ON A PRINCIPLE OF OGUS: THE HASSE INVARIANT'S ORDER OF VANISHING AND "FROBENIUS AND THE HODGE FILTRATION"

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1. Introduction

This work is inspired by the following result by Ogus in [Ogu01b] and [Ogu01a]. Let k be an algebraically closed field of characteristic $p > 0$. Let $X \to S$ be a family of Calabi-Yau[1] varieties over k. Under some technical conditions on $X \to S$, including degeneration of the Hodge spectral sequence at E_1, Ogus proves that the order of vanishing of the Hasse invariant on S is equal to the "conjugate line position", i.e. the largest piece of the Hodge filtration containing the line of the conjugate filtration. We call this equality *Ogus' principle*.

1.1. **Main results.** We introduce a group-theoretical Ogus' principle associated to triples (G, μ, r), where G is a connected, reductive $\mathbf{F}_p$-group, $\mu\colon \mathbf{G}_m \to G_k$ is a cocharacter, and r is a representation of G. To such a triple we associate a global section $\mathrm{Ha}(G, \mu, r)$ of a line bundle on $G\text{-}\mathtt{Zip}^\mu$, the stack of G-zips of type μ, which generalizes the classical Hasse invariant defined via de Rham cohomology. We say that Ogus' principle is satisfied if the vanishing order of $\mathrm{Ha}(G, \mu, r)$ equals the conjugate line position of the pushforward along r of the universal G-zip of type μ over $G\text{-}\mathtt{Zip}^\mu$. We prove Ogus' principle for several triples (G, μ, r).

Theorem 1.1.1. *Ogus' principle holds for the following triples (G, μ, r):*

(a) $\mathsf{type}(G) = \mathsf{A}_1^d$, *$\mu$ is regular and r is faithful of dimension $2d$*
(b) $G = \mathrm{GL}(n)$, $\mathsf{type}(L) = \mathsf{A}_{n-2}$ *and* $r = \wedge^n(\mathrm{Std} \oplus \mathrm{Std}^\vee)$,
(c) $G = \mathrm{GSp}(2n)$, $\mathsf{type}(L) = \mathsf{A}_{n-1}$ *and* $r = \wedge^n \mathrm{Std}^\vee$,
(d) $G = \mathrm{SO}(2n+1)$, $\mathsf{type}(L) = \mathsf{B}_{n-1}$ *and* $r = \mathrm{Std}$
(e) $G = \mathrm{SO}(2n)$, $\mathsf{type}(L) = \mathsf{D}_{n-1}$ *and* $r = \mathrm{Std}$
(f) $G = \mathrm{GL}(4)$, $\mathsf{type}(L) = \mathsf{A}_1 \times \mathsf{A}_1$ *and* $r = \wedge^2(\mathrm{Std})$.

Remark 1.1.2. We also give an explicit value of the vanishing order of the Hasse invariant $\mathrm{Ha}(G, \mu, r)$ on each Ekedahl-Oort and Bruhat stratum of $G\text{-}\mathtt{Zip}^\mu$. This is also done for (G, μ, r) where G is a spin similitude group of type B_m (resp. D_m), $\mathsf{type}(L) = \mathsf{B}_{m-1}$ (resp. D_{m-1}) and r is the spin representation.

Let $(\mathbf{G}, \mathbf{X})$ be an abelian-type Shimura datum. Assume that $\mathbf{G}$ is unramified at p. Let $K_p \subset \mathbf{G}(\mathbf{Q}_p)$ be a hyperspecial maximal compact subgroup. By the work of Kisin [Kis10] and Vasiu [Vas99], as K^p ranges over open, compact subgroups of $\mathbf{G}(\mathbf{A}_f^p)$, the associated projective system of Shimura varieties admits an integral canonical model $(\mathscr{S}_{K_pK^p}(\mathbf{G}, \mathbf{X}))_{K^p}$ in the sense of Milne [Mil92]. Set $K := K_pK^p$ and let S_K be the special k-fiber of $\mathscr{S}_{K_pK^p}(\mathbf{G}, \mathbf{X})$. Let G be the reductive $\mathbf{F}_p$-group deduced from the $\mathbf{Q}$-group $\mathbf{G}$ and let $\mu \in X_*(G)$ be a representative of the conjugacy class of cocharacters deduced from the Hermitian symmetric space $\mathbf{X}$. By Zhang [Zha18] there is a smooth morphism $\zeta_K\colon S_K \to G\text{-}\mathtt{Zip}^\mu$. We deduce the following.

[1] An n-dimensional variety X is Calabi-Yau if $\dim H^0(X, \mathcal{O}_X) = \dim H^n(X, \mathcal{O}_X) = 1$ and $H^i(X, \mathcal{O}_X) = 0$ for all $i \neq 0, n$. The vanishing of $H^1(X, \mathcal{O}_X)$ excludes abelian varieties.

