# Fundamental groups of curves in positive characteristic as local moduli

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**Abstract**

In the present survey, we explain the author's insights and investigations concerning anabelian phenomena of curves over algebraically closed fields of characteristic $p > 0$ which says that the geometry of moduli spaces of curves can be understood from the geometric fundamental groups of curves.

Keywords: pointed stable curve, admissible fundamental group, anabelian geometry, moduli space, positive characteristic.

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## 1 Grothendieck's anabelian geometry over arithmetic fields

In the 1980s, A. Grothendieck suggested a theory of arithmetic geometry called anabelian geometry ([G]). Roughly speaking, this theory focuses on the following question:

> How much information about the geometry of a variety can be carried out group-theoretically from various versions of its algebraic fundamental group?

The varieties that can be completely determined by their fundamental groups are called "anabelian varieties" by Grothendieck. To classify the anabelian varieties in all dimensions over all fields is called "anabelian dream" of him. In the particular case of dimension 1, he conjectured that smooth pointed stable curves (or hyperbolic curves) defined over certain fields are anabelian varieties.

## 1.1

Let

$$X^\bullet = (X, D_X)$$

be a pointed stable curve of type $(g_X, n_X)$ over a field $k$, where $X$ denotes the underlying curve which is a semi-stable curve over $k$, $D_X$ denotes the (finite) set of marked points satisfying [K, Definition 1.1 (iv)], $g_X$ denotes the genus of $X$, and $n_X \stackrel{\text{def}}{=} \#(D_X)$. Here, $\#(-)$ denotes the cardinality of $(-)$.

Suppose that $X^\bullet$ is smooth over $k$. When $k$ is an "arithmetic" field (e.g. a number field, a $p$-adic field, a finite field, etc.), Grothendieck's anabelian conjectures for curves (or the Grothendieck conjectures for short), roughly speaking, are formulated based on the following *anabelian philosophy* ([G]):

> **Hom-version:** The sets of dominant morphisms of smooth pointed stable curves can be determined group-theoretically from the sets of open continuous homomorphisms of their algebraic fundamental groups.

In particular, Hom-version implies the following important versions:

> **Isom-version:** The sets of isomorphisms of smooth pointed stable curves can be determined group-theoretically from the sets of isomorphisms of their algebraic fundamental groups.

> **Weak Isom-version:** The isomorphism class of $X^\bullet$ can be determined group-theoretically from the isomorphism class of its algebraic fundamental group.

Note that we have "Hom-version"$\Rightarrow$"Isom-version"$\Rightarrow$"Weak Isom-version".

## 1.2

Grothendieck's anabelian conjectures have been proven in many cases. For instance, we have the following important results:

When $k$ is a number field, the conjecture was proved by H. Nakamura (weak Isom-version if $g_X = 0$) ([N1], [N2]), A. Tamagawa (Isom-version if $n_X \neq 0$) ([T1]), and S. Mochizuki (Isom-version if $n_X = 0$) ([M1]).

When $k$ is a sub-$p$-adic field (i.e. a subfield of a finitely generated field over $\mathbb{Q}_p$), *the Hom-version of the Grothendieck conjecture has been completely solved by Mochizuki ([M2]) which is one of the highest achievements in the theory of anabelian geometry.*

When $k$ is a finitely generated field over the finite field $\mathbb{F}_p$, the Isom-version of the Grothendieck conjecture was proved by Tamagawa ([T1] if $n_X \neq 0$), and by Mochizuki ([M3] if $n_X = 0$).

All the proofs of the Grothendieck conjectures for curves over arithmetic fields mentioned above require the use of *the non-trivial outer Galois representations* induced by the fundamental exact sequences of fundamental groups.

## 2 Beyond the arithmetic actions

We maintain the notation introduced in 1.1. Moreover, we assume that $k$ is an algebraically closed field of characteristic $p \geq 0$.

### 2.1

By choosing a suitable base point $x$ of $X^{\rm sm} \setminus D_X$, where $X^{\rm sm}$ denotes the smooth locus of $X$, we have the admissible fundamental group $\pi_1^{\rm adm}(X^\bullet, x)$ of $X^\bullet$ (see [Y5, Section 2.1.5]). For simplicity, we shall write

$$\Pi_{X^\bullet}$$

for $\pi_1^{\rm adm}(X^\bullet, x)$, since we only focus on the isomorphism class of $\pi_1^{\rm adm}(X^\bullet, x)$. In particular, if $X^\bullet$ is smooth over $k$, then $\Pi_{X^\bullet}$ is naturally isomorphic to the tame fundamental group $\pi_1^{\rm t}(X^\bullet)$.

