AROUND THE MANTOVAN FORMULA

WANSU KIM

Since we obtained the Hodge-type Mantovan formula in joint work with Hamacher [9], there have been many new developments. Axiom A in [9], which was known in some special cases, has been verified by Gleason–Lim–Xu [8]. The theory of integral models of global and local Shimura varieties has been reshaped in terms of Scholze's theory of $\mathfrak{p}$-adic shtukas [28], and integral models have been constructed in greater generalities than before (*cf.* [23], [18], [4], [24]).

In the most recent joint work with Hamacher [10, §2], we constructed *central leaves* and *Igusa covers* for any Witt vector local shtuka over a perfect base (as well as their function field analogues). The construction can be applied to the perfect special fibre of an abelian-type Shimura variety provided that there is a "Witt vector $\mathcal{G}^c$-shtuka" on it. Combining this with recent developments on integral models of global and local Shimura varieties, it seems that all the ingredients for the abelian-type Mantovan formula are in place. In this article, we attempt (perhaps unsuccessfully) to describe "upgrade patches" to [9] from recent developments.

After discussing the modular curve case in §1, we review the Hodge-type Mantovan formula [9] (with "upgrade patches") in §2. We then sketch some of the new developments in the abelian-type case in §3.

Contents

1. Warm-up: Almost product structure for modular curves

Let p be a prime, and fix another integer N coprime to p. (We assume $N \geqslant 3$ for simplicity.) For any $r \geqslant 0$ let $Y(Np^r)$ denote the (open)[1] modular curve over $\operatorname{Spec}\mathbb{Z}_p$ with full level Np^r structure in the sense of Drinfeld. We set

$$Y(Np^r)_{\overline{\mathbb{Q}}_p} \coloneqq Y(Np^r)\times_{\operatorname{Spec}\mathbb{Z}_p}\operatorname{Spec}\overline{\mathbb{Q}}_p \quad \text{and} \quad \overline{Y}(Np^r) \coloneqq \big(Y(Np^r)\times_{\operatorname{Spec}\mathbb{Z}_p}\operatorname{Spec}\overline{\mathbb{F}}_p\big)_{\mathrm{red}},$$

which are (disconnected) geometric generic and reduced special fibres at p.

The bad reduction of $Y(Np^r)$ indicates that ordinary and supersingular loci in $\overline{Y}(Np^r)$ behave quite differently. We will now describe explicitly the limit as $r \to \infty$ of the formal completion $Y(Np^r)^{\wedge}_{/\overline{Y}(Np^r)^b}$ for $b \in \{\mathrm{ord}, \mathrm{ss}\}$ (with $\overline{Y}(Np^r)^b$ denoting the ordinary or supersingular locus).

Let us set up some notation. Let $\breve{\mathbb{Z}}_p$ denote the p-completed maximal unramified extension of $\mathbb{Z}_p$, with $\breve{\mathbb{Q}}_p \coloneqq \operatorname{Frac}(\breve{\mathbb{Z}}_p)$. We define $\breve{\mathbb{Z}}_p^{\mathrm{cyc}}$ to be the p-adic completion of $\breve{\mathbb{Z}}_p[\mu_{p^\infty}]$, and write $\breve{\mathbb{Q}}_p^{\mathrm{cyc}} \coloneqq \breve{\mathbb{Z}}_p^{\mathrm{cyc}}[\frac{1}{p}]$. Let $\mathcal{E}$ denote the universal elliptic curve over $Y(N)$, and for any (formal) scheme S over $Y(N)$ we let $\mathcal{E}_S$ denote the pullback of $\mathcal{E}$ over S. We may even write $\mathcal{E}$ for $\mathcal{E}_S$ if there is no risk of confusion.

1.1. Perfect Igusa curves on the ordinary locus.

The following definition of *ordinary Igusa variety* is due to Caraiani–Scholze [1, §4.3]. (The definition below is a slight variant of it, following [9, §6.1].)

Definition 1.1. We define $\mathfrak{Ig}_N^{\mathrm{ord}} \coloneqq \operatorname{Isom}_{Y(N)^{\wedge}_{/\overline{Y}(N)^{\mathrm{ord}}}}(\mu_{p^\infty}\times\mathbb{Q}_p/\mathbb{Z}_p, \mathcal{E}[p^\infty])$, and let $\mathrm{Ig}_N^{\mathrm{ord}} \coloneqq \mathfrak{Ig}_N^{\mathrm{ord}} \times_{\operatorname{Spf}\breve{\mathbb{Z}}_p}\operatorname{Spec}\overline{\mathbb{F}}_p$. We call $\mathrm{Ig}_N^{\mathrm{ord}}$ the *(perfect) ordinary Igusa variety.*

To explain the terminology, $\mathrm{Ig}_N^{\mathrm{ord}}$ turns out to be the perfection of the limit of the ordinary part of the Igusa curves in the sense of Harris–Taylor and Mantovan; *cf.* [1, Prop 4.3.8]. In particular, $\mathrm{Ig}_N^{\mathrm{ord}}$ is the perfection of a pro-finite étale cover of $\overline{Y}(N)^{\mathrm{ord}}$, and $\mathfrak{Ig}_N^{\mathrm{ord}}$ is its canonical p-adic lift.

We will now construct a pro-finite étale covering of formal schemes

$$(\mathfrak{Ig}_N^{\mathrm{ord}} \times_{\operatorname{Spf}\breve{\mathbb{Z}}_p}\operatorname{Spf}\breve{\mathbb{Z}}_p^{\mathrm{cyc}}) \times \underline{\operatorname{Surj}(\mathbb{Z}_p^2,\mathbb{Z}_p)} \twoheadrightarrow \varprojlim_r Y(Np^r)^{\wedge}_{/\overline{Y}(Np^r)^{\mathrm{ord}}}, \tag{1}$$

which is a torsor under the action of $\operatorname{Aut}(\mu_{p^\infty}\times\mathbb{Q}_p/\mathbb{Z}_p) \cong \mathbb{Z}_p^\times\times\mathbb{Z}_p^\times$. Here, $\underline{\operatorname{Surj}(\mathbb{Z}_p^2,\mathbb{Z}_p)}$ is the formal scheme attached to the profinite set of surjective maps $\mathbb{Z}_p^2 \twoheadrightarrow \mathbb{Z}_p$. Indeed, we interpret $\operatorname{Spf}\breve{\mathbb{Z}}_p$ as the moduli of infinite Drinfeld level structures on μ_{p^∞}, so a point of $\operatorname{Spf}\breve{\mathbb{Z}}_p \times \underline{\operatorname{Surj}(\mathbb{Z}_p^2,\mathbb{Z}_p)}$ defines an infinite Drinfeld level on $\mathcal{E}[p^\infty]$. Conversely, if E is an ordinary elliptic curve over a $\breve{\mathbb{Z}}_p$-scheme S where p is locally nilpotent equipped with an infinite Drinfeld level structure $\alpha\colon \mathbb{Z}_p^2 \to \varprojlim_r E[p^r]$, then there is an isomorphism $\beta\colon \mu_{p^\infty}\times\mathbb{Q}_p/\mathbb{Z}_p \xrightarrow{\sim} E[p^\infty]$, defining an S-point of $\mathfrak{Ig}_N^{\mathrm{ord}}$. (Indeed, $\alpha \bmod p^r$ defines a splitting of connected-étale sequence of $E[p^r]$ and forces $E[p^r]^{\text{ét}}$ to be constant.) Now, α restricted to $E[p^\infty] \xleftarrow[\beta^\circ]{\cong} \mu_{p^\infty}$ defines a point of $\operatorname{Spf}\breve{\mathbb{Z}}_p$, and we also have a surjection $\mathbb{Z}_p^2 \xrightarrow{\alpha} \varprojlim_r E[p^r] \twoheadrightarrow T_pE[p^\infty]^{\text{ét}} \xleftarrow[\beta^{\text{ét}}]{\cong} \mathbb{Z}_p$. As $\operatorname{Aut}(\mu_{p^\infty}\times\mathbb{Q}_p/\mathbb{Z}_p)$ acts simply transitively on the choices of β, we obtain the claim.

