a model over $\mathbb{Z}$ of the $\mathbb{Q}$-group $\mathbf{G}$, denoted again by $\mathbf{G}$. We define a compact open subgroup $\mathsf{K}^p(N)$ of $\mathbf{G}(\mathbb{A}_f^p)$ as the kernel of the reduction mod N map $\mathbf{G}(\widehat{\mathbb{Z}}^p) \to \mathbf{G}(\widehat{\mathbb{Z}}^p/N\widehat{\mathbb{Z}}^p) = \mathbf{G}(\mathbb{Z}/N\mathbb{Z})$. We set $\mathsf{K}_p = \mathbf{G}(\mathbb{Z}_p)$ and $\mathsf{K} = \mathsf{K}_p \cdot \mathsf{K}^p(N) \subset \mathbf{G}(\mathbb{A}_f)$. Let $\mathbf{M}_\mathsf{K} = \mathbf{M}_\mathsf{K}(\mathscr{D})$ be the contravariant functor from the category of locally Noetherian schemes over $\mathbb{Z}_{(p)}$ to the category of sets which takes a connected scheme S over $\mathbb{Z}_{(p)}$ to the set of isomorphism classes of tuples $(A, \lambda, \iota, \overline{\eta})$ where

- (A, λ, ι) is a principally polarized O_B-abelian scheme over S as in Definition 3 which satisfies the determinant condition.
- $\overline{\eta}$ is a $\pi_1(S, \overline{s})$-invariant $\mathsf{K}^p(N)$-orbit of $O_B \otimes \widehat{\mathbb{Z}}^p$-linear isomorphisms $\eta : \Lambda \otimes \widehat{\mathbb{Z}}^p \xrightarrow{\sim} T^p(A_{\overline{s}})$ which preserve the pairings

  $$\psi : \Lambda \otimes \widehat{\mathbb{Z}}^p \times \Lambda \otimes \widehat{\mathbb{Z}}^p \to \widehat{\mathbb{Z}}^p \quad \text{and} \quad \langle \, , \, \rangle_\lambda : \widehat{T}^p(A_{\overline{s}}) \times \widehat{T}^p(A_{\overline{s}}) \to \widehat{\mathbb{Z}}^p(1)$$

  up to a scalar in $(\widehat{\mathbb{Z}}^p)^\times$. Here, $\overline{s}$ is a geometric point of S, $A_{\overline{s}}$ is the fiber of A over $\overline{s}$, $\widehat{T}^p(A_{\overline{s}})$ is its prime-to-p Tate module, and $\langle \, , \, \rangle_\lambda$ is the alternating pairing induced by λ.

Two tuples $(A, \lambda, \iota, \overline{\eta})$ and $(A', \lambda', \iota', \overline{\eta}')$ are said to be isomorphic if there exists an O_B-linear isomorphism of abelian schemes $f : A \xrightarrow{\sim} A'$ such that $\lambda = f^t \circ \lambda' \circ f$ and $\overline{\eta}' = \overline{f \circ \eta}$.

By [14] and [15, Ch.2], the functor $\mathbf{M}_\mathsf{K}$ is represented by a quasi-projective scheme (denoted again by) $\mathbf{M}_\mathsf{K}$ over $\mathbb{Z}_{(p)}$. We remark that $\mathbf{M}_\mathsf{K}$ is isomorphic to the moduli problem of prime-to-p isogeny classes of abelian schemes with a $\mathbb{Z}_{(p)}^\times$-polarization which was studied in [14], under the assumption that Λ is self-dual ([15, Prop. 1.4.3.4]). Since the $\mathbb{Q}$-group $\mathbf{G}$ satisfies the Hasse Principle, the generic fiber $\mathbf{M}_\mathsf{K} \otimes_{\mathbb{Z}_{(p)}} \mathbb{Q}$ is isomorphic to the canonical model $\mathrm{Sh}_\mathsf{K}(\mathbf{G}, X)$ (rather than a finite union of them).