If $p = 0$, since $\Pi_{X^\bullet}$ is isomorphic to the profinite completion of the topological fundamental group of a Riemann surface of type $(g_X, n_X)$, the anabelian geometry of curves does not exist in this situation since $\Pi_{X^\bullet}$ depends only on the type $(g_X, n_X)$. On the other hand, if $p > 0$, the situation is quite different from that in characteristic 0. The admissible fundamental group $\Pi_{X^\bullet}$ is very mysterious and its structure is no longer known.

### Assumptions

In the remainder of the present survey, we assume that $k$ is an algebraically closed field of characteristic $p > 0$ unless indicated otherwise.

### 2.2

After M. Raynaud and D. Harbater proved Abhyankar's conjecture, Harbater asked whether or not the geometric information of a curve over $k$ can be carried out from its geometric fundamental groups ([Ha1], [Ha2]). In the late 1990s, Tamagawa discovered surprisingly that *the anabelian phenomena also exist for curves over algebraically closed fields of positive characteristic*. In this situation, the arithmetic fundamental group coincides with the geometric fundamental group, thus there is a total absence of a Galois action of the base field. This kind of anabelian phenomenon is the reason why we do not have an explicit description of the geometric fundamental group of any pointed stable curve in positive characteristic.

Let us explain this kind of anabelian phenomenon in detail. Let $W^\bullet$ be a pointed stable curve over an algebraically closed field $l$. We shall call $W^{\bullet,\rm min}$ *the minimal model of* $W^\bullet$ *over* $l^{\rm min}$ if $l^{\rm min} \subseteq l$ is the minimal algebraically closed subfield of $l$ such that there exists a pointed stable curve $W^{\bullet,\rm min}$ over $l^{\rm min}$ satisfying

$$W^\bullet \cong W^{\bullet,\rm min} \times_{l^{\rm min}} l.$$

Based on Grothendieck's anabelian philosophy mentioned in 1.1, we have the following important conjecture which was formulated by Tamagawa ([T4]) for smooth pointed stable curves, and by the author ([Y4]) for arbitrary pointed stable curves:

**Weak Isom-version Conjecture .** *Let $X_i^\bullet$, $i \in \{1, 2\}$, be a pointed stable curve of type $(g_{X_i}, n_{X_i})$ over an algebraically closed field $k_i$ of characteristic $p > 0$. Then $X_1^{\bullet,\min}$ is Frobenius equivalent ([Y4, Definition 3.4]) to $X_2^{\bullet,\min}$ if and only if*

$$\Pi_{X_1^\bullet} \cong \Pi_{X_2^\bullet}.$$

*In particular, if $X_i^\bullet$, $i \in \{1, 2\}$, is smooth over $k_i$, then we have*

$$X_1^{\bullet,\min} \cong X_2^{\bullet,\min}$$

*as schemes if and only if*

$$\Pi_{X_1^\bullet} \cong \Pi_{X_2^\bullet}.$$

At present, the weak Isom-version conjecture has only been proved in the following cases: If $X_i^\bullet$, $i \in \{1, 2\}$, is smooth over $\overline{\mathbb{F}}_p$, the weak Isom-version conjecture has been proved by Tamagawa ([T4]) when $g_X = 0$, and by A. Sarashina ([S]) when $(g_X, n_X) = (1, 1)$ and $p > 2$. If $X_i^\bullet$, $i \in \{1, 2\}$, is an arbitrary pointed stable curve, the weak Isom-version conjecture has been proved by the author ([Y4]) under certain assumptions concerning types of smooth pointed stable curves associated to irreducible components of $X_i^\bullet$.

On the other hand, Tamagawa ([T3]) also proposed the so-called "*Isom-version conjecture*" based on Grothendieck's anabelian philosophy mentioned in 1.1. However, there are no results on this conjecture yet. While the weak Isom-version conjecture and the Isom-version conjecture are still wild open problems at present, Tamagawa has nonetheless achieved an important result known as the "*finiteness theorem*":

> *Over $\overline{\mathbb{F}}_p$, only finitely many isomorphism classes of smooth pointed stable curves have the same tame fundamental groups.*

This result was partially proved by Raynaud ([R]), F. Pop-Saïdi ([PS]), and was completely proved by Tamagawa ([T5]). *Tamagawa's theorem is one of the highest achievements in the theory of anabelian geometry, and was included in the new edition of SGA1 as one of the most important results since Grothendieck's introduction of the theory of algebraic fundamental groups.*