[1] In this note, we will not consider the questions related to the compactifications of Shimura varieties, so we will ignore compactified modular curves as well.

1.2. **Supersingular (0-dimensional) Igusa variety.** Let us briefly discuss the *supersingular Igusa variety*, which turns out to be 0-dimensional. We pick a point in $x \in \overline{Y}(N)^{\mathrm{ss}}$ and let $\mathcal{E}_x$ denote the corresponding supersingular elliptic curve over $\overline{\mathbb{F}}_p$. Write $D \coloneqq \mathrm{End}(E_x) \otimes \mathbb{Q}$ and $\mathscr{O}_D \coloneqq \mathrm{End}(E_x)$, which are respectively the quaternion division algebra over $\mathbb{Q}$ ramified only at $\{p, \infty\}$ and its maximal order. Then the following description is well known[2]:

$$\overline{Y}(N)^{\mathrm{ss}} \cong \underline{D^\times \backslash (D \otimes_{\mathbb{Q}} \mathbb{A}^\infty)^\times / (1 + N\widehat{\mathscr{O}}_D)^\times}, \tag{2}$$

where the chosen point x corresponds to the trivial double coset on the right hand side.

Write $\mathbb{X}_{\mathrm{ss}} \coloneqq \mathcal{E}_x[p^\infty]$, which is independent of the choice of $x \in \overline{Y}(N)^{\mathrm{ss}}$ up to isomorphism. We define the *(perfect) supersingular Igusa variety* as follows

$$\mathrm{Ig}(Np^\infty)^{\mathrm{ss}} \coloneqq \mathrm{Isom}_{\overline{Y}(N)^{\mathrm{ss}}}(\mathbb{X}^{\mathrm{ss}}, \mathcal{E}[p^\infty]|_{\overline{Y}(N)^{\mathrm{ss}}}), \tag{3}$$

which is a *right* torsor over $\overline{Y}(N)^{\mathrm{ss}}$ for the action of $\mathrm{Aut}(\mathbb{X}_{\mathrm{ss}}) \cong \mathscr{O}_{D,p}^\times$, where $\mathscr{O}_{D,p}$ is the p-adic completion of $\mathscr{O}_D$. One can even make it more explicit as follows:

$$\mathrm{Ig}(Np^\infty)^{\mathrm{ss}} \cong \underline{\widehat{D^\times} \backslash (D \otimes_{\mathbb{Q}} \mathbb{A}^\infty)^\times / (1 + N\widehat{\mathscr{O}}_D^p)^\times}, \tag{4}$$

where $\widehat{D^\times}$ is the closure of $D^\times$ in $(D \otimes_{\mathbb{Q}} \mathbb{A}^\infty)^\times$. Finally, let $\mathfrak{Ig}(Np^\infty)^{\mathrm{ss}}$ denote its lift over $\mathrm{Spf}\,\breve{\mathbb{Z}}_p$. It should be warned that this 0-dimensional Igusa variety $\mathrm{Ig}(Np^\infty)^{\mathrm{ss}}$ has nothing to do with the Igusa curve on the ordinary locus.

1.3. **Interpretation as the almost product structure.** For $b \in \{\mathrm{ord}, \mathrm{ss}\}$, set $\mathbb{X}_b \coloneqq \mathcal{E}_x[p^\infty]$ for some (equivalently, any) $x \in \overline{Y}(N)^b$. For example, we may set $\mathbb{X}_{\mathrm{ord}} = \mu_{p^\infty} \times \mathbb{Q}_p/\mathbb{Z}_p$. Let RZ^b denote the (special) formal scheme over $\mathrm{Spf}\,\breve{\mathbb{Z}}_p$ parametrising deformations of $\mathbb{X}_b$ up to quasi-isogeny; see Rapoport–Zink [26, Def 2.15] for the precise definition. As $\mathbb{X}_b$ is 1-dimensional, RZ^b turns out to be a countable disjoint union of the universal deformation space of $\mathbb{X}_b$. (The set of connected components is identified with $\mathbb{Z}$ via the degree of quasi-isogeny if $b = \mathrm{ss}$, and $\mathbb{Z} \times \mathbb{Z}$ via the degree of the connected and étale parts of quasi-isogeny if $b = \mathrm{ord}$.) Let RZ^b_r denote the finite cover of RZ^b given by the Drinfeld level p^r structure, and write $\mathrm{RZ}^b_\infty \coloneqq \varprojlim_r \mathrm{RZ}^b_r$ (where the limit is as a formal scheme). Then we have a natural Hecke $\mathrm{GL}_2(\mathbb{Q}_p)$-action on RZ^b_∞.

The formal schemes RZ^b, RZ^b_∞ and $\mathfrak{Ig}^b_N$ admits a natural (right) action of the "self quasi-isogeny group" $\mathrm{Qisg}(\mathbb{X}_b)$, constructed by Caraiani–Scholze [1, §4.2], which can be represented by a formal group scheme over $\mathrm{Spf}\,\breve{\mathbb{Z}}_p$ with perfect special fibre. See *loc. cit.* for further details, where $\mathrm{Qisg}(\mathbb{X}_b)$ is denoted by $\mathrm{Aut}(\widetilde{\mathbb{X}}_b)$ instead. We now interpret the proceeding discussions as the following $\mathrm{GL}_2(\mathbb{Q}_p)$ -equivariant isomorphism, called the "almost product structure":

$$\left(\mathrm{RZ}^b_\infty \times_{\mathrm{Spf}\,\breve{\mathbb{Z}}_p} \mathfrak{Ig}^b_N\right) / \mathrm{Qisg}(\mathbb{X}_b) \xrightarrow{\cong} \varprojlim_r Y(Np^r)^\wedge_{/\overline{Y}(Np^r)^b}. \tag{5}$$

1.3.1. *Almost product structure: supersingular locus.* We have $\mathrm{Qisg}(\mathbb{X}_{\mathrm{ss}}) = \underline{D_p^\times}$, a constant locally profinite group, where D_p is the p-adic completion of D as in §1.2. The profinite open subgroup $\mathscr{O}_{D,p}^\times$ corresponds to the automorphism group of $\mathbb{X}_{\mathrm{ss}}$, and the Frobenius isogeny corresponds to a uniformiser in D_p. This tells us how to extend the (right) $\mathscr{O}_{D,p}^\times$-action on the deformation space to the $D_p^\times$-action on $\mathrm{RZ}^{\mathrm{ss}}$. The natural right action of $D_p^\times$ on $\mathfrak{Ig}^{\mathrm{ss}}_N$ is via the *right multiplication* on (4).

[2] The Shimura curve case was obtained by Carayol [2, §11], and the same proof should work.

Now, fix a "base point" $x \in \overline{Y}(N)^{ss}$ and identify $Y(N)^{\wedge}_x$ with the neutral component of RZ^{ss}. Then one can deduce (5) for $b = ss$ from (2).

1.3.2. *Almost product structure: ordinary locus.* In the ordinary case, the main difference is the structure of $\mathrm{Qisg}(\mathbb{X}_{\mathrm{ord}})$ (*cf.* [1, Rmk 4.2.12]); namely, we have $\mathrm{Qisg}(\mathbb{X}_{\mathrm{ord}}) \cong (\mathbb{Q}_p^\times)^2 \ltimes \widetilde{\mu}_{p^\infty}$, where $\widetilde{\mu}_{p^\infty} := \varprojlim_p \mu_{p^\infty}$. Then the deformation space of $\mathbb{X}_{\mathrm{ord}}$ with "infinite Drinfeld level structure" turns out to be

$$\mathrm{Spf}\, \breve{\mathbb{Z}}_p^{\mathrm{cyc}}[\![q^{p^{-\infty}} - 1]\!] \times \underline{\mathrm{Surj}(\mathbb{Z}_p^2, \mathbb{Z}_p)},$$

where q is the Serre–Tate local coordinate in the sense of [14, §3.1], and the p-power root of q comes from iterating the pullback by the canonical lift of Frobenius (*cf.* [14, Lemma 4.1.2]). In particular, the Serre–Tate local coordinate gives a canonical identification of the neutral component $\mathrm{Spf}\, \breve{\mathbb{Z}}_p^{\mathrm{cyc}}[\![q^{p^{-\infty}} - 1]\!]$ with the neutral component $(\widetilde{\mu}_{p^\infty})_{\breve{\mathbb{Z}}_p^{\mathrm{cyc}}}$ of $\mathrm{Qisg}(\mathbb{X}_{\mathrm{ord}})_{\breve{\mathbb{Z}}_p^{\mathrm{cyc}}}$, with the canonical lift corresponding to the identity section.