4. The superspecial and supersingular locus

We write $\mathcal{M}_\mathsf{K} := \mathbf{M}_\mathsf{K} \otimes_{\mathbb{Z}_{(p)}} \overline{\mathbb{F}}_p$ for the geometric special fiber of $\mathbf{M}_\mathsf{K}$. It is known that the ordinary locus of $\mathcal{M}_\mathsf{K}$ is non-empty if and only if either m is even or every place v of F lying over p is unramified in B (see [31, Proposition 2.2]). Here we consider the opposite extreme case. We write

$$\mathcal{M}_{\mathsf{K}}^{\mathrm{sp}} \subset \mathcal{M}_{\mathsf{K}}^{\mathrm{ss}} \subset \mathcal{M}_{\mathsf{K}}$$

for the superspecial and supersingular locus: the largest reduced closed subschemes such that

$$\mathcal{M}_{\mathsf{K}}^{\mathrm{sp}}(\overline{\mathbb{F}}_p) = \{(A, \lambda, \iota, \bar{\eta}) \in \mathcal{M}_{\mathsf{K}}(\overline{\mathbb{F}}_p) \mid A \text{ is superspecial}\},$$
$$\mathcal{M}_{\mathsf{K}}^{\mathrm{ss}}(\overline{\mathbb{F}}_p) = \{(A, \lambda, \iota, \bar{\eta}) \in \mathcal{M}_{\mathsf{K}}(\overline{\mathbb{F}}_p) \mid A \text{ is supersingular}\}.$$

Theorem 5. *The superspecial locus $\mathcal{M}_{\mathsf{K}}^{\mathrm{sp}}$ is non-empty.*

Note that there is no assumption on p in Theorem 5. A main step of the proof is to construct a principally polarized Dieudonné $O_B \otimes \mathbb{Z}_p$-module satisfying the determinant condition. This requires the equivalent conditions in Theorem 4.

To describe our next result, we assume that p is unramified in B. Then $\mathsf{K}_p = \mathbf{G}(\mathbb{Z}_p) \subset \mathbf{G}(\mathbb{Q}_p)$ is a hyperspecial parahoric subgroup and $\mathcal{M}_{\mathsf{K}}$ is a smooth algebraic variety over k. In this case, an exact formula for the cardinality of the superspecial locus $\mathcal{M}_{\mathsf{K}}^{\mathrm{sp}}$ was given in [30, Theorem 1.3], using Shimura's mass formula [23]. However, in [30] it is implicitly assumed that there exists a self-dual O_B-lattice Λ (Theorem 4) and that the superspecial locus $\mathcal{M}_{\mathsf{K}}^{\mathrm{sp}}$ is non-empty (Theorem 5).

In [9], Hamacher gave a formula for the dimension of Newton strata on the reduction of PEL Shimura varieties (of type A or C) with hyperspecial level at p. In the moduli scheme $\mathcal{M}_{\mathsf{K}}$ of type C, the unique closed Newton stratum (called the basic locus) is precisely the supersingular locus $\mathcal{M}_{\mathsf{K}}^{\mathrm{ss}}$: It is equi-dimensional of dimension

$$\dim \mathcal{M}_{\mathsf{K}}^{\mathrm{ss}} = \sum_{v|p} \left(\lfloor f_v/2 \rfloor \frac{m(m+1)}{2} + (f_v - 2\lfloor f_v/2 \rfloor) \cdot \lfloor m^2/4 \rfloor \right), \tag{4.1}$$

where v runs over the places of F over p and f_v is the inertia degree of v.

We give an explicit formula for the number of irreducible components of $\mathcal{M}_{\mathsf{K}}^{\mathrm{ss}}$. Let $D_{p,\infty}$ denote the unique quaternion $\mathbb{Q}$-algebra ramified precisely at $\{p, \infty\}$, and D the unique quaternion F-algebra such that $B \otimes_{\mathbb{Q}} D_{p,\infty} \simeq \mathrm{Mat}_2(D)$. Let Δ' be the discriminant of D over F. For a finite place v of F, let $q_v := p^{f_v}$ be the cardinality of the residue field of v.

Theorem 6. *Assume that $p > 2$ is unramified in B. Then the number of irreducible components of the supersingular locus $\mathcal{M}_{\mathsf{K}}^{\mathrm{ss}}$ is equal to*

$$|\mathbf{G}(\mathbb{Z}/N\mathbb{Z})| \cdot \prod_{v|p} \binom{f_v}{\lfloor f_v/2 \rfloor}^m \cdot \frac{(-1)^{dm(m+1)/2}}{2^{md}} \cdot \prod_{j=1}^{m} \zeta_F(1-2j) \cdot \prod_{v|\Delta'} \lambda_v,$$

where $\zeta_F(s)$ is the Dedekind zeta function of F, and for $v \mid \Delta'$,

$$\lambda_v = \begin{cases} \prod_{i=1}^{m}(q_v^i + (-1)^i) & \textit{if } m \textit{ is odd, or } v \nmid p \textit{ and } \mathrm{ord}_{\Pi_v}(\gamma) \textit{ is odd;} \\ \prod_{i=1}^{m/2}(q_v^{4i-2} - 1) & \textit{otherwise.} \end{cases}$$

Theorem 6, together with Equation (4.1) and [30, Theorem 1.3], generalizes Theorem 1 to good reduction of an arbitrary type C family of Shimura varieties.