Corollary 1.1.3. Ogus' principle holds for S_K if the $\mathbf{Q}$-group $\mathbf{G}$ is either of the following:

(a) A restriction of scalars $\mathrm{Res}_{F/\mathbf{Q}}(\mathrm{GL}(2))$ for a totally real extension $F/\mathbf{Q}$.
(b) A unitary group associated to an imaginary quadratic field $K/\mathbf{Q}$ such that the base change $\mathbf{G}_\mathbf{R}$ to $\mathbf{R}$ is a unitary similitude group of signature either $(n-1,1)$ or $(2,2)$ and p splits in K.
(c) The $\mathbf{Q}$-split symplectic group (the Siegel case).
(d) A $\mathbf{Q}$-form of the orthogonal group $\mathrm{SO}(m)$ such that $\mathbf{G}_\mathbf{R} \cong \mathrm{SO}(m-2,2)$. If m is even, assume that G is $\mathbf{F}_p$-split.

Remark 1.1.4. In upcoming joint work with Jean-Stefan Koskivirta we extend (b) to arbitrary signatures at inert primes.

We make the following conjecture.

Conjecture 1.1.5. *Suppose that $(\mathbf{G},\mathbf{X})$ is of Hodge-type. Let Y/S_K be the abelian variety obtained from a symplectic embedding of $(\mathbf{G},\mathbf{X})$. Let $n \coloneqq \dim(Y/S_K)$. Assume that the conjugacy class of μ is defined over $\mathbf{F}_p$. For every k-point x of S_K,*

$$\mathrm{ord}_x \mathrm{Ha}(H^n_{\mathrm{dR}}(Y/S_K)) = \mathsf{clp}_x H^n_{\mathrm{dR}}(Y/S_K).$$

Finally, using Madapusi-Pera's extension of the Kuga-Satake construction to mixed characteristic [Mad15] we also recover Ogus' result for K3-surfaces.

Corollary 1.1.6. Let $M^\circ_{2d,K}$ be the moduli space of K3 surfaces with level K and a polarization of degree $2d$. Then Ogus' principle holds for the universal family $Y/M^\circ_{2d,K}$.

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2. Ogus principle for families mod p

Let $f\colon X \to S$ be a smooth proper morphism over k of dimension n. Let $F_S\colon S \to S$ denote the absolute Frobenius morphism, let $X^{(p)} \coloneqq X \times_{S,F_S} S$ with structure morphism $f^{(p)}\colon X^{(p)} \to S$.

The de Rham cohomology $H^i_{\mathrm{dR}}(X/S)$ comes equipped with two spectral sequences degenerating to it; the Hodge spectral sequence

$$E^{a,b}_{\mathrm{Hdg},1} = R^b f_* \Omega^a_{X/S} \implies H^{a+b}_{\mathrm{dR}}(X/S)$$

and the conjugate spectral sequence[2]

$$E^{a,b}_{\mathrm{conj},2} \cong R^a f^{(p)}_* \Omega^b_{X^{(p)}/S} \implies H^{a+b}_{\mathrm{dR}}(X/S).$$

We make the following assumptions.

dR.1 For all $a,b \geq 0$, the sheaves $R^a\pi_*\Omega^b_{X/S}$ are locally free;
dR.2 the Hodge spectral sequence degenerates at E_1; and
dR.3 the sheaf $R^n f_* \mathscr{O}_X$ is a line bundle.

[2]The conjugate spectral sequence as written here is not the one obtained from definition of H_{dR} as a hypercohomology. Rather, we use here the Cartier isomorphisms.

The assumptions **dR.1** and **dR.2** above implies that the construction of the sheaves $R^a\Omega^b_{X/S}$ commutes with arbitrary base change and that the conjugate spectral sequence degenerates at the second page. A list of families satisfying assumption **dR.1** and **dR.2** is copied from [MW04]:

(a) Any abelian scheme $A \to S$.
(b) Any smooth proper curve $C \to S$.
(c) Any K3-surface $X \to S$.
(d) Every smooth complete intersection in $\mathbb{P}^n_S$.

In particular, in the applications to Shimura varieties the assumptions are satisfied.

Let $\mathrm{Fil}^\bullet_{\mathrm{Hdg}}$ denote the decreasing Hodge filtration on $H^n_{\mathrm{dR}}(X/S)$ and let $\mathrm{Fil}_{\mathrm{conj},\bullet}$ denote the increasing conjugate filtration. For all i the Cartier isomorphism induces an isomorphism

$$\varphi_i \colon (\mathrm{Gr}^i \mathrm{Fil}^\bullet_{\mathrm{Hdg}})^{(p)} \to \mathrm{Gr}_i \mathrm{Fil}_{\mathrm{conj},\bullet}\,.$$

See [Kat70] for details.