Now, fixing a "base point" $x \in \overline{Y}(N)^{\mathrm{ord}}$ and repeating the construction in 1.3.1 using (1), we deduce (5) for $b = \mathrm{ord}$.

1.3.3. *Remark on the generalisation to the Siegel case.* The above discussion can be directly extended for Harris–Taylor Shimura varieties [12]. But for the Siegel case, some of the features do not generalise: we do not have nice integral models with arbitrary level at p so we only generalise the perfectoid generic fibre of (5), and the relevant Rapoport–Zink spaces and Igusa varieties are much less explicit.

2. Hodge-type case: Review and recent progress

The main result of joint work with Hamacher [9] is to extend the construction of *integral local Shimura varieties* (or *Rapoport–Zink spaces*) and *Igusa varieties* in the tamely ramified Hodge-type setting, as well as the almost product structure of Newton strata. We also deduce the Mantovan formula expressing the cohomology of Hodge-type Shimura varieties in terms of local Shimura varieties and Igusa varieties. The unramified PEL case is due to Mantovan [20, 21], built upon the work of Harris–Taylor [12] and Oort [22]. For the Hodge-type case, we use the Siegel case as input.

The construction of Hodge-type Rapoport–Zink spaces and Igusa varieties in [9] involves the auxiliary choice of an *integral model of Shimura variety*, whose construction (by Kisin–Pappas [17]) also involves auxiliary choices. In this section, we first review the main result of [9] and try to describe "upgrade patches" via recent developments.

2.1. **Review of Kisin–Pappas integral models and Witt vector shtukas.** Fix a prime $p > 2$, and let $(G, \mathfrak{D})$ be a *Hodge-type* Shimura datum such that $G_{\mathbb{Q}_p}$ splits over a tame extension and we have $p \nmid |\pi_1(G^{\mathrm{der}})|$. Let $E := E(G, \mathfrak{D})$ denote the reflex field, and we fix a place $v|p$ of E. The prototypical non-PEL Hodge-type example is the GSpin *Shimura datum* attached to a rational quadratic space (V, q) with signature $(n, 2)$ that splits over a tame extension of $\mathbb{Q}_p$. Then we get a Shimura datum by setting $G := \mathrm{GSpin}(V, q)$ and $\mathfrak{D}$ to be the space of oriented negative definite 2-planes in $V_{\mathbb{R}}$. (Here, we have $E(G, \mathfrak{D}) = \mathbb{Q}$.)

Let $\mathcal{G}/\mathbb{Z}_p$ denote the Bruhat–Tits integral model of $G_{\mathbb{Q}_p}$ corresponding to the *full stabiliser* of some facet of the Bruhat–Tits building, and set $K_p := \mathcal{G}(\mathbb{Z}_p)$. Then for any small enough open compact subgroup $K^p \subset G(\mathbb{A}^{p\infty})$ (with $K := K^pK_p$), Kisin–Pappas [17] constructed a normal integral model $\mathscr{S}_K$ over $\mathscr{O}_{E,(v)}$ of $\mathrm{Sh}_K = \mathrm{Sh}_K(G, \mathfrak{D})$ with number of natural properties. The construction of $\mathscr{S}_{\mathsf{K}}$ involves auxiliary choices, while it is believed to be independent of them. (We will discuss further at §2.3.)

As a consequence of (the auxiliary choice made for) the construction, one gets a finite unramified morphism $\mathscr{S}_{\mathsf{K}} \to \mathscr{S}'_{\mathsf{K}',\mathscr{O}_{E,(v)}}$ (inducing a closed immersion on the generic fibre), where $\mathscr{S}'_{\mathsf{K}'}/\mathbb{Z}_{(p)}$ is the integral canonical model of the Siegel modular variety with K' hyperspecial at p. In particular, we get a "universal abelian scheme" $\mathscr{A}_{\mathsf{K}}$ over $\mathscr{S}_{\mathsf{K}}$, equipped with a (group-theoretically produced) family of absolute Hodge tensors (t_α) over Sh_{K}. To say a word about tensors, the choice of $\mathscr{S}_{\mathsf{K}} \to \mathscr{S}'_{\mathsf{K}',\mathscr{O}_{E,(v)}}$ also induces a closed immersion $\mathcal{G} \hookrightarrow \mathrm{GSp}_{2g} \subset \mathrm{GL}_{2g}$ over $\mathbb{Z}_p$ where $g = \dim_{\mathscr{S}_{\mathsf{K}}} \mathscr{A}_{\mathsf{K}}$, and (t_α) comes from the choice of tensors $(s_\alpha) \subset (\mathbb{Z}_p^{2g})^{\otimes}$ whose pointwise stabiliser is $\mathcal{G}$.

Furthermore, the absolute Hodge cycle (t_α) has "good reduction" in the following sense. We have Frobenius-invariant tensors $(t_\alpha^{\mathbb{D}})$ of the (contravariant) Dieudonné module $\mathbb{D}(\mathscr{A}_{\mathsf{K}}|_{\overline{\mathscr{S}}_{\mathsf{K}}^{\mathrm{perf}}})$ (where $\overline{\mathscr{S}}_{\mathsf{K}}^{\mathrm{perf}}$ is the perfection of the geometric special fibre of $\mathscr{S}_{\mathsf{K}}$) which are "pointwise compatible" with the étale components $(t_\alpha^{\text{ét}})$ via the crystalline comparison isomorphism; that is, given $x \in \mathscr{S}_{\mathsf{K}}(R)$ for a mixed characteristic complete dvr R with residue field $\overline{\mathbb{F}}_p$, the crystalline comparison isomorphism takes the fibre of $(t_\alpha^{\mathbb{D}})$ at the special point of x to the fibre of $(t_\alpha^{\text{ét}})$ at the generic point of x. This property was obtained in [9, Cor 4.11] and [16, Prop 1.3.12, 1.3.7] by different methods.

We can group-theoretically package the good reduction of tensors as follows. Let $L^+\mathcal{G}$ and LG respectively denote the Witt vector positive loop group and the Witt vector loop group attached to $\mathcal{G}$, in the sense of [32, §1.1.1]. By *Witt vector $\mathcal{G}$-shtuka*[3] over a perfect scheme S we mean a right $L^+\mathcal{G}$-torsor $\mathcal{P}$ over S equipped with an isomorphism of LG-torsors $\varphi\colon \sigma^*\mathcal{L}\mathcal{P} \xrightarrow{\sim} \mathcal{L}\mathcal{P}$, where σ is the Frobenius and $\mathcal{L}\mathcal{P} \coloneqq \mathcal{P} \times^{L^+\mathcal{G}} LG$.

Proposition 2.1. *We have a Witt vector $\mathcal{G}$-shtuka $\underline{\mathcal{P}} = (\mathcal{P}, \varphi)$ over $\overline{\mathscr{S}}_{\mathsf{K}}^{\mathrm{perf}}$, where*

$$\mathcal{P} \coloneqq \mathrm{Isom}\left([\mathbb{Z}_p^{2g}, (s_\alpha)]_{W(\overline{\mathscr{S}}_{\mathsf{K}}^{\mathrm{perf}})}, [\mathbb{D}(\mathscr{A}_{\mathsf{K}}|_{\overline{\mathscr{S}}_{\mathsf{K}}^{\mathrm{perf}}}), (t_\alpha^{\mathbb{D}})]\right)$$

and φ is induced by the crystalline Frobenius on $\mathbb{D}(\mathscr{A}_{\mathsf{K}}|_{\overline{\mathscr{S}}_{\mathsf{K}}^{\mathrm{perf}}})$.