Here we give a sketch of the proof of Theorem 6. Let L be the field of fractions of the ring $W(\overline{\mathbb{F}}_p)$ of Witt vectors over $\overline{\mathbb{F}}_p$. The absolute Frobenius endmorphism is denoted by σ. One can associate an affine Deligne-Lusztig variety $X_\mu(b)$ with an element $b \in \mathbf{G}(L)$ and the cocharacter μ_h as in (2.4). This is a locally closed subscheme of the Witt vector partial affine flag variety $\mathrm{Gr}_{\mathbf{G}_{\mathbb{Q}_p}}$ ([1, 34]). We choose an element b such that b is basic ([13, Section 5]) and that $X_\mu(b)$ is non-empty. Further we define a $\mathbb{Q}_p$-group J_b by

$$J_b(R) = \{g \in G(L \otimes_{\mathbb{Q}_p} R) \mid g^{-1} b \sigma(g) = b\}$$

for any commutative $\mathbb{Q}_p$-algebra R. Then $J_b(\mathbb{Q}_p)$ naturally acts on $X_\mu(b)(\overline{\mathbb{F}}_p)$ by left multiplication. We write $\mathrm{Irr}(X_\mu(b))$ for the set of irreducible components of $X_\mu(b)$. By the work of Nie [19] and Zhou-Zhu [33], the set of orbits of $\mathrm{Irr}(X_\mu(b))$ under the action of $J_b(\mathbb{Q}_p)$ is in natural bijection with the "Mirkovic-Vilonen basis" of a certain weight space of a representation of the dual group of $\mathbf{G}_{\mathbb{Q}_p}$. We first compute the dimension of this weight space and obtain a formula for the number of $J_b(\mathbb{Q}_p)$-orbits of $\mathrm{Irr}(X_\mu(b))$:

$$|J_b(\mathbb{Q}_p) \backslash \mathrm{Irr}(X_\mu(b))| = \prod_{v \mid p} \binom{f_v}{\lfloor f_v/2 \rfloor}^m. \tag{4.2}$$

Next we describe the supersingular locus via the p-adic uniformization theorem of Rapoport and Zink. Take a point $x \in \mathcal{M}_{\mathsf{K}}(\overline{\mathbb{F}}_p)$, and let (A, λ, ι) denote the principally polarized O_B-abelian variety over $\overline{\mathbb{F}}_p$ corresponding to x. We write $\mathrm{End}_B^0(A) := \mathrm{End}_{O_B}(A) \otimes \mathbb{Q}$, and define a $\mathbb{Q}$-group I by

$$I(R) = \{g \in (\mathrm{End}_B^0(A) \otimes_{\mathbb{Q}} R)^\times \mid \exists c(g) \in R^\times \text{ s.t. } g' \cdot g = \mathrm{id} \otimes c(g)\}$$

for any commutative $\mathbb{Q}$-algebra R. Here, $g \mapsto g'$ is the Rosati involution induced by λ. Then the $\mathbb{Q}$-group I is an inner form of $\mathbf{G}_{\mathbb{Q}}$, and such that $I(\mathbb{R})$ is compact modulo center. Further, there are natural identifications

$$I_{\mathbb{Q}_\ell} = \begin{cases} \mathbf{G}_{\mathbb{Q}_\ell} & \text{if } \ell \neq p; \\ J_b & \text{if } \ell = p. \end{cases} \tag{4.3}$$

By Theorem 5, the supersingular locus $\mathcal{M}_{\mathsf{K}}^{\mathrm{ss}}$ is non-empty. Further, the supersingular locus is precisely the basic locus ([26, Definition 8.2 and Example 8.3]). Moreover, the group $\mathbf{G}_{\mathbb{Q}}$ satisfies the Hasse principle. Hence the p-adic uniformization theorem applies to the supersingular locus: There is an isomorphism of perfect schemes

$$I(\mathbb{Q})\backslash X_\mu(b) \times \mathbf{G}(\mathbb{A}_f^p)/\mathsf{K}^p(N) \xrightarrow{\sim} \mathcal{M}_{\mathsf{K}}^{\mathrm{ss,pfn}}, \tag{4.4}$$

where $\mathcal{M}_{\mathsf{K}}^{\mathrm{ss,pfn}}$ denotes the perfection of $\mathcal{M}_{\mathsf{K}}^{\mathrm{ss}}$ ([21, Theorem 6.30], [27, Corollary 7.2.16], [11, Proposition 5.2.2]). This step is the place where the assumption $p > 2$ is used.