2.1. **The Hasse invariant and the conjugate line position.** The *Hasse invariant* is defined as the composition

$$\mathrm{Ha}_S \colon (\mathrm{Gr}^0 \mathrm{Fil}^\bullet_{\mathrm{Hdg}})^{(p)} \xrightarrow{\varphi_i} \mathrm{Gr}_0 \mathrm{Fil}_{\mathrm{conj},\bullet} \hookrightarrow H^n_{\mathrm{dR}}(X/S) \twoheadrightarrow \mathrm{Gr}^0 \mathrm{Fil}^\bullet_{\mathrm{Hdg}}\,.$$

By **dR.3** $\mathrm{Gr}_0 \mathrm{Fil}^\bullet_{\mathrm{Hdg}} = R^n f_* \mathscr{O}_X$ is a line bundle and thus Ha_S can be seen as a section of ω^{p-1}, where $\omega := \det f_*\Omega_{X/S}$.

Remark 2.1.1. If $X \to S$ is the universal principally polarized abelian variety (with some level structure), then the non-vanishing set of the Hasse invariant consists exactly of the ordinary abelian varieties (i.e., those whose p-rank is maximal). For cases of Shimura varieties, the Hasse invariant also satisfies other desirable properties such as being compatible with varying the prime to p level. Several authors have constructed generalisations of the classical Hasse invariant enjoying similar properties to those described above (see e.g., [GN17], [KW18], [Gor01]). Thanks to these properties, the Hasse invariant and its generalisations have been used succesfully to produce congruences between automorphic forms and automorphic Galois representations in the Langlands program (see e.g., [ERX17], [Tay91], [GK19a]). The idea of using the Hasse invariant to this end goes back to the work of Deligne–Serre in the 1970's on modular forms of weight 1 ([DS74]). The theory of Hasse invariants has also been used to study the geometry of Shimura varieties mod p. For instance, in [GK19a] it is shown that the Ekedahl–Oort (E–O) stratification of a Hodge-type Shimura variety is uniformly principally pure (see [GK19b] for a definition), and that each stratum in the E–O stratification of the minimal compactification is affine.

For any k-point s in S, the *conjugate line position*[3] is defined as

$$\mathsf{clp}_s(H^n_{\mathrm{dR}}(X/S)) := \min\{i : \mathrm{Fil}_{\mathrm{conj},0,s} \subset \mathrm{Fil}^i_{\mathrm{Hdg},s}\}.$$

2.1.2. *Ogus' principle for families mod p.* We say that Ogus' principle holds for $X \to S$ if, for all k-points s in S, we have that

$$\mathrm{ord}_s(\mathrm{Ha}_S) = \mathsf{clp}_s(H^n_{\mathrm{dR}}(X/S)).$$

Ogus proves that this holds for families of Calabi-Yau varieties satisfying the following assumptions.

(a) Same as assumption **dR.2**.
(b) The Kodaira-Spencer map[4] $\Theta \colon T_{S/k} = (\Omega^1_{S/k})^\vee \to R^1\pi_* T_{X/S}$ is surjective.

[3] This is sometimes referred to as the a-number. We prefer the notion conjugate line position as it is more descriptive.

[4] Defined e.g. as the coboundary map of $R^1\pi_*$ applied to the short exact seqquence $0 \to T_{X/S} \to T_{X/k} \to \pi^* T_{S/k} \to 0$.

(c) For all pair of integers $j \leq m < p$ the maps

$$\mathrm{Sym}^j R^1\pi_*T_{X/S} \otimes \pi_*\Omega^1_{X/S} \to R^j\pi_*\Omega^{n-j}_{X/S}$$

induced by cup product and interior multiplication, are surjective.

3. Group theoretic Ogus' principle

3.1. **Group theoretic notation.** Fix a connected, reductive $\mathbf{F}_p$-group G and a cocharacter $\mu\colon \mathbf{G}_m \to G_k$. It determines a pair of opposite parabolics P, P^+ of G_k intersecting in a common Levi factor $L := P \cap P^+ = \mathrm{Cent}_{G_k}(\mu)$. Let $Q := (P^+)^{(p)}$ with Levi quotient M.

Let $\sigma : k \to k$ denote the arithmetic Frobenius $a \mapsto a^p$. Given a k-scheme X, let $X^{(p)} := X \otimes_{k,\sigma} k$ be its Frobenius twist and let $\varphi : X \to X^{(p)}$ be the relative Frobenius morphism. The unipotent radical of an algebraic k-group H is denoted $R_u(H)$. If $g \in G$ and $H \subset G$, then let ${}^gH := gHg^{-1}$.