Proof. This can be read off from the proof of Cor 4.12 in [9]; indeed, $\mathcal{P}$ is an $L^+\mathcal{G}$-torsor as it trivialises over the perfection of the complete local ring of $\overline{\mathscr{S}}_{\mathsf{K}}$ at each closed point (*cf.* [17, §3.2]), and φ is well defined by Frobenius-invariance of $(t_\alpha^{\mathbb{D}})$. ☐

Note that the pair $(\mathcal{L}\mathcal{P}, \varphi)$ naturally gives rise to an "F-isocrystal with G-structure", so we get the Newton stratification $\overline{\mathscr{S}}_{\mathsf{K}}^{b}$ of $\overline{\mathscr{S}}_{\mathsf{K}}$ indexed by the neutral acceptable set $b \in B(G_{\mathbb{Q}_p}, \mu)$; *cf.* [9, Cor 4.12] , [16, Cor 1.3.13].

2.2. **Hodge-type almost product structure.** We fix an $\overline{\mathbb{F}}_p$-point x of a Newton stratum $\overline{\mathscr{S}}_{\mathsf{K}}^{b}$, and set $\mathbb{X}_b \coloneqq \mathscr{A}_{\mathsf{K},x}[p^\infty]$.[4] We next define $\mathrm{Qisg}_G(\mathbb{X}_b)$, RZ^b and $\mathfrak{Ig}^b_{\mathsf{K}^p}$, and show the following isomorphism of formal schemes over $\mathrm{Spf}\, \breve{\mathbb{Z}}_p$ depending on x (called the

[3]A Witt vector $\mathcal{G}$-shtuka coincides with the p-adic case of *local $\mathcal{G}$-shtuka* [10, Def 2.2(1)], but we use this terminology to avoid potential confusions with other notions of local shtukas.

[4]To be pedantic we should require some analogue of *complete slope divisibility* for the Witt vector $\mathcal{G}$-shtuka $\underline{\mathcal{P}}_x$, but let us ignore this subtle point. (Indeed, the construction of Igusa varieties $\mathrm{Ig}^b_{\mathsf{K}^p}$ in [9] required complete slope divisibility, but such a condition is removed in the later collaboration [10, §2] as we explain in §3.2.)

level-K *almost product structure*):

$$(6)\qquad (\mathrm{RZ}^{\mathsf{b}} \times_{\mathrm{Spf}\, \breve{\mathbb{Z}}_p} \mathfrak{Ig}^{\mathsf{b}}_{\mathsf{K}^p})/\operatorname{Qisg}_{\mathsf{G}}(\mathbb{X}_{\mathsf{b}}) \xrightarrow{\cong} (\mathscr{S}_{\mathsf{K}})^{\wedge}_{/\overline{\mathscr{S}}^{\mathsf{b}}_{\mathsf{K}}}.$$

Here, RZ^{b} is a certain Hodge-type Rapoport–Zink space, $\mathfrak{Ig}^{\mathsf{b}}_{\mathsf{K}^p}$ is the p-adic lift of a Hodge-type Igusa variety, and $\operatorname{Qisg}_{\mathsf{G}}(\mathbb{X}_{\mathsf{b}})$ is the group of tensor-preserving self quasi-isogeny group of $[\mathbb{X}_{\mathsf{b}}, (t^{\mathbb{D}}_{\alpha,x})]$.

In the unramified PEL case (such as in the Siegel case), the isomorphism (6) was obtained in Caraiani–Scholze [1], built upon Rapoport–Zink [26], Mantovan [20] and Scholze–Weinstein [27]. The strategy of [9], which was inspired by Howard–Pappas [13], is to "pull back" the Siegel case of (6) to obtain the general Hodge-type case.

Let $\overline{\mathscr{S}}'^{\mathsf{b}'}_{\mathsf{K}'}$ denote the Newton stratum in $\overline{\mathscr{S}}'_{\mathsf{K}'}$ where $\overline{\mathscr{S}}^{\mathsf{b}}_{\mathsf{K}}$ maps to. We write $\mathbb{X}_{\mathsf{b}'}$ for the underlying quasi-polarised p-divisible group of $\mathbb{X}_{\mathsf{b}}$, and $\operatorname{Qisg}_{\mathsf{G}'}(\mathbb{X}_{\mathsf{b}'})$ for the group of quasi-polarised self quasi-isogenies. Finally, let $\mathrm{RZ}^{\mathsf{b}'}$ and $\mathfrak{Ig}^{\mathsf{b}'}_{\mathsf{K}'^p}$ denote the Siegel Rapoport–Zink space and the lift of Igusa varieties thus obtained.

Firstly, a formal closed subgroup scheme $\operatorname{Qisg}_{\mathsf{G}}(\mathbb{X}_{\mathsf{b}}) \subset \operatorname{Qisg}_{\mathsf{G}'}(\mathbb{X}'_{\mathsf{b}})$ whose points are self quasi-isogenies preserving $(t^{\mathbb{D}}_{\alpha,x})$ was constructed by the author [15] (which has a gap, corrected by D'Addezio–van Hoften [3, §4]). As in the PEL case, we have $\operatorname{Qisg}_{\mathsf{G}}(\mathbb{X}_{\mathsf{b}}) = \underline{\mathsf{J}_{\mathsf{b}}(\mathbb{Q}_p)} \ltimes \operatorname{Qisg}^{\circ}_{\mathsf{G}}(\mathbb{X}_{\mathsf{b}})$ for some lift of perfect formal group $\operatorname{Qisg}^{\circ}_{\mathsf{G}}(\mathbb{X}_{\mathsf{b}})$, and the structure of $\operatorname{Qisg}^{\circ}_{\mathsf{G}}(\mathbb{X}_{\mathsf{b}})$ as well as its action on the formal neighbourhood of a closed point in $\overline{\mathscr{S}}^{\mathsf{b}}_{\mathsf{K}}$ is analysed in [15].

Next, in Def 5.9 (resp., Def/Lem 6.1) in [9] we defined RZ^{b} (resp., $\mathfrak{Ig}^{\mathsf{b}}_{\mathsf{K}^p}$) as the closed union of connected components of $\mathrm{RZ}^{\mathsf{b}'} \times_{\mathscr{S}_{\mathsf{K}'}} \mathscr{S}_{\mathsf{K}}$ (resp., $\mathfrak{Ig}^{\mathsf{b}'}_{\mathsf{K}'^p} \times_{\mathscr{S}_{\mathsf{K}'}} \mathscr{S}_{\mathsf{K}}$) cut out by the condition that the tensors $(t^{\mathbb{D}}_{\alpha})$ are preserved by the quasi-isogeny rigidification (resp., Igusa level structure); indeed, to show that the condition on tensors cuts out a closed union of connected components, we use a slight generalisation of Katz' result on "parallel transport by Frobenius" for morphisms of constant F-crystals over perfect schemes; see [9, Prop A.1] for the details.