Then we relate the the number of irreducible components of the supersingular locus to the *mass* of I with respect to an open compact subgroup U of $I(\mathbb{A}_f)$. Here, the mass of I with respect to U is defined as a weighted cardinality of the double coset space $I(\mathbb{Q})\backslash I(\mathbb{A}_f)/U$:

$$\mathrm{Mass}(I, U) := \sum_{[g]\in I(\mathbb{Q})\backslash I(\mathbb{A}_f)/U} \frac{1}{|I(\mathbb{Q}) \cap g^{-1}Ug|}.$$

For each irreducible component Y of $X_\mu(b)$, let $J_b(Y)$ denote the stabilizer of Y in $J_b(\mathbb{Q}_p)$. The result of He-Zhou-Zhu [11] shows that $J_b(Y)$ is a parahoric subgroup of $J_b(\mathbb{Q}_p)$ having maximum volume. We fix identifications $\mathbf{G}(\mathbb{A}_f^p) = I(\mathbb{A}_f^p)$ and $J_b(\mathbb{Q}_p) = I(\mathbb{Q}_p)$ as in (4.3), and we regard $J_b(Y)\mathbf{G}(\widehat{\mathbb{Z}}^p)$ as a subgroup of $I(\mathbb{A}_f)$. The action of $J_b(\mathbb{Q}_p)$ on the set $\mathrm{Irr}(X_\mu(b))$ induces a bijection

$$\coprod_{[Y]\in J_b(\mathbb{Q}_p)\backslash \mathrm{Irr}(X_\mu(b))} J_b(\mathbb{Q}_p)/J_b(Y) \xrightarrow{\sim} \mathrm{Irr}(X_\mu(b)).$$

This bijection together with the isomorphism in (4.4) induces a bijection

$$\coprod_{[Y]\in J_b(\mathbb{Q}_p)\backslash \mathrm{Irr}(X_\mu(b))} I(\mathbb{Q})\backslash I(\mathbb{A}_f)/J_b(Y)\mathsf{K}^p(N) \xrightarrow{\sim} \mathrm{Irr}(\mathcal{M}_{\mathsf{K}}^{\mathrm{ss}}). \tag{4.5}$$

The assumption $N \geq 3$ implies $I(\mathbb{Q}) \cap (g^{-1}J_b(Y)\mathsf{K}^p(N)g) = 1$ for any $g \in I(\mathbb{A}_f)$ (cf. [18, Lemma, p. 207]). Hence we have that

$$\begin{aligned} &|I(\mathbb{Q})\backslash I(\mathbb{A}_f)/J_b(Y)\mathsf{K}^p(N)| \\ =&\,\mathrm{Mass}(I, J_b(Y)\mathsf{K}^p(N)) \\ =&\,\mathrm{Mass}(I, J_b(Y)\mathbf{G}(\widehat{\mathbb{Z}}^p)) \cdot |\mathbf{G}(\mathbb{Z}/N\mathbb{Z})|. \end{aligned} \tag{4.6}$$

Finally, we use the arithmetic mass formula of Gan, Hanke, J.-K. Yu for quaternionic unitary groups [8, Section 9] and compute

$$\mathrm{Mass}(I, J_b(Y)\mathbf{G}(\widehat{\mathbb{Z}}^p)) = \frac{(-1)^{dm(m+1)/2}}{2^{md}} \cdot \prod_{j=1}^{m} \zeta_F(1-2j) \cdot \prod_{v|\Delta'} \lambda_v. \tag{4.7}$$

Here, λ_v for a place $v \mid \Delta'$ is the reciprocal of a volume of a certain parahoric subgroup of a quaternionic unitary group over $\mathbb{Q}_\ell$ when $v \mid \ell$. The theorem follows from (4.2), (4.5), (4.6), and (4.7).

We note that our method also applies to the basic locus of a $\mathrm{GU}(r,s)$ Shimura variety (of type A) attached to an imaginary quadratic field [25].

5. Bad reduction of Shimura curves

In this section we discuss the case where $m = 1$ and $d = 1$ (that is, $F = \mathbb{Q}$), and p is allowed to be ramified in B. Thus, we restrict the datum $(B, *, O_B)$ to the case where B is an indefinite quaternion $\mathbb{Q}$-algebra. We write Δ for the discriminant of B over $\mathbb{Q}$. Let $\mathbf{M}$ (resp. $\mathbf{M}_{\mathsf{K}}$) denote the moduli scheme over $\mathbb{Z}_{(p)}$ of principally polarized O_B-abelian surfaces (A, λ, ι) (resp. with level N-structure) satisfying the determinant condition. Further, we relax the condition on the moduli scheme $\mathbf{M}$ by removing the determinant condition, and write $\widetilde{\mathbf{M}}$ for the coarse moduli scheme over $\mathbb{Z}_{(p)}$ of principally polarized O_B-abelian surfaces. Note that we have $\mathbf{M}_{\mathbb{Q}} = \widetilde{\mathbf{M}}_{\mathbb{Q}}$.