3.2. ***G*-zips.** The theory of G-zips were developed by Pink-Wedhorn-Ziegler in [PWZ11] and [PWZ15] and builds on the notion of F-zips, introduced and studied by Moonen-Wedhorn in [MW04]. Thereafter, Goldring-Koskivirta further developed the theory in [GK19b]. We present here the main definitions needed to state our results.

3.2.1. *F-zips.* Let $\mathscr{V}$ be a locally free sheaf of rank n over a k-scheme S. By a descending filtration $\mathrm{Fil}^\bullet$ on $\mathscr{V}$ we mean a sequence of sub $\mathscr{O}_S$-modules $\mathrm{Fil}^\bullet = (\mathrm{Fil}^i)_{i\in\mathbf{Z}}$ such that Fil^i is a local direct summand of Fil^{i-1} for all i, and such that $\mathrm{Fil}^i = \mathscr{V}$ for all i small enough and $\mathrm{Fil}^i = 0$ for all i large enough. We remark in particular that the graded pieces $\mathrm{Gr}^i = \mathrm{Fil}^i / \mathrm{Fil}^{i-1}$ are locally free sheaves. We define an ascending filtration analogously and write Gr_i for its graded pieces. An *F-zip over S* is a tuple $\underline{\mathscr{V}} = (\mathscr{V}, \mathrm{Fil}^\bullet_{\mathrm{Hdg}}, \mathrm{Fil}^{\mathrm{conj}}_\bullet, \varphi_\bullet)$ where $\mathrm{Fil}^\bullet_{\mathrm{Hdg}}$ is a descending filtration on $\mathscr{V}$, $\mathrm{Fil}^{\mathrm{conj}}_\bullet$ is an ascending filtration on $\mathscr{V}$ and for all i, $\varphi_i\colon (\mathrm{Gr}^i)^{(p)} \xrightarrow{\sim} \mathrm{Gr}_i$ is an isomorphism between graded pieces. Given a function $\gamma\colon \mathbf{Z} \to \mathbf{N}$ we say that an F-zip has type γ if $\gamma(i) = \dim \mathrm{Gr}^i$ for all $i \in \mathbf{Z}$. To give a type γ is equivalent to give a (conjugacy class of a) cocharacter μ_γ.

Example 3.2.2. As we saw, for all i the de Rham cohomology $H^i_{\mathrm{dR}}(X/S)$ has the structure of an F-zip.

3.2.3. *G-zips of type μ.* A *G-zip of type μ* over S is a tuple (I, I_P, I_Q, ι) where I is a G_k-torsor over S, $I_P \subset I$ is a P-torsor over S, $I_Q \subset I^{(p)}$ is a Q-torsor over S and

$$\iota : I_P^{(p)}/R_u(P)^{(p)} \xrightarrow{\cong} I_Q/R_u(Q)$$

is an isomorphism of M-torsors. A morphism of G-zips $(I, I_P, I_Q, \iota) \to (I', I'_P, I'_Q, \iota')$ is a morphism of G_k-torsors $I \to I'$ compatible with the reductions to P and Q and compatible with the morphisms ι, ι'.

Example 3.2.4. If $G = \mathrm{GL}(n)$ then to give an F-zip of type γ is equivalent to give a G-zip of type μ_γ.

Let $G\text{-}\mathtt{Zip}^\mu(S)$ denote the category of G-zips (resp. G-zip flags) of type μ over S. This construction give rise to a smooth algebraic k-stack $G\text{-}\mathtt{Zip}^\mu$.

3.2.5. *F-zips of Calabi-Yau type.* Let $\underline{\mathscr{V}}$ be an F-zip over S, and let i_0 be the largest integer such that $\mathrm{Fil}^{i_0}_{\mathrm{Hdg}} = \mathscr{V}$. We say that $\underline{\mathscr{V}}$ is of *CY-type* if $\mathrm{Gr}^{i_0}_{\mathrm{Hdg}} := \mathrm{Fil}^{i_0}_{\mathrm{Hdg}} / \mathrm{Fil}^{i_0+1}_{\mathrm{Hdg}}$ is a line bundle.

3.3. **The Hasse invariant and conjugate line position of a triple (G,μ,r).** Let $\underline{\mathscr{V}}$ be an F-zip of Calabi-Yau type. We define a Hasse invariant analogously as to the classical case: the Hasse invariant of $\underline{\mathscr{V}}$ is defined as

$$\mathrm{Ha}(\underline{\mathscr{V}}) : (\mathrm{Gr}^{i_0})^{(p)} \xrightarrow{\varphi^i} \mathrm{Gr}_{i_0} \to \mathscr{V} \to \mathrm{Gr}^{i_0}.$$

If $\underline{\mathscr{V}}$ is not of Calabi-Yau type, then let $d := \dim \mathrm{Gr}^{i_0}$ and define $\mathrm{Ha}(\underline{\mathscr{V}})$ as $\mathrm{Ha}(\wedge^d \underline{\mathscr{V}})$.