# 3 Admissible fundamental groups as local moduli

## 3.1

In the 1990s, when Tamagawa tried to formulate a "Hom-version conjecture" for the *tame* fundamental groups of smooth pointed stable curves over algebraically closed fields of characteristic $p$ based on Grothendieck's anabelian philosophy mentioned in

1.1, he noted that the following phenomenon exists in positive characteristic ([T3, Remark 1.34]): There are smooth pointed stable curves $X_1^\bullet$, $X_2^\bullet$ of type $(g_X, n_X)$ such that

$$\mathrm{Hom}^{\mathrm{dom}}(X_1^\bullet, X_2^\bullet) = \emptyset \text{ but } \mathrm{Hom}^{\mathrm{op}}_{\mathrm{pg}}(\Pi_{X_1^\bullet}, \Pi_{X_2^\bullet}) \neq \emptyset$$

hold. Here, $\mathrm{Hom}^{\mathrm{dom}}(X_1^\bullet, X_2^\bullet)$ denotes the set of dominate morphisms of curves, and $\mathrm{Hom}^{\mathrm{op}}_{\mathrm{pg}}(\Pi_{X_1^\bullet}, \Pi_{X_2^\bullet})$ denotes the set of open continuous homomorphisms of profinite groups. The above phenomenon means that

> *the "Hom-version" conjecture for curves over algebraically closed fields of characteristic $p$ does not exist if we only consider anabelian philosophy suggested originally by Grothendieck mentioned in 1.1 .*

### 3.2

The author considered the following the fundamental question:

> Does there exist *anabelian explanation* for $\mathrm{Hom}^{\mathrm{op}}_{\mathrm{pg}}(\Pi_{X_1^\bullet}, \Pi_{X_2^\bullet})$?

Moreover, the author obtained the following insight:

> The admissible fundamental groups of pointed stable curves over algebraically closed fields of characteristic $p$ can be regarded as an analogue of *local moduli spaces* of the curves.

Roughly speaking, this means that for $q, q' \in \overline{M}_{g,n}$ arbitrary points and $X_q^\bullet$, $X_{q'}^\bullet$ the curves corresponding to geometric points over $q, q'$, respectively:

1. We can group-theoretically detect, "up to Frobenius", whether or not $X_{q'}^\bullet$ is a deformation of $X_q^\bullet$ from $\mathrm{Hom}^{\mathrm{op}}(\Pi_{X_{q'}^\bullet}, \Pi_{X_q^\bullet})$ – i.e. "up to Frobenius", $q$ is a specialization (in the sense of moduli space) of $q'$ if and only if $\mathrm{Hom}^{\mathrm{op}}(\Pi_{X_{q'}^\bullet}, \Pi_{X_q^\bullet}) \neq \emptyset$.

2. The deformations of $X_q^\bullet$ can be reconstructed group-theoretically from

$$\mathrm{Hom}^{\mathrm{op}}(-, \Pi_{X_q^\bullet})$$

with certain addition conditions.

In the remainder of the present survey, we explain some higher non-trivial results obtained by the author that provide strong evidence for the above insight.

## 4 Moduli spaces of fundamental groups

### 4.1

In [HYZ], the author discovered a new kind of anabelian phenomenon that shows that *the sets of open continuous homomorphisms of admissible fundamental groups* *contains deformation information of curves.* This phenomenon can be precisely captured by the theory of "*moduli spaces of fundamental groups*" and its main conjecture "*the homeomorphism conjecture*" established by the author.

Roughly speaking, in [Y6], the author introduced a topological space

$$\overline{\Pi}_{g,n}$$

which we call *the moduli space of admissible fundamental groups of type* $(g, n)$. The underlying set is the set of the isomorphism classes of admissible fundamental groups of curves of type $(g, n)$ over algebraically closed fields of characteristic $p$, and the topology can be completely determined by the set of finite quotients of admissible fundamental groups.