Since $\mathscr{S}_{\mathsf{K}}$ does *not* admit a nice moduli interpretation as the PEL case, it is non-trivial to show that $\operatorname{Qisg}_{\mathsf{G}}(\mathbb{X}_{\mathsf{b}})$ acts on RZ^{b} and $\mathfrak{Ig}^{\mathsf{b}}_{\mathsf{K}^p}$ via restriction of $\operatorname{Qisg}_{\mathsf{G}'}(\mathbb{X}_{\mathsf{b}'})$-action on $\mathrm{RZ}^{\mathsf{b}'}$ and $\mathfrak{Ig}^{\mathsf{b}'}_{\mathsf{K}'^p}$. The action of $\operatorname{Qisg}^{\circ}_{\mathsf{G}}(\mathbb{X}_{\mathsf{b}})$ on RZ^{b} and $\mathfrak{Ig}^{\mathsf{b}}_{\mathsf{K}^p}$ can be deduced from its action on the formal neighbourhood of $\overline{\mathscr{S}}^{\mathsf{b}}_{\mathsf{K}}$ at each closed point studied in [15], but the action of the "component group" $\mathsf{J}_{\mathsf{b}}(\mathbb{Q}_p)$ is subtle, which roughly asserts that any tensor-preserving quasi-isogeny of p-divisible group $[\mathbb{X}_{\mathsf{b}}, (t^{\mathbb{D}}_{\alpha,x})] \dashrightarrow [\mathbb{Y}, (u^{\mathbb{D}}_{\alpha})]$ gives rise to a closed point $y \in \overline{\mathscr{S}}^{\mathsf{b}}_{\mathsf{K}}$ with $[\mathbb{Y}, (u^{\mathbb{D}}_{\alpha})] \cong (\mathscr{A}_{\mathsf{K},y}[p^{\infty}], (t^{\mathbb{D}}_{\alpha,y})]$; *cf.* Axiom A in [9, §1]. When [9] was written this property was known only in some special cases, but it is now proven in complete generality by Gleason–Lim–Xu [8]. Finally, the almost product structure (6) can be deduced from the Siegel case via considering the "almost product structure" on the formal completion $(\overline{\mathscr{S}}^{\mathsf{b}}_{\mathsf{K}})^{\wedge}_{y}$ at each closed point [15, §5.2].

Set $\mathcal{S}^{\mathsf{b}}_{\mathsf{K},\mathsf{C}} := (\mathscr{S}_{\mathsf{K}})^{\wedge\,\mathrm{ad}}_{/\overline{\mathscr{S}}^{\mathsf{b}}_{\mathsf{K}}} \times_{\mathrm{Spa}\,\breve{\mathscr{O}}_{\mathsf{E},v}} \mathrm{Spa}\,\mathsf{C}$ and $\mathrm{RZ}^{\mathsf{b}}_{\mathsf{C}} := \mathrm{RZ}^{\mathsf{b},\mathrm{ad}} \times_{\mathrm{Spa}\,\breve{\mathscr{O}}_{\mathsf{E},v}} \mathrm{Spa}\,\mathsf{C}$, where $\mathsf{C} := \widehat{\overline{\breve{\mathsf{E}}}}_{v}$. Let $\mathcal{S}^{\mathsf{b}}_{\infty,\mathsf{C}}$ denote the preimage of $\mathcal{S}^{\mathsf{b}}_{\mathsf{K},\mathsf{C}}$ in the perfectoid limit $\varprojlim_{\mathsf{K}'\subset\mathsf{K}}(\mathsf{Sh}_{\mathsf{K}'} \times_{\mathrm{Spec}\,\mathsf{E}} \mathrm{Spec}\,\mathsf{C})^{\mathrm{an}}$, and $\mathrm{RZ}^{\mathsf{b}}_{\infty,\mathsf{C}}$ the perfectoid limit of the coverings $\{\mathrm{RZ}^{\mathsf{b}}_{\mathsf{K}'_p,\mathsf{C}}\}_{\mathsf{K}'_p\subset\mathsf{K}_p}$ with K'_p-level structure. And lastly, set $\mathfrak{Ig}^{\mathsf{b}}_{\infty,\mathsf{C}} := \varprojlim_{\mathsf{K}'^p\subset\mathsf{K}^p} \mathfrak{Ig}^{\mathsf{b},\mathrm{ad}}_{\mathsf{K}'^p} \times_{\mathrm{Spa}\,\breve{\mathscr{O}}_{\mathsf{E},v}} \mathrm{Spa}\,\mathsf{C}$. Then $\mathcal{S}^{\mathsf{b}}_{\infty,\mathsf{C}}$, $\mathrm{RZ}^{\mathsf{b}}_{\infty,\mathsf{C}}$ and $\mathfrak{Ig}^{\mathsf{b}}_{\infty,\mathsf{C}}$ are perfectoid spaces equipped with a natural action of $\mathsf{G}(\mathbb{A}^{\infty})$, $\mathsf{G}(\mathbb{Q}_p) \times \operatorname{Qisg}_{\mathsf{G}}(\mathbb{X}_{\mathsf{b}})_{\mathsf{C}}$ and

$\mathrm{Qisg}_G(\mathbb{X}_b)_C$, respectively. (Note that $\mathrm{Qisg}_G(\mathbb{X}_b)_C$ is a perfectoid group; that is, a group object in the category of perfectoid spaces over C.)

Now from (6) we get a following $G(\mathbb{A}^\infty)$-equivariant isomorphism

$$\left(\mathrm{RZ}^b_{\infty,C} \times_{\mathrm{Spa}\, C} \mathfrak{Ig}^b_{\infty,C}\right) / \mathrm{Qisg}_G(\mathbb{X}_b)_C \xrightarrow{\cong} \mathcal{S}^b_{\infty,C}, \tag{7}$$

which we call the *infinite level almost product structure.* (Compare with (5).)

2.3. **Remark on canonicity and p-adic shtukas.** Our construction of RZ^b *a priori* depends on the auxiliary choice of the integral model $\mathscr{S}_K$, whose construction *a priori* depends on yet another auxiliary choice of a carefully chosen embedding into some Siegel modular variety. There have been important progresses on making these constructions *canonical* via Scholze's theory of p-adic shtukas [28], which we briefly introduce.

Regarding $\mathscr{S}_K$, Pappas–Rapoport [23, §4.2] defined the notion of *canonical* integral models of Shimura varieties defined in terms of the family of $\mathcal{G}$-shtukas when K_p is parahoric. This definition is extended to the case when K_p is *quasi-parahoric* in a recent preprint by P. Daniels–P. van hoften–D. Kim–M. Zhang; *cf.* [4, Def 4.1.2]. Furthermore, one can show that the Kisin–Pappas integral model in §2.1 is canonical, and the special fibre of the $\mathcal{G}$-shtuka recovers the Witt vector $\mathcal{G}$-shtuka $\underline{\mathcal{P}}$ in Prop 2.1. (This claim can be extracted from the statement and the proof of [4, Th 4.1.12]. Note that p-adic $\mathcal{G}$-shtukas in characteristic p coincide with Witt vector $\mathcal{G}$-shtukas by [23, Th 1.1.3].) Furthermore, the main result of [4] shows that canonical integral models exist for any Hodge-type Shimura varieties with any level that is *quasi-parahoric* at p (even allowing $p = 2$). One may consider removing the assumptions on G for the Hodge-type almost product structure (i.e., (6) and (7)), but we will discuss this in the *abelian-type setting* in §3.

Regarding RZ^b, it can be read off from Prop 1.3.1 and Th 1.3.3 in [23] that RZ^b coincides with the integral local Shimura variety (constructed purely locally in [24] as the moduli of p-adic $\mathcal{G}$-shtukas).

Lastly, the construction of $\mathfrak{Ig}^b_{K^p}$ implies

$$\mathrm{Ig}^b_{K^p} \coloneqq \mathfrak{Ig}^b_{K^p} \times_{\mathrm{Spf}\, \breve{\mathbb{Z}}_p} \mathrm{Spec}\, \overline{\mathbb{F}}_p \cong \mathrm{Isom}_\varphi(\underline{\mathcal{P}}_x, \underline{\mathcal{P}}), \tag{8}$$

where $\underline{\mathcal{P}}$ is defined in Prop 2.1 and the rightmost term is the isom-sheaf of Witt vector $\mathcal{G}$-shtukas. In the unramified PEL case, the analogue of (8) for $\mathfrak{Ig}^b_{K^p,C}$ (in terms of p-adic $\mathcal{G}$-shtukas on C) and the almost product structure (7) are incorporated in the "fibre product formula" of Caraiani–Scholze [1, §4.3] and the theory of Igusa stacks of M. Zhang [31]. The Hodge-type generalisation is expected to be available soon; *cf.* [5].