The conjugate line position is defined for any k-point s as

$$\mathsf{clp}_s(\underline{\mathscr{V}}) := \max\{j \in \mathbf{Z} | \mathrm{Gr}_{i_0,s} \subset \mathrm{Gr}^j_s\} - i_0.$$

3.3.1. We say that a representation $r\colon G \to \mathrm{GL}(V)$ is of *Calabi-Yau* type if the highest μ-weight of V_k has multiplicity 1. Let $\underline{\mathscr{I}}$ denote the universal G-zip of type μ over the stack $G\text{-}\mathtt{Zip}^{\mu}$. By the associated product construction, the representation r produces an F-zip $\underline{\mathscr{I}}(r)$ on $G\text{-}\mathtt{Zip}^{\mu}$ which is of Calabi-Yau type since r is. We thus define the *Hasse invariant* of the triple (G,μ,r) as

$$\mathrm{Ha}(G,\mu,r) := \mathrm{Ha}(\underline{\mathscr{I}}(r)),$$

and similarly the *conjugate line position* of (G,μ,r) is defined as

$$\mathsf{clp}_x(G,\mu,r) := \mathsf{clp}_x(\underline{\mathscr{I}}(r)),$$

for all k-points x of $G\text{-}\mathtt{Zip}^{\mu}$.

3.4. **The group theoretic Ogus' principle.** Let r be of CY-type. We say that the triple (G,μ,r) satisfies Ogus' principle if for all k-points x of $G\text{-}\mathtt{Zip}^{\mu}$, we have that

$$\mathrm{ord}_x(\mathrm{Ha}(G,\mu,r)) = \mathsf{clp}_x(\underline{\mathscr{I}}(r)).$$

Example 3.4.1. By definition, the de Rham cohomology on $X \to S$ induces a morphism $\zeta\colon S \to G\text{-}\mathtt{Zip}^{\mu}$ where $G = \mathrm{GL}(\dim H_{\mathrm{dR}}(X/S))$ and μ is the character obtained from the Hodge filtration. By construction, $\mathrm{Ha}_S = \zeta^*\mathrm{Ha}(G,\mu,\mathrm{id})$. Thus, if ζ is smooth, then the group theoretic Ogus' principle implies the Ogus' principle for the family $X \to S$.

3.5. **Relation to Shimura varieties.** Let $(\mathbf{G},\mathbf{X})$ be an abelian-type Shimura datum. Assume that $\mathbf{G}$ is unramified at p. Let $K_p \subset \mathbf{G}(\mathbf{Q}_p)$ be a hyperspecial maximal compact subgroup. Let K^p range over open, compact subgroups of $\mathbf{G}(\mathbf{A}_f^p)$. By the work of Kisin [Kis10] and Vasiu [Vas99] the associated projective system of Shimura varieties admits an integral canonical model $(\mathscr{S}_{K_pK^p}(\mathbf{G},\mathbf{X}))_{K^p}$ in the sense of Milne [Mil92]. Set $K := K_pK^p$ and let S_K be the special k-fiber of $\mathscr{S}_{K_pK^p}(\mathbf{G},\mathbf{X})$.

Assume that $(\mathbf{G},\mathbf{X})$ is of Hodge type. For $g \geq 1$, let $(\mathrm{GSp}(2g),\mathbf{X}_g)$ be the Siegel datum, consisting of the $\mathbf{Q}$-split symplectic similitude group $\mathrm{GSp}(2g)$ and the Siegel double half-space $\mathbf{X}_g$. Given a symplectic embedding

$$(\mathbf{G},\mathbf{X}) \hookrightarrow (\mathrm{GSp}(2g),\mathbf{X}_g),$$

for all sufficiently small K^p there exists a level $K' \subset \mathrm{GSp}(2g,\mathbf{A}_f)$ and an induced finite map from S_K to the special k-fiber of the Siegel-type Shimura variety $S_{g,K'}$ [Kis10, (2.3.3)]. Let Y/S_K be the resulting family of abelian schemes. The Zip period map associated to $H^1_{\mathrm{dR}}(Y/S_K)$ factors through a smooth (Zhang [Zha18]) surjective (Kisin-Madapusi Pera-Shin [KMS]) morphism

$$\zeta : S_K \to G\text{-}\mathtt{Zip}^{\mu}, \tag{3.5.1}$$

where G is the reductive $\mathbf{F}_p$-group deduced from the $\mathbf{Q}$-group $\mathbf{G}$ and $\mu \in X_*(G)$ is a representative of the conjugacy class of cocharacters deduced from the Hermitian symmetric space $\mathbf{X}$. See [GK19a, Section 4.1-4.2] for more details. When $(\mathbf{G},\mathbf{X})$ is not of Hodge-type, there still exists a smooth, surjective morphism (3.5.1) by [SZ22], but it no longer arises from an F-Zip of the form $H^1_{\mathrm{dR}}(Y/S_K)$.