Let $\overline{M}_{g,n}$ be the coarse moduli spaces of the moduli stack over $\overline{\mathbb{F}}_p$ classifying pointed stable curves of type $(g, n)$. By introducing the so-called "Frobenius equivalence $\sim_{fe}$" on $\overline{M}_{g,n}$ ([Y4]), the author proved that there exists a *continuous* map

$$\pi^{\mathrm{adm}}_{g,n} : \overline{\mathfrak{M}}_{g,n} \overset{\mathrm{def}}{=} \overline{M}_{g,n}/\sim_{fe} \twoheadrightarrow \overline{\Pi}_{g,n}, \ [q] \mapsto [\Pi_q],$$

where $[q]$ denotes the equivalence class of $q \in \overline{M}_{g,n}$, $\Pi_q$ denotes the admissible fundamental group of a curve corresponding to a geometric point over $q$, and $[\Pi_q]$ denotes the isomorphic class of $\Pi_q$. The main conjecture of the theory of moduli spaces of fundamental groups is the following:

**Homeomorphism conjecture**: *The continuous map* $\pi^{\mathrm{adm}}_{g,n} : \overline{\mathfrak{M}}_{g,n} \twoheadrightarrow \overline{\Pi}_{g,n}$ *is a homeomorphism.*

Note that the weak Isom-version conjecture only says that $\pi^{\mathrm{adm}}_{g,n}$ is a bijection as *sets*. This conjecture generalizes all the known conjectures concerning tame and admissible anabelian geometry of curves over algebraically closed fields of characteristic $p$. Furthermore, it supplies a viewpoint to consider anabelian geometry of curves over algebraically closed fields of characteristic $p$ based on the following *new anabelian philosophy*:

> *The* anabelian properties *concerning pointed stable curves over algebraically closed fields of characteristic* $p$ *are equivalent to the* topological properties *of moduli spaces of admissible fundamental groups.*

The above philosophy tells us *what are the anabelian phenomena that we can reasonably expect* for pointed stable curves over algebraically closed fields of characteristic $p$. This means that the homeomorphism conjecture is a *dictionary* between the geometry of pointed stable curves (or moduli spaces of curves) and the anabelian properties of pointed stable curves. It has raised a host of new questions and new conjectures concerning anabelian phenomena in positive characteristic which cannot be seen if we only consider the weak Isom-version conjecture.

*The author believes that the theory of moduli spaces of fundamental groups and the homeomorphism conjecture offer an approach towards Grothendieck's anabelian dream for curves over algebraically closed fields of characteristic* $p > 0$.

## 4.2 The homeomorphism conjecture for 1-dimensional moduli spaces

At present, the author obtained the following result ([Y6], [Y7]):

**Theorem 4.1.** *Let* $[q] \in \overline{\mathfrak{M}}_{g,n}$ *be an arbitrary point. Suppose that* $[q]$ *is closed, and that* $(g,n)$ *is equal to either* $(0,n)$ *or* $(1,1)$ *when* $p > 2$*. Then* $\pi^{\rm adm}_{g,n}([q]) = [\Pi_q]$ *is closed in* $\overline{\Pi}_{g,n}$*. In particular, the homeomorphism conjecture holds if* $(g,n)$ *is equal to either* $(0,4)$ *or* $(1,1)$ *when* $p > 2$*.*

The above result is equivalent to the following result by using fundamental groups.

**Theorem 4.2.** *Let* $q_i \in \overline{M}_{g,n}$*,* $i \in \{1,2\}$*, be an arbitrary point. Suppose that* $q_1$ *is closed, and that* $(g,n)$ *is equal to either* $(0,n)$ *or* $(1,1)$ *when* $p > 2$*. Then the set*

$$\mathrm{Hom}^{\rm op}_{\rm pg}(\Pi_{q_1}, \Pi_{q_2})$$

*is not empty if and only if* $q_1$ *is Frobenius equivalent to* $q_2$*.*

We maintain the notation introduced in Theorem 4.2. If $q_i \in M_{g,n}$, $i \in \{1,2\}$, the above results was proved in [HYZ]. If $q_i$, $i \in \{1,2\}$, is an arbitrary point of $\overline{M}_{g,n}$, the proof of the above results are much more difficult than in the case where $q_i \in M_{g,n}$, and the main ingredients are two group-theoretical formulas concerning average and maximum generalized Hasse-Witt invariant of prime-to-$p$ cyclic admissible coverings of pointed stable curves obtained by the author by using the theory of Raynaud-Tamagawa theta divisor ([Y3], [Y5]).

# 5 Group-theoretical specialization conjecture

## 5.1 Combinatorial data, topological data, and geometric data

Let $X_i^\bullet$, $i \in \{1,2\}$, be an arbitrary pointed stable curve of type $(g_X, n_X)$ over $k_i$ of characteristic $p > 0$, $\Gamma_{X_i^\bullet}$ the dual semi-graph of $X_i^\bullet$ ([Y8, Section 2.2.1]), and $\Pi_{X_i^\bullet}$ the admissible fundamental group of $X_i^\bullet$.