2.4. **Cohomological consequence: Mantovan formula.** From now on, fix a prime $\ell \neq p$ as well as an isomorphism $\mathbb{C} \cong \overline{\mathbb{Q}}_\ell$. The (suitably defined) compact support cohomology $R\Gamma_c(\mathrm{RZ}^b_{\infty,C}, \overline{\mathbb{Q}}_\ell)$ admits a natural smooth action of $G(\mathbb{Q}_p) \times J_b(\mathbb{Q}_p) \times W_{E_v}$, induced by the action of $G(\mathbb{Q}_p) \times \mathrm{Qisg}_G(\mathbb{X}_b)_C$ and the Weil descent datum on $\mathrm{RZ}_{\infty,C}$ [9, Lem 5.14].

Definition 2.2. For a locally profinite group H, let $\mathrm{Groth}(H)$ denote the Grothendieck group of admissible H-representations. Given $\tau \in \mathrm{Groth}(J_b(\mathbb{Q}_p))$ we set

$$\mathrm{Mant}_{b,\mu}(\tau) \coloneqq \sum_{i=1}^{2d} (-1)^i \, \mathrm{Ext}^{-2d+i}_{J_b(\mathbb{Q}_p)}\left(R\Gamma_c(\mathrm{RZ}^b_{\infty,C}, \overline{\mathbb{Q}}_\ell), \tau\right)(-d) \in \mathrm{Groth}(G(\mathbb{Q}_p) \times W_{E_v}),$$

where $\mathrm{d} \coloneqq \langle 2\rho, \mu \rangle$ is the dimension of Sh_K.[5]

As $\mathrm{RZ}^b_{\infty,C}$ is the infinite-level local Shimura variety, $\mathrm{Mant}_{b,\mu}$ is expected to "encode" the local Langlands correspondence and Jacquet–Langlands correspondence. The precise conjecture due to Kottwitz and Harris–Viehmann can be found in [25].

Let $\mathscr{L}_\xi$ denote the automorphic $\overline{\mathbb{Q}}_\ell$-local system on $\mathscr{S}_K$, and we pull it back to $\mathfrak{Ig}^b_{K^p}$. (For example, one may choose $\mathscr{L}_\xi = \overline{\mathbb{Q}}_\ell$.) Note that $\mathrm{R}\Gamma_c(\mathrm{Ig}^b_\infty, \mathscr{L}_\xi) \coloneqq \varinjlim_{K^p} \mathrm{R}\Gamma_c(\mathrm{Ig}^b_{K^p}, \mathscr{L}_\xi)$ is a bounded complex of smooth $\mathsf{G}(\mathbb{A}^{p\infty}) \times J_b(\mathbb{Q}_p)$-representations with admissible cohomology, where $\mathrm{Ig}^b_{K^p}$ is the special fibre of $\mathfrak{Ig}^b_{K^p}$.

We extend $\mathrm{Mant}_{b,\mu}$ to $\mathrm{Groth}(\mathsf{G}(\mathbb{A}^{p\infty}) \times J_b(\mathbb{Q}_p)) \to \mathrm{Groth}(\mathsf{G}(\mathbb{A}^\infty) \times W_{E_v})$. We also abusively let $\mathrm{R}\Gamma_c(\mathrm{Ig}^b_\infty, \mathscr{L}_\xi) \in \mathrm{Groth}(\mathsf{G}(\mathbb{A}^{p\infty}) \times J_b(\mathbb{Q}_p))$ denote the alternating sum of the cohomology. We are now ready to state the *Hodge-type Mantovan formula.*

Theorem 2.3. *Suppose that the Hodge-type Shimura variety* Sh_K *is* proper[6] *in addition to the assumptions in §2.1. Then the following equality holds*

$$\mathrm{R}\Gamma_c(\mathsf{Sh}_{\infty,C}, \mathscr{L}_\xi) = \sum_{b \in B(G_{\mathbb{Q}_p}, \mu)} \mathrm{Mant}_{b,\mu}\left(\mathrm{R}\Gamma_c(\mathrm{Ig}^b_\infty, \mathscr{L}_\xi)\right)$$

in $\mathrm{Groth}(\mathsf{G}(\mathbb{A}^\infty) \times W_{E_v})$*, where the sum is over the index of the Newton stratification.*

This formula offers an extremely useful global tool for local results; indeed, one can study $\mathrm{R}\Gamma_c(\mathrm{Ig}^b_\infty, \mathscr{L}_\xi)$ via trace formula techniques, and use it to deduce interesting consequences on $\mathrm{Mant}_{b,\mu}$; *cf.* Harris–Taylor [12] and Shin [30].

Let us discuss a heuristic behind the proof. By excision, one is reduced to showing

$$\mathrm{R}\Gamma_c(\mathcal{S}^b_{\infty,C}, \mathscr{L}_\xi) \cong \mathrm{R}\,\mathrm{Hom}_{J_b(\mathbb{Q}_p)}\left(\mathrm{R}\Gamma_c(\mathrm{RZ}^b_{\infty,C}, \overline{\mathbb{Q}}_\ell), \mathrm{R}\Gamma_c(\mathrm{Ig}^b_\infty, \mathscr{L}_\xi)\right)(-\mathrm{d})[-2\mathrm{d}]. \tag{9}$$

(Note that the RHS defines $\mathrm{Mant}_{b,\mu}$-term in the Grothendieck group.) We want to interpret it as the "Künneth formula" for the infinite-level almost product structure (7), which asserts that $\mathrm{RZ}^b_{\infty,C} \times_{\mathrm{Spa}\, C} \mathfrak{Ig}^b_{\infty,C} \to \mathcal{S}^b_{\infty,C}$ is a torsor under the action of a perfectoid group $\mathrm{Qisg}_G(\mathbb{X}_b)_C$. Employing "topological heuristics", one can regard $\mathfrak{Ig}^b_{\infty,C}$ (resp., $\mathrm{Qisg}_G(\mathbb{X}_b)_C$-torsor) as "deformation-retracting" to Ig^b_∞ (resp., to a $J_b(\mathbb{Q}_p)$-torsor). This heuristic was made precise for $\mathfrak{Ig}^b_{\infty,C}$ via formal nearby cycles [1, Lem 4.4.3], which seems plausible to generalise for perfectoid $\mathrm{Qisg}^\circ_G(\mathbb{X}_b)$-torsors.

Unfortunately, we failed to turn it into a proof then, so we had to work with auxiliary integral models at arbitrary deeper levels at p in [9, §7]. The main obstacle was that the cohomological tools that we had then (*cf.* [9, §2]) require the presence of special formal integral models. Fast forwarding to the present time, we now have tools to directly handle the "infinite-level cohomology" thanks to Fargues–Scholze [7]. Furthermore, the theory of Igusa stacks by M. Zhang [31] (at least in the unramified PEL case) has beautifully transformed the almost product structure (7) (and the aforementioned heuristic) so that the Fargues–Scholze machinery can be applied directly. This already has an application to the mod ℓ analogue of the Mantovan formula [11, §3], and it would not be so surprising if the ℓ-adic Mantovan formula could be obtained similarly.[7] Note that we are expected to

[5]The admissibility of $\mathrm{Mant}_{b,\mu}(\tau)$ follows as RZ^b_C satisfies the conditions for [25, Prop 6.1].

[6]In the proof, the only role of properness is the nearby cycle spectral sequence for Shimura varieties with automorphic coefficient sheaves. However, many non-compact Shimura varieties admits the nearby cycle spectral sequence with automorphic coefficient sheaves (*cf.* [19]), so in [9] we assumed "Axiom B" instead of properness. See [9, Rmk 7.9] for further discussions.

[7]Note that one cannot trivially deduce the ℓ-adic case from the torsion case, as the projective limit does not preserve smoothness of the group action.

have the Hodge-type generalisation of the theory of Igusa stacks and the mod ℓ analogue of the Mantovan formula in the forthcoming work of P. Daniels, P. van Hoften, D. Kim and M. Zhang [5].