The F-Zip $H^g_{\mathrm{dR}}(Y/S_K) = \wedge^g H^1_{\mathrm{dR}}(Y/S_K)$ is of CY-type. It arises from the representation r which is the gth exterior power of the dual of $G \to \mathrm{GSp}(2g) \to \mathrm{GL}(2g)$ deduced from φ. Since ζ is smooth, we see that Ogus' principle for S_K is implied by the group theoretic Ogus' principle for (G, μ, r).

4. Method of proof

Keep the notation of Section 3.1 and let r be a representation of CY-type. We compute the vanishing order of $\mathrm{Ha}(G, \mu, r)$ and $\mathsf{clp}_x(G, \mu, r)$ separated. The latter is done via [MW04]. The main technical part is the computation of the former. In short, we use the theory of G-zips to reduce the question of the vanishing order to a study of highest weight sections on the flag variety G/B. In this section, we sketch how to obtain this reduction and how to compute the vanishing order of the highest weight sections.

4.1. **Root data.** Let T be a maximal torus contained in a Borel $B \subset P$. Let $(X^*(T), \Phi, X_*(T), \Phi^\vee)$ denote the root system associated to (G, T), and let Δ denote the simple roots determined by B^+, the Borel opposite to B. Let $W = W(G_k, T_k)$ denote the Weyl group. It is a Coxeter group with length function $l\colon W \to \mathbf{N}$ and longest element w_0. We denote both the element $w \in W$ and each suitable[5] lift in $N_G(T)$ by the same letter w. For a subset $K \subset \Delta$, let $W_K := \langle s_\alpha : \alpha \in K\rangle \subset W$. Let ${}^K W$ (resp. W^K) denote the elements of minimal length in the cosets of $W_K \backslash W$ (resp W/W_K). Let $w_{0,K}$ denote the element of maximal length in W_K. Let $I, J \subset \Delta$ be the type of P and Q respectively. Let $z := w_0 w_{0,J}$, the longest element in W^J.

4.2. **Realization of G-$\mathtt{Zip}^\mu$ as a quotient stack.** Let S be a k-scheme and let $g \in G(S)$. Let I_g and $I_{g,P}$ be the trivial torsors $S \times G$ respectively $S \times P$, let $I_{g,Q}$ be the image of $S \times Q$ in $S \times G$ under left multiplication by g and let ι_g be the isomorphism induced by left multiplication by g. Let $\underline{I}_g$ be the G-zip $\underline{I}_g = (I_q, I_{g,P}, I_{g,Q}, \iota_g)$. Let $\underline{I}$ be an arbitary G-zip of type μ over S. Since G, P, Q and L are smooth groups, there is a $g \in G(S)$ such that $\underline{I}$ is étale-locally isomorphic to the G-zip $\underline{I}_g$ (see [PWZ11, Lemma 3.5]).

Let $E := P \times_M Q$, and let it act on G_k by $(a, b) \cdot g = agb^{-1}$. Then we have an isomorphism

$$[E\backslash G] \xrightarrow{\cong} G\text{-}\mathtt{Zip}^\mu$$

which, roughly speaking, is induced by the map $g \mapsto \underline{I}_g$ (see [PWZ11, Section 3.4] for details).

4.2.1. *The Ekedahl-Oort stratification.* For each $w \in W$, let G_w denote the E-orbit of wz^{-1}. By [PWZ11, Theorem 7.5 and Theorem 11.3], the morphism $w \mapsto G_w$ induces a homeomorphism of the underlying topological space of G-$\mathtt{Zip}^\mu$ with ${}^I W$. Here ${}^I W$ is equipped with the topology obtained from the order given by

$$w \preceq w' \iff \text{there exists } x \in W_I \text{ such that } xwz\varphi(x^{-1})z^{-1} \leq w',$$

where φ is the Frobenius action. From this one obtains a stratification whose strata closure relations are given by the order above; the closure of w' consists of all w such that $w \preceq w'$. For each $w \in {}^I W$ let $[E\backslash G_w]$ denote the corresponding stratum of G-$\mathtt{Zip}^\mu$. It is a smooth locally closed substack, called the *zip stratum*, or the *Ekedahl-Oort stratum*, corresponding to w. See [PWZ11] for details.