In [Y8], the author introduced the following sets

$$\mathrm{Com}(\Gamma_{X_i^\bullet}),\ \mathrm{Typ}(X_i^\bullet),\ \mathrm{Geo}(\Pi_{X_i^\bullet})$$

which we call the combinatorial data associated to $\Gamma_{X_i^\bullet}$, the topological data associated to $X_i^\bullet$, and the geometric data associated to $\Pi_{X_i^\bullet}$, respectively (see [Y8, Section 2.3 and Definition 2.5] for precise definitions). Roughly speaking, $\mathrm{Com}(\Gamma_{X_i^\bullet})$ consists of various sub-semi-graphs of $\Gamma_{X_i^\bullet}$ which encodes the *gluing data* of various pointed stable sub-curves of $X_i^\bullet$, $\mathrm{Typ}(X_i^\bullet)$ consists of the *topological types* of various pointed stable sub-curves of $X_i^\bullet$, and $\mathrm{Geo}(\Pi_{X_i^\bullet})$ consists of the closed subgroups of $\Pi_{X_i^\bullet}$ which are isomorphic to the admissible fundamental groups of various pointed stable sub-curves of $X_i^\bullet$.

## 5.2

Motivated by the homeomorphism conjecture, the author formulated the following conjecture (see [Y8, Section 3.1.3] for a more precise formulation):

**Group-theoretical Specialization Conjecture .** *Let* $\phi \in \mathrm{Hom}^{\mathrm{op}}_{\mathrm{pg}}(\Pi_{X_1^\bullet}, \Pi_{X_2^\bullet})$ *be an arbitrary open continuous homomorphism. Then we have*

$$\phi(\mathrm{Geo}(\Pi_{X_1^\bullet})) \subseteq \mathrm{Geo}(\Pi_{X_2^\bullet}).$$

Let us explain the anabelian phenomena concerning the group-theoretical specialization conjecture. Let $q_1, q_2 \in \overline{M}_{g,n}$ be arbitrary points such that $q_2$ is contained in $V(q_1)$, where $V(q_1)$ denotes the topological closure of $q_1$ in $\overline{M}_{g,n}$. Then there exist a complete discrete valuation ring $R$ and a morphism $\mathrm{Spec}\, R \to \overline{\mathcal{M}}_{g,n} \to \overline{M}_{g,n}$ such that the image of the morphism is $\{q_1, q_2\}$. Let $\overline{\eta}$ and $\overline{s}$ be a geometric generic point and a geometric closed point over the generic point and the closed point of $\mathrm{Spec}\, R$, respectively. Write $\mathcal{X}^\bullet$ for the pointed stable curve over $R$ determined by the morphism $\mathrm{Spec}\, R \to \overline{\mathcal{M}}_{g,n}$, $\mathcal{X}^\bullet_\eta$ for the generic fiber, $\mathcal{X}^\bullet_s$ for the special fiber, $X^\bullet_{q_1}$ for $X^\bullet_{\overline{\eta}} \overset{\mathrm{def}}{=} \mathcal{X}^\bullet_\eta \times_\eta \overline{\eta}$, and $X^\bullet_{q_2}$ for $X^\bullet_{\overline{s}} \overset{\mathrm{def}}{=} \mathcal{X}^\bullet_s \times_s \overline{s}$.

By the general theories of log geometry and admissible fundamental groups, we obtain a specialization surjective homomorphism of admissible fundamental groups (=an open continuous homomorphism of admissible fundamental groups arising from scheme theory)

$$sp^{\mathrm{adm}}_R : \Pi_{X^\bullet_{q_1}} \twoheadrightarrow \Pi_{X^\bullet_{q_2}}.$$