We call a positive involution $*$ *principal* if there exists an element $\gamma \in B^\times$ that satisfies (2.1) and $\gamma^2 = -\Delta$. In this case, every O_B-abelian surface (A, ι) with the determinant condition admits a unique O_B-linear principal polarization λ ([6, Proposition 4.3] and [2, Proposition 3.3]). The geometry of $\mathbf{M}$ in this case has been studied and is well-understood; see [20, 3]. In particular, the Cherenik-Drinfeld theorem [6, Section 4] states that the formal completion of $\mathbf{M} \otimes W(\overline{\mathbb{F}}_p)$ along the special fiber at $p \mid \Delta$ admits a p-adic unformization by one-dimensional Deligne's formal scheme $\widehat{\Omega}^{\mathrm{nr}} = \widehat{\Omega} \hat{\otimes} W(\overline{\mathbb{F}}_p)$.

From now on, we no longer assume that $*$ is principal. We set

$$S := \{\text{primes } \ell : \ell \mid \Delta,\ \mathrm{ord}_{\Pi_\ell}(\gamma) \text{ is even}\},$$

where Π_ℓ denotes a uniformizer of $B \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$. Note that the set S depends only on $(B, *, O_B)$ but not the choice of γ. Denote by $\mathcal{M} := \mathbf{M} \otimes \overline{\mathbb{F}}_p$, $\mathcal{M}_{\mathsf{K}} := \mathbf{M}_{\mathsf{K}} \otimes \overline{\mathbb{F}}_p$, and $\widetilde{\mathcal{M}} := \widetilde{\mathbf{M}} \otimes \overline{\mathbb{F}}_p$ the geometric special fibers. Using the methods of this article, we obtain the following explicit results.

Theorem 7. *Assume that $m = 1$ and $d = 1$.*

(1) *The following statements are equivalent:*
  (a) *The generic fiber $\mathbf{M}_{\mathbb{Q}}$ is non-empty.*
  (b) *The special fiber $\mathcal{M}$ is non-empty.*
  (c) *The set S is empty.*
  (d) *The involution $*$ is principal.*

(2) *Assume that the conditions in* (1) *hold and that* $p \mid \Delta$. *Then the special fiber* $\mathcal{M}_{\mathsf{K}}$ *has*

$$|\mathbf{G}(\mathbb{Z}/N\mathbb{Z})|\cdot \frac{1}{12} \cdot \prod_{\ell \mid (\Delta/p)} (\ell - 1)$$

irreducible components.

(3) *The moduli scheme* $\widetilde{\mathbf{M}}$ *is non-empty if and only if* $S \subset \{p\}$.

(4) *When* $S = \{p\}$, *the moduli scheme* $\widetilde{\mathbf{M}}$ *is contained in the special fiber* $\widetilde{\mathbf{M}} \otimes \mathbb{F}_p$. *Moreover, the geometric special fiber* $\widetilde{\mathcal{M}}$ *is finite and we have the mass formula*

$$\mathrm{Mass}(\widetilde{\mathcal{M}}(\overline{\mathbb{F}}_p)) := \sum_{[(A,\lambda,\iota)] \in \widetilde{\mathcal{M}}(\overline{\mathbb{F}}_p)} \frac{1}{|\operatorname{Aut}(A,\lambda,\iota)|} = \frac{1}{12} \prod_{\ell \mid (\Delta/p)} (\ell - 1). \tag{5.1}$$

Remark 8. A triple $(B, *, O_B)$ satisfying $S = \{p\}$ can be obtained as follows: Take a prime $\ell \neq p$ such that p is inert or ramified in $K := \mathbb{Q}(\sqrt{-\ell})$. Let B be the indefinite quaternion $\mathbb{Q}$-algebra ramified exactly at $\{p, \ell\}$. Then there exists an embedding $K \hookrightarrow B$ of $\mathbb{Q}$-algebras, and hence an element $\gamma \in B^\times$ such that $\gamma^2 = -\ell$. Define a positive involution $*$ on B by $b \mapsto b^* = \gamma \bar{b} \gamma^{-1}$. Further, choose a maximal order O_B of B containing γ. Then the triple $(B, *, O_B)$ satisfies the desired property.

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