4.3. ***G*-zip flags.** A *G-zip flag of type* μ over a k-scheme S is a tuple $(\underline{I}, I_B)$, where $\underline{I}$ is a G-zip of type μ and $I_B \subset I$ is a sub B-torsor. The forgetful map $(\underline{I}, I_B) \mapsto \underline{I}$ induces a smooth morphism

$$\pi_{\mathtt{Flag},\mathtt{Zip}}\colon G\text{-}\mathtt{ZipFlag}^\mu \to G\text{-}\mathtt{Zip}^\mu.$$

[5]For each $\dot{w} \in W$ we choose a lift $w \in N_{G_k}(T_k)$ such that $w_1 \dot{w}_2 = \dot{w}_1 \dot{w}_2$ whenever $l(w_1 \dot{w}_2) = l(\dot{w}_1) + l(\dot{w}_2)$. This is possible by [DG70, Exp. XXIII, Section 6].

4.3.1. *Realization as a quotient stack.* Let $E' \coloneqq E \cap (B_k \times G_k)$. We have that $G\text{-}\mathtt{ZipFlag}^\mu \cong [E' \backslash G]$ where the action of E' is the one induced from the action of E (see [GK19b]).

4.4. **Reduction to a flag variety and the Bruhat stratification.** Recall that the classical Bruhat decomposition of G is given by

$$G = \coprod_{w \in W} BwB.$$

Let $\mathrm{Sbt} \coloneqq [B\backslash G/B]$ where $B \times B$ acts on G by $(a,b)\cdot agb^{-1}$. This is called the *Schubert stack*. It inherits a stratification from the Bruhat decomposition; $\mathrm{Sbt} = \coprod_{w\in W} \mathrm{Sbt}_w$, where $\mathrm{Sbt}_w \coloneqq [B\backslash BwB/B]$. The morphism $g \mapsto gz$ induces an isomorphism $[B\backslash G/{}^zB] \to \mathrm{Sbt}$. Composed with the projection $[E'\backslash G] \to [B\backslash G/{}^zB]$ we obtain a smooth morphism

$$\pi_{\mathtt{Flag},\mathrm{Sbt}} \colon G\text{-}\mathtt{ZipFlag}^\mu \to \mathrm{Sbt}\,.$$

Analogously to the Schubert stack, the *Bruhat stack* is the double quotient $\mathcal{B} \coloneqq [P\backslash G/Q]$, studied by Wedhorn in depth in [Wed14a],[Wed14b]. It too has a stratification induced by Bruhat decomposition; $\mathcal{B} = \coprod_{w\in {}^IW^J} \mathcal{B}_w$. The identity map $G \to G$ induces a smooth surjection $\pi_{\mathtt{Zip},\mathcal{B}} \colon G\text{-}\mathtt{Zip}^\mu \to \mathcal{B}$. By taking preimages, this induces the *Bruhat stratification* on $G\text{-}\mathtt{Zip}^\mu$, which is coarser than the EO-stratification.

4.4.1. We thus have the following diagram of smooth morphisms

$$\begin{array}{ccccc} G\text{-}\mathtt{Zip}^\mu & \xleftarrow{\pi_{\mathtt{Flag},\mathtt{Zip}}} & G\text{-}\mathtt{ZipFlag}^\mu & & \\ \Big\downarrow{\scriptstyle \pi_{\mathtt{Zip},\mathcal{B}}} & & \Big\downarrow{\scriptstyle \pi_{\mathtt{Flag},\mathrm{Sbt}}} & & \\ \mathcal{B} & & \mathrm{Sbt} & \longleftarrow & G/B \end{array} \tag{4.4.2}$$

4.4.3. *Line bundles on the stacks.* Any pair of characters $\lambda_1, \lambda_2 \colon B \to \mathbf{G}_m$ induce a line bundle $\mathscr{L}_{\mathrm{Sbt}}(\lambda_1,\lambda_2)$ on Sbt. By [GK19a, Theorem 2.2.1] $\mathscr{L}_{\mathrm{Sbt}}(\lambda_1,\lambda_2)$ admits nontrivial global sections if and only if λ_1 is dominant and $\lambda_2 = -w_0\lambda_1$. Given a character λ of B, via the projection $E' \to B$ we obtain also a character on E', still denoted λ. This yields a line bundle $\mathscr{L}_{G\text{-}\mathtt{ZipFlag}}(\lambda)$ on $G\text{-}\mathtt{ZipFlag}^\mu$. By [GK19a, Lemma 3.1.1] we have that

$$\pi^*_{\mathtt{Flag},\mathrm{Sbt}}\mathscr{L}_{\mathrm{Sbt}}(\lambda, -w_0\lambda) = \mathscr{L}_{G\text{-}\mathtt{ZipFlag}}(D_{w_0}(\lambda)),$$

where $D_{w_0} \colon X^*(T) \to X^*(T)$ is the map

$$D_{w_0} \colon \lambda \mapsto \lambda - p^{\sigma^{-1}}(zw_0^{-1}\lambda).$$

If $\lambda \colon L \to \mathbf{G}_m$ is a character of L, then via the projections $E \to P \to L$, we obtain a character of E, which yields a line bundle $\mathscr{L}_{G\text{-}\mathtt{Zip}}(\lambda)$ on $G\text{-}\mathtt{Zip}^\mu$.