Since $\mathcal{X}^\bullet_s$ is a reduction of $\mathcal{X}^\bullet_\eta$, the deformation theory of admissible coverings of $\mathcal{X}^\bullet$ implies that

$$sp^{\mathrm{adm}}_R(\mathrm{Geo}(\Pi_{X^\bullet_{q_1}})) \subseteq \mathrm{Geo}(\Pi_{X^\bullet_{q_2}}),$$

where $\mathrm{Geo}(\Pi_{X^\bullet_{q_i}})$, $i \in \{1,2\}$, denotes the geometric data associated to $\Pi_{X^\bullet_{q_i}}$. For instance, let $\Pi_1 \in \mathrm{Geo}(\Pi_{X^\bullet_{q_1}})$ be a closed subgroup of $\Pi_{X^\bullet_{q_1}}$ associated to the pointed stable sub-curve $\widetilde{X}^\bullet_{v_1}$ determined by an irreducible component $X_{v_1}$ of $X_{\overline{\eta}} = X_{q_1}$ (see [Y8, Section 2.2.4] for $\widetilde{X}^\bullet_{v_1}$). Then $sp^{\mathrm{adm}}_R(\Pi_1)$ is a closed subgroup of $\Pi_{X^\bullet_{q_2}}$ associated to the pointed stable sub-curve determined by the *degeneration* (or *reduction*) of $X_{v_1}$ in $X_{\overline{s}} = X_{q_2}$. This means that we have the following *geometric phenomena*:

- The combinatorial data $\mathrm{Com}(\Gamma_{X^\bullet_{q_2}})$ and the topological data $\mathrm{Typ}(X^\bullet_{q_2})$ can be controlled by the combinatorial data $\mathrm{Com}(\Gamma_{X^\bullet_{q_1}})$ and the topological data $\mathrm{Typ}(X^\bullet_{q_1})$ via the "*deformation*" $\mathcal{X}^\bullet$ of $X^\bullet_{q_2}$ over $R$ *arising from scheme theory.*

- The geometric data $\mathrm{Geo}(\Pi_{X^\bullet_{q_2}})$ of $X^\bullet_{q_2}$ can be controlled by the geometric data $\mathrm{Geo}(\Pi_{X^\bullet_{q_1}})$ of $X^\bullet_{q_1}$ via an open continuous homomorphism $sp^{\mathrm{adm}}_R$ of admissible fundamental groups *arising from scheme theory.*

On the other hand, the group-theoretical specialization conjecture mean that there should exist the following *anabelian phenomena*:

- The combinatorial data $\mathrm{Com}(\Gamma_{X^\bullet_{q_2}})$ and the topological data $\mathrm{Typ}(X^\bullet_{q_2})$ can be controlled by the combinatorial data $\mathrm{Com}(\Gamma_{X^\bullet_{q_1}})$ and the topological data $\mathrm{Typ}(X^\bullet_{q_1})$ via the "*deformation*" $\mathrm{Hom}^{\mathrm{op}}_{\mathrm{pg}}(\Pi_{X^\bullet_{q_1}}, \Pi_{X^\bullet_{q_2}})$ explained in 3.2 which is *arose from group theory.*

- The geometry data $\mathrm{Geo}(\Pi_{X^\bullet_{q_2}})$ of $X^\bullet_{q_2}$ can be controlled by the geometry data $\mathrm{Geo}(\Pi_{X^\bullet_{q_1}})$ of $X^\bullet_{q_1}$ via an arbitrary open continuous homomorphism $\phi$ of admissible fundamental groups which is *arose from group theory.*

## 5.3

The group-theoretical specialization conjecture is highly non-trivial even in the simplest case where $X^\bullet_i$, $i \in \{1, 2\}$, is *smooth* over $k_i$, $\mathrm{Hom}^{\mathrm{op}}_{\mathrm{pg}}(\Pi_{X^\bullet_1}, \Pi_{X^\bullet_2}) = \mathrm{Isom}_{\mathrm{pg}}(\Pi_{X^\bullet_1}, \Pi_{X^\bullet_2})$ (this condition is equivalent to $\mathrm{Isom}_{\mathrm{pg}}(\Pi_{X^\bullet_1}, \Pi_{X^\bullet_2}) \neq \emptyset$), and $\phi \in \mathrm{Isom}_{\mathrm{pg}}(\Pi_{X^\bullet_1}, \Pi_{X^\bullet_2})$ is an *isomorphism*, where $\mathrm{Isom}_{\mathrm{pg}}(\Pi_{X^\bullet_1}, \Pi_{X^\bullet_2})$ denotes the set of isomorphisms of admissible fundamental groups. In this special case, the group-theoretical specialization conjecture was proved by Tamagawa which are the main results of [T4] (see [T4, Theorem 0.1 and Theorem 5.2]).