3. Abelian-type case: Sketch

Many interesting *proper* Shimura varieties are of abelian type but not of Hodge type (eg., non-PEL Shimura curves, type C examples). To give a GSpin example, choose a totally real number field $\mathsf{F} \neq \mathbb{Q}$, and an F-quadratic space (V, q) with signature $(n, 2)$ at one infinite place and positive definite elsewhere. Then $\mathsf{G} = \mathrm{Res}_{\mathsf{F}/\mathbb{Q}}\, \mathrm{GSpin}(\mathsf{V}, \mathsf{q})$ admits a proper Shimura variety that is of abelian type, but not of Hodge type. This gives enough reason to desire for the abelian-type Mantovan formula.

In this section, we sketch (very roughly) the ingredients needed to generalise the Hodge-type almost product structure in [9] to the abelian-type case.

3.1. **Canonical integral models.** Let us begin with an extremely brief review of recent progresses on integral models of abelian-type Shimura varieties (*cf.* [18], [6]). Fix a prime $p > 2$. Let $(\mathsf{G}, \mathfrak{D})$ be an *abelian-type* Shimura datum, and $\mathcal{G}/\mathbb{Z}_p$ a *parahoric* integral model with $\mathsf{K}_p \coloneqq \mathcal{G}(\mathbb{Z}_p)$. Let G^c denote the quotient of G by the maximal $\mathbb{Q}$-anisotropic $\mathbb{R}$-split central torus. (Note that $\mathsf{G} = \mathsf{G}^c$ in the Hodge-type case.) Then $\mathcal{G}$ determines the parahoric integral model $\mathcal{G}^c$ of G^c (resp. $\mathcal{G}^{\mathrm{ad}}$ of G^{ad}), which are obtained as a suitable quotient of $\mathcal{G}$.

Finally, suppose that there exists a *canonical* integral model $\mathscr{S}_{\mathsf{K}}$ of $\mathsf{Sh}_{\mathsf{K}} = \mathsf{Sh}_{\mathsf{K}}(\mathsf{G}, \mathfrak{D})$ for $\mathsf{K} = \mathsf{K}^p\mathsf{K}_p$ in the sense of [23, Conj 4.2.2] and [4, Def 4.1.2]. In particular, we get a Witt vector $\mathcal{G}^c$ -shtuka $\underline{\mathcal{P}}$ on the perfected geometric special fibre $\overline{\mathscr{S}}_{\mathsf{K}}^{\mathrm{perf}}$ (resp. $\mathcal{G}^{\mathrm{ad}}$-shtuka $\underline{\mathcal{P}}^{\mathrm{ad}}$ on $\overline{\mathscr{S}}_{\mathsf{K}^{\mathrm{ad}}}^{\mathrm{ad,perf}}$). This assumption is known to be satisfied in the following cases:

(1) $(\mathsf{G}, \mathfrak{D})$ is of Hodge type; *cf.* P. Daniels–P. van Hoften–D. Kim–M. Zhang [4, Th I].
(2) $p > 2$ and G^{ad} is *essentially tame*[8]; *cf.* Daniels–Youcis [6] built upon Kisin–Zhou [18, §5] and [4, Th I]. Note that essential tameness is automatic if $p \geqslant 5$. (See §5.3, esp. Rmk 5.3.1, in [23].)

Let us elaborate the latter case further. In this case, Kisin–Zhou [18] extended the construction of Kisin–Pappas integral models as follows. Choose a Hodge-type Shimura datum $(\mathsf{G}', \mathfrak{D}')$ *ad-isomorphic* to $(\mathsf{G}, \mathfrak{D})$ satisfying a list of properties in [18, Prop 5.2.6]. Applying [18, Th 5.2.12]), we get an integral model of Sh_{K} for any small enough $\mathsf{K} = \mathsf{K}^p\mathsf{K}_p$, as well as integral models $\mathscr{S}_{\mathsf{K}'}$ and $\mathscr{S}^{\mathrm{ad}}_{\mathsf{K}^{\mathrm{ad}}}$ for the Hodge-type and adjoint Shimura varieties (with suitable parahoric levels at p), equipped with the following finite maps

$$\mathscr{S}'_{\mathsf{K}'} \longrightarrow \mathscr{S}^{\mathrm{ad}}_{\mathsf{K}^{\mathrm{ad}}} \longleftarrow \mathscr{S}_{\mathsf{K}} \tag{10}$$

corresponding to $(\mathsf{G}', \mathfrak{D}') \to (\mathsf{G}^{\mathrm{ad}}, \mathfrak{D}^{\mathrm{ad}}) \leftarrow (\mathsf{G}, \mathfrak{D})$, such that the map on the "geometric components" $\mathscr{S}'^{+}_{\mathsf{K}'} \to \mathscr{S}^{\mathrm{ad},+}_{\mathsf{K}^{\mathrm{ad}}}$ factors through $\mathscr{S}^{+}_{\mathsf{K}}$. Furthermore, Daniels–Youcis [6] showed that these integral models are *canonical* in the sense of Pappas–Rapoport [23, Conj 4.2.2].

3.2. **Central leaves and Igusa covers.** Let $\mathcal{H}$ be an affine smooth scheme over $\mathbb{Z}_p$ with connected fibres. Let $\underline{\mathcal{Q}}$ be a Witt vector $\mathcal{H}$ -shtuka on a perfect $\overline{\mathbb{F}}_p$-scheme S, and fix $x \in \mathsf{S}(\overline{\mathbb{F}}_p)$. In recent joint work with Hamacher [10, §2.14*ff*], we showed that the *central leaf* $\mathscr{C}^{/x}$ – the locus where the geometric fibre of $\underline{\mathcal{Q}}$ is isomorphic to $\underline{\mathcal{Q}}_x$ – is locally closed

[8]The essential tameness condition is called *acceptability* of $(\mathsf{G}, \mathfrak{D})$ in [18].

in S, and the "Igusa cover" $\mathrm{Isom}_\varphi(\underline{\mathcal{Q}}_x, \underline{\mathcal{Q}})$ is pro-finite étale over $\mathscr{C}^{/x}$. (This extends the classical results when $\underline{\mathcal{Q}}$ comes from a completely slope divisible p-divisible group over S.) Applying this construction to $\underline{\mathcal{P}}$ over $\overline{\mathscr{S}}_K^{\mathrm{perf}}$ and an $\overline{\mathbb{F}}_p$-point x, we get a *(canonical) central leaf* $\mathscr{C}^b$ and the *(perfect) Igusa variety* $\mathrm{Ig}^b_{K^p}$ over it. We define an *adjoint central leaf* $\mathscr{C}^{b,\mathrm{ad}} \subset \overline{\mathscr{S}}_K^{\mathrm{perf}}$ as a central leaf for the pull back of $\underline{\mathcal{P}}^{\mathrm{ad}}$; or equivalently, the preimage of a central leaf in $\overline{\mathscr{S}}_{K^{\mathrm{ad}}}^{\mathrm{ad,perf}}$. We have the following comparison between $\mathscr{C}^b$ and $\mathscr{C}^{b,\mathrm{ad}}$.

Proposition 3.1. *If the centre $Z_{\mathcal{G}^c}$ of $\mathcal{G}^c$ is a torus then we have $\mathscr{C}^{b,\mathrm{ad}} = \mathscr{C}^b$. In general, $\mathscr{C}^{b,\mathrm{ad}}$ is a finite (open and closed) disjoint union of central leaves.*

Note that the centre of a split GSpin_{n+2} for $n > 0$ is a torus if n is odd; otherwise, the maximal central torus is of index 2 in the centre. So for GSpin Shimura varieties, adjoint central leaves may differ from canonical central leaves.

Prop 3.1 was announced in Shen–Zhang [29, §5.4.5] in the good reduction case. The alternative proof in [10, Prop 2.20] uses the good reduction hypothesis only for constructing $\underline{\mathcal{P}}$, hence the latter approach actually works and proves Prop 3.1. Note that the proof also describes the set of central leaves appearing in $\mathscr{C}^{b,\mathrm{ad}}$.