Suppose that $r \colon G \to GL(V)$ is a representation of CY-type with highest weight λ_r. Let

$$\eta_r \coloneqq -\lambda_r.$$

The composition $r \circ \mu$ induces a filtration on V_k whose graded pieces are stable under the action of L via r. Since r is of CY-type, λ_r restricts to a characater of L and we have that $\mathrm{Ha}(G,\mu,r)$ is a section of the line bundle $\mathscr{L}_{G\text{-}\mathtt{Zip}}((p-1)(\eta_r))$. When (G,μ,r) arises from a Shimura variety of Hodge type, then the character η_r is often called the *Hodge character*. It satisfies the equation $\omega \cong \zeta_K^*\mathscr{L}(\eta_r)$.

Solving the equation

$$(p-1)\eta_r = D_{w_0}(\lambda)$$

gives a character λ such that

$$\pi^*_{\texttt{Flag,Zip}}\mathscr{L}_{G\text{-}\texttt{Zip}}((p-1)\eta_r) = \pi^*_{\texttt{Flag},\mathrm{Sbt}}\mathscr{L}_{\mathrm{Sbt}}(\lambda, -w_0\lambda)). \tag{4.4.4}$$

If $\mathrm{Ha}(G,\mu,r)$ is not identically zero, then there is a global section $\mathrm{Ha}_{\mathrm{Sbt}}$ of $\mathscr{L}(\lambda, -w_0\lambda)$ such that $\pi^*_{\texttt{Flag},\mathrm{Sbt}}\mathrm{Ha}_{\mathrm{Sbt}} = \pi^*_{\texttt{Flag,Zip}}\mathrm{Ha}(G,\mu,r)$. Hence, for any $w \in W$, the vanishing order of $\mathrm{Ha}_{\mathrm{Sbt}}$ on Sbt_w equals the vanishing order of $\mathrm{Ha}(G,\mu,r)$ on all Ekedahl-Oort strata contained in $\pi_{\texttt{Flag,Zip}}\left(\pi^{-1}_{\texttt{Flag},\mathrm{Sbt}}(\mathrm{Sbt}_w)\right)$.

4.4.5. *Reduction to a flag variety.* Let $\lambda \in X^*(T)$ be a dominant character. Let $H^0(\lambda) \coloneqq H^0(G/B, \mathscr{L}_{G/B}(\lambda))$ and let f_λ be a highest weight vector of $H^0(\lambda)$.

4.4.6. *A classical construction of f_λ.* For all $u^+tu \in U^+B$ define let $f_\lambda(u^+tu) = \lambda(t)^{-1}$. For any $\alpha \in \Delta$, s_α normalizes $\prod_{\beta\in\Phi^+\setminus\{\alpha\}} U_\beta$. Hence, by using relations regarding root group morphisms we can coordinate shift any element in $s_\alpha U^+B$ to lie U^+B. We obtain thus an extension of f_λ to $s_\alpha U^+B$. The subscheme $\bigcup_{\alpha\in\Delta} s_\alpha U^+B$ has codimension greater than 1, hence f_λ extends to G (see [Jan03, Section II.2.6.] for details).

4.4.7. *Extending the classical construction.* Suppose that (G,μ) arises from a Shimura variety of abelian type as in Section 3.5. We extend the classical construction of f_λ to subsets of the form wU^+B for all $w \in {}^IW$. Hence, we can study the vanishing order of f_λ on all points in wU^+B.

We identify wU^+B with $k^{|\Phi^+|} \times B$. Under this identification, B^+wB is the zero-locus of the last $l(w)$ coordinates of $k^{|\Phi^+|}$. Since $Bw_0wB = w_0B^+wB \subset wU^+B$, we see that the vanish order of f_λ on B^+wB equals that of $w_0 \cdot f_\lambda$ on Bw_0wB.

4.4.8. By a direct computation, one finds that for all dominant characters $\lambda \in X^*(T)$, we have that

$$H^0(\mathrm{Sbt}, \mathscr{L}_{\mathrm{Sbt}}(\lambda, -w_0\lambda) = H^0(-w_0\lambda)_{-\lambda}$$

Let λ be such that Equation (4.4.4) holds. The order of vanishing of $f_{-w_0\lambda}$ on any point in B^+wB gives the vanishing order of $w_0 \cdot f_{-w_0\lambda} = \mathrm{Ha}_{\mathrm{Sbt}}$ on any point in Bw_0wB. We obtain thus the vanishing order of $\mathrm{Ha}(G,\mu,r)$ on all points in $\pi_{\texttt{Flag,Zip}}(\pi^{-1}_{\texttt{Flag},\mathrm{Sbt}}(\mathrm{Sbt}_{w_0w}))$.

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