If we assume that

$$\mathrm{Hom}^{\mathrm{op}}_{\mathrm{pg}}(\Pi_{X^\bullet_1}, \Pi_{X^\bullet_2}) = \mathrm{Isom}_{\mathrm{pg}}(\Pi_{X^\bullet_1}, \Pi_{X^\bullet_2}),$$

and that $\phi \in \mathrm{Isom}_{\mathrm{gp}}(\Pi_{X^\bullet_1}, \Pi_{X^\bullet_2})$ is an isomorphism, then the group-theoretical specialization conjecture is equivalent to the so-called "*combinatorial Grothendieck conjecture*" which is the main conjecture in the theory of combinatorial anabelian geometry developed by Y. Hoshi and Mochizuki (e.g. [HM1], [HM2], [M4]) in characteristic 0, and by the author in characteristic $p$ ([Y1], [Y2]). Thus, the group-theoretical specialization conjecture can be regarded as *the ultimate generalization of the combinatorial Grothendieck conjecture in characteristic p.*

We have the following result was obtained by the author ([Y9]):

**Theorem 5.1.** *The group-theoretical specialization conjecture holds.*

## 5.4 Contribution to moduli spaces of fundamental groups

Let $\Gamma_{X^\bullet}$ be the dual semi-graph of a pointed stable curve $X^\bullet$ of type $(g, n)$ and $\omega_{X^\bullet} : v(\Gamma_{X^\bullet}) \to \{g_v\}_{v \in v(\Gamma_{X^\bullet})}$ defined as $v \mapsto g_v$, where $g_v$ denotes the genus of smooth pointed stable curve associated to irreducible component corresponding to $v$. We put $C_g(\Gamma, \omega) \overset{\mathrm{def}}{=} \{q \in \overline{M}_{g,n} \mid (\Gamma_{X^\bullet_q}, \omega_{X^\bullet_q}) = (\Gamma, \omega)\}$. Then there is the so-called "*combinatorial stratification of* $\overline{M}_{g,n}$" defined as follows, which plays an important role in the comparation between moduli spaces of algebraic and tropical curves:

$$\overline{M}_{g,n} = \bigsqcup_{(\Gamma,\omega)} C_g(\Gamma, \omega),$$

and

$$C_g(\Gamma_2, \omega_2) \subseteq \overline{C_g(\Gamma_1, \omega_1)}$$

if and only if $(\Gamma_2, \omega_2)$ is a "degeneration" of $(\Gamma_1, \omega_1)$, where $\overline{(-)}$ denotes the topological closure of $(-)$. We put $\mathfrak{C}_g(\Gamma, \omega) \stackrel{\text{def}}{=} C_g(\Gamma, \omega)/\sim_{fe}$. Then we have

$$\overline{\mathfrak{M}}_{g,n} = \bigsqcup_{(\Gamma,\omega)} \mathfrak{C}_g(\Gamma, \omega),$$

and $\mathfrak{C}_g(\Gamma_2, \omega_2) \subseteq \overline{\mathfrak{C}_g(\Gamma_1, \omega_1)}$ if and only if $(\Gamma_2, \omega_2)$ is a "degeneration" of $(\Gamma_1, \omega_1)$.

On the other hand, by the combinatorial Grothendieck conjecture in positive characteristic obtained by the author ([Y1], [Y2]), we can also define the combinatorial stratification for $\overline{\Pi}_{g,n}$ in a group-theoretical way, namely, we have (*as sets*)

$$\overline{\Pi}_{g,n} = \bigsqcup_{(\Gamma,\omega)} \mathfrak{C}_g^{\text{gp}}(\Gamma, \omega),$$

where "gp" means "group-theoretical".

Then the group-theoretical specialization conjecture (i.e. Theorem 5.1) implies the following result:

**Corollary 5.2.** *We have that $\mathfrak{C}_g^{\text{gp}}(\Gamma_2, \omega_2) \subseteq \overline{\mathfrak{C}_g^{\text{gp}}(\Gamma_1, \omega_1)}$ if and only if $(\Gamma_2, \omega_2)$ is a "degeneration" of $(\Gamma_1, \omega_1)$. Moreover, the continuous map $\pi_{g,n}^{\text{adm}} : \overline{\mathfrak{M}}_{g,n} \to \overline{\Pi}_{g,n}$ induces the following:*

$$\pi_{g,n}^{\text{adm}}(\mathfrak{C}_g(\Gamma, \omega)) = \mathfrak{C}_g^{\text{gp}}(\Gamma, \omega),$$
$$\pi_{g,n}^{\text{adm}}(\overline{\mathfrak{C}_g(\Gamma, \omega)}) = \overline{\mathfrak{C}_g^{\text{gp}}(\Gamma, \omega)}.$$