3.3. **Almost product structure: sketch of the idea.** Pappas–Rapoport [24] gave a canonical (and local) construction of abelian-type integral local Shimura varieties RZ^b as normal special formal schemes. Now repeating the proof of Th 1.3.3 in [23] one should be able to deduce the abelian-type *Rapoport–Zink uniformisation*, which enjoy some natural functorial properties between ad-isomorphic Shimura varieties. Likewise, the functoriality for canonical integral models and p-adic shtukas on them should imply the similar functoriality for Igusa varieties.

Applying this functoriality to the diagram (10) with Prop 3.1 in mind, one should be able to descend the Hodge-type almost product structure for $\mathscr{S}'_{K'}$ (*cf.* (6), (7)) to the adjoint-type Shimura variety $\mathscr{S}^{\mathrm{ad}}_{K^{\mathrm{ad}}}$, and in turn "lift" it to the abelian-type Shimura variety $\mathscr{S}_K$. In the last step, one should take care of the gap between $\mathscr{C}^{b,\mathrm{ad}}$ and $\mathscr{C}^b$ if $Z_{\mathcal{G}^c}$ is not a torus, but this can be explicitly controlled. Finally, one can deduce the *abelian-type* Mantovan formula following the proof in the Hodge-type case [9, §7].

It should be noted that a more novel alternative approach to the abelian-type almost product structure and Mantovan's formula via Igusa stacks will be available in some near future as an application of the Hodge-type case in the forthcoming work [5]. The Igusa stack technique is expected to have more powerful applications, such as the mod ℓ analogue of the Mantovan formula and torsion vanishing results. On the other hand, the idea outlined in this section is a straightforward extension of the proof in the Hodge-type case in the joint work with Hamacher [9], necessitating only a few new elements and circumventing extensive foundational efforts. The sketchy idea outlined in this section shall be written up in details in the near future.

Acknowledgement. We thank the organisers of *Algebraic Number Theory and Related Topics 2023* and RIMS for their hospitality. This note was written while the author is visiting Graduate School of Mathematical Sciences, University of Tokyo, hosted by Naoki Imai. We sincerely thank their hospitality. We also thank Patrick Daniels and Alex Youcis for offering further details on [6] and [5]. This work was supported by the National Research Foundation of Korea(NRF) grant funded by the Korea government(MSIT) (No. RS-2023-00208018)

References

[1] Ana Caraiani and Peter Scholze. On the generic part of the cohomology of compact unitary Shimura varieties. *Ann. of Math. (2)*, 186(3):649–766, 2017.

[2] Henri Carayol. Sur la mauvaise réduction des courbes de Shimura. *Compositio Math.*, 59(2):151–230, 1986.

[3] Marco D'Addezio and Pol van Hoften. Hecke orbits on Shimura varieties of Hodge type. *preprint*.

[4] Patrick Daniels, Pol van Hoften, Dongryul Kim, and Mingjia Zhang. On a conjecture of Pappas and Rapoport. *preprint*.

[5] Patrick Daniels, Pol van Hoften, Dongryul Kim, and Mingjia Zhang. Igusa stacks for Shimura varieties of Hodge type. *in preparation*.

[6] Patrick Daniels and Alex Youcis. Canonical integral models of Shimura varieties of abelian type. *preprint*.

[7] Laurent Fargues and Peter Scholze. Geometrization of the local Langlands correspondence. *preprint*.

[8] Ian Gleason, Dong Gyu Lim, and Yujie Xu. The connected components of affine Deligne–Lusztig varieties.

[9] Paul Hamacher and Wansu Kim. $\mathfrak{l}$-adic étale cohomology of Shimura varieties of Hodge type with non-trivial coefficients. *Math. Ann.*, 375(3-4):973–1044, 2019.

[10] Paul Hamacher and Wansu Kim. Point counting on Igusa varieties for function fields. *preprint*.

[11] Linus Hamann and Si Ying Lee. Torsion vanishing for some Shimura varieties. *preprint*.

[12] Michael Harris and Richard Taylor. *The geometry and cohomology of some simple Shimura varieties*, volume 151 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 2001. With an appendix by Vladimir G. Berkovich.

[13] Benjamin Howard and Georgios Pappas. Rapoport-Zink spaces for spinor groups. *Compos. Math.*, 153(5):1050–1118, 2017.

[14] Nicholas M. Katz. Serre-Tate local moduli. In *Algebraic surfaces (Orsay, 1976–78)*, volume 868 of *Lecture Notes in Math.*, pages 138–202. Springer, Berlin, 1981.

[15] Wansu Kim. On central leaves of Hodge-type Shimura varieties with parahoric level structure. *Math. Z.*, 291(1-2):329–363, 2019.

[16] Mark Kisin, Keerthi Madapusi Pera, and Sug Woo Shin. HondaTate theory for Shimura varieties. *Duke Mathematical Journal*, 171(7), May 2022.

[17] Mark Kisin and Georgios Pappas. Integral models of Shimura varieties with parahoric level structure. *Publ. Math. Inst. Hautes Études Sci.*, First Online:98 pages, 12 2018.

[18] Mark Kisin and Rong Zhou. Independence of ℓ for Frobenius conjugacy classes attached to abelian varieties. *preprint*, https://www.dpmms.cam.ac.uk/~rz240/l-indep_v2.pdf.

[19] Kai-Wen Lan and Benoît Stroh. Nearby cycles of automorphic étale sheaves. *Compos. Math.*, 154(1):80–119, 2018.

[20] Elena Mantovan. On the cohomology of certain PEL-type Shimura varieties. *Duke Math. J.*, 129(3):573–610, 2005.

[21] Elena Mantovan. $\mathfrak{l}$-adic étale cohomology of PEL type Shimura varieties with non-trivial coefficients. In *WIN—women in numbers*, volume 60 of *Fields Inst. Commun.*, pages 61–83. Amer. Math. Soc., Providence, RI, 2011.

[22] Frans Oort. Foliations in moduli spaces of abelian varieties. *J. Amer. Math. Soc.*, 17(2):267–296, 2004.

[23] Georgios Pappas and Michael Rapoport. $\mathfrak{p}$-adic shtukas and the theory of global and local Shimura varieties. *Camb. J. Math.*, 12(1):1–164, 2024.

[24] Georgios Pappas and Michael Rapoport. On integral local Shimura varieties. *preprint*.

[25] Michael Rapoport and Eva Viehmann. Towards a theory of local Shimura varieties. *Münster J. Math.*, 7(1):273–326, 2014.

[26] Michael Rapoport and Thomas Zink. *Period spaces for $\mathfrak{p}$-divisible groups*, volume 141 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 1996.

[27] Peter Scholze and Jared Weinstein. Moduli of $\mathfrak{p}$-divisible groups. *Camb. J. Math.*, 1(2):145–237, 2013.

[28] Peter Scholze and Jared Weinstein. Berkeley lectures on $\mathfrak{p}$-adic geometry, notes from lecture course at UC Berkeley, Fall 2014. *Preprint*, 2017.

[29] Xu Shen and Chao Zhang. Stratifications in good reductions of Shimura varieties of abelian type. *Asian Journal of Mathematics*, 26(2):167–226, 2022.
[30] Sug Woo Shin. On the cohomology of Rapoport-Zink spaces of EL-type. *Amer. J. Math.*, 134(2):407–452, 2012.
[31] Mingjia Zhang. A PEL-type Igusa stack and the $\mathfrak{p}$-adic geometry of Shimura varieties. *preprint*.
[32] Xinwen Zhu. Affine Grassmannians and the geometric Satake in mixed characteristic. *Ann. of Math. (2)*, 185(2):403–492, 2017.

Department of Mathemetical Sciences, KAIST, 291 Daehak-ro, Yuseong-gu, Daejeon, 34141, Republic of Korea. // Graduate School of Mathematical Sciences, University of Tokyo, 3-8-1 Komaba, Meguro-ku, Tokyo 153-8914, Japan.
Email address: wansu.math@kaist.ac.kr