*Namely, $\pi_{g,n}^{\text{adm}}$ preserves "combinatorial stratifications".*

# 6 Grothendieck's anabelian conjecture for curves over sub-$p$-adic fields

Let $R$ be a strict henselization of the ring of integers of a $p$-adic local field, $K$ the quotient field of $R$, $k$ the residue field of $R$, $\overline{K}$ an algebraic closure of $K$, and $G_K$ the absolute Galois group of $K$. Let $\mathcal{X}_i$, $i \in \{1, 2\}$, be a stable curve over $R$ such that the generic fiber $\mathcal{X}_{i,\eta} \stackrel{\text{def}}{=} \mathcal{X}_i \times_R K$ is smooth over $K$. We denote by

$$\pi_1(\mathcal{X}_{i,\eta}),\ \pi_1(\mathcal{X}_{i,\overline{\eta}}),\ \pi_1^{\text{adm}}(\mathcal{X}_{i,s})$$

the étale fundamental group of the generic fiber $\mathcal{X}_{i,\eta}$, the étale fundamental group of the geometric generic fiber $\mathcal{X}_{i,\overline{\eta}} \stackrel{\text{def}}{=} \mathcal{X}_{i,\eta} \times_K \overline{K}$, and the admissible fundamental group of the special fiber $\mathcal{X}_{i,s} \stackrel{\text{def}}{=} \mathcal{X}_i \times_R k$.

Note that we have the fundamental exact sequence of fundamental groups

$$1 \to \pi_1(\mathcal{X}_{i,\overline{\eta}}) \to \pi_1(\mathcal{X}_{i,\eta}) \stackrel{pr_i}{\to} G_K \to 1.$$

Let $\phi_\eta \in \text{Isom}(\pi_1(\mathcal{X}_{1,\eta}), \pi_1(\mathcal{X}_{2,\eta}))$ be an arbitrary isomorphism such that $pr_2 \circ \phi_\eta = pr_1$. Then $\phi_\eta$ induces a $G_K$-equivariant isomorphism $\phi_{\overline{\eta}} : \pi_1(\mathcal{X}_{1,\overline{\eta}}) \xrightarrow{\sim} \pi_1(\mathcal{X}_{2,\overline{\eta}})$. In [HY], we proved the following result:

**Theorem 6.1.** *We maintain the notation introduced above. Moreover, we suppose that the following conditions are satisfied:*

- *The isomorphism $\phi_s$ is induced by an $k$-isomorphism $\mathcal{X}_{1,s} \overset{\sim}{\to} \mathcal{X}_{2,s}$*
- *The diagram*

$$\begin{array}{ccc} \pi_1(\mathcal{X}_{1,\overline{\eta}}) & \xrightarrow{sp_1} & \pi_1^{\mathrm{adm}}(\mathcal{X}_{1,s}) \\ \phi_{\overline{\eta}} \downarrow & & \phi_s \downarrow \\ \pi_1(\mathcal{X}_{2,\overline{\eta}}) & \xrightarrow{sp_2} & \pi_1^{\mathrm{adm}}(\mathcal{X}_{2,s}) \end{array}$$

  *is commutative, where $sp_i$ denotes the specialization homomorphism.*

*Then $\mathcal{X}_{1,\eta}$ is isomorphic to $\mathcal{X}_{2,\eta}$ as $K$-schemes.*

By using certain techniques of anabelian geometry and geometry of curves, as a corollary, we obtain a completely new proof of the following Mochizuki's famous result ([M2]):

**Corollary 6.2.** *The Isom-version of Grothendieck's anabelian conjecture for smooth pointed stable curves over sub-p-adic fields holds.*

The method given in the proof of Theorem 6.1 is completely different from Mochizuki's approach (i.e., without using Faltings' $p$-adic Hodge theory), and depends mainly on the techniques of algebraic geometry in positive characteristic and fundamental groups of curves in positive characteristic. Roughly speaking, the main ingredients are the following:

- Faltings-Chai-Lan's constructions about toroidal compactification of PEL-type Shimura varieties, (log) local Torelli for generalized semi-Prym schemes, Mumford-Faltings-Chai's theory of degeneration of abelian varieties, Serre-Tate theory, Tate's theorem about extensions of $p$-divisible groups.
- Raynaud-Tamagawa theta divisor, combinatorial anabelian geometry in positive characteristic, Tamagawa's theorem about Isom-verison of Grothendieck's anabelian conjecture for affine curves over finite fields.

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