FROM EXCEPTIONAL SETS TO NON-FREE SECTIONS

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Abstract. This is a report of the author's talk at RIMS workshop Algebraic Number Theory and Related Topics 2023 which was held at RIMS Kyoto University during December 11th–15th 2023. We discuss recent results on exceptional sets in Manin's conjecture as well as their applications to moduli spaces of sections of Fano fibrations. This paper is an English translation of the author's extended abstract [Tan23].

1. Introduction

One of goals of algebraic geometry is to understand algebraic varieties, i.e., geometric objects defined by finitely many polynomial equations in several variables. In this broad subject, there is an area called as diophantine geometry which studies rational points on algebraic varieties, i.e., points whose coordinates are rational numbers. For example, one of the biggest achievement in diophantine geometry is the Fermat's last theorem which is proved by Andrew Wiles. This states that when $n \geq 3$, the equation $x^n + y^n = 1$ has only trivial rational solutions.

There are various types of problems in diophantine geometry, but the author often considers the distribution of rational points on algebraic varieties in the situation that there are infinitely many rational points. For example, we consider density of the set of rational points in various topologies or we count the number of rational points of bounded height and consider the asymptotic formula for such a counting function when the height of rational points grows.

Manin's conjecture is a conjecture on this asymptotic formula. We consider a class of Fano varieties which generalizes low degree hypersurfaces in the projective spaces, and we study the asymptotic formula for the counting function of rational points on a Fano variety. When you count rational points, it is possible that rational points are accumulating along subvarieties. Thus it is important to consider exceptional sets and remove the contribution from an exceptional set when you count rational points on a Fano variety so that the asymptotic formula reflects the global geometry of the underlying variety. With Brian Lehmann and others, the author has been studying birational geometry of exceptional sets in Manin's conjecture using tools from higher dimensional algebraic geometry such as the minimal model program which has been initiated by Shigefumi Mori in 1980's.

Moreover Manin's conjecture is also considered over the global function fields such as $\mathbb{F}_q(t)$. Through the function field version of Manin's conjecture, we have applied

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the study of exceptional sets in Manin's conjecture to moduli spaces of curves on Fano varieties. In this note, we discuss these recent developments on birational geometry of Manin's conjecture as well as their applications to the study of moduli spaces of curves on Fano varieties.

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2. Manin's conjecture

In this section, we consider Manin's conjecture over number fields.

2.1. **Height functions.** In Manin's conjecture, we consider the "size" of rational points and count the number of rational points of bounded size. A concept of size of rational points is provided by the notion of height functions which measures geometric and arithmetic complexities of rational points. Here is an example of height functions for projective spaces:

Example 2.1. Let $\mathbb{P}^n_{\mathbb{Q}}$ be the projective space defined over $\mathbb{Q}$. In this case, the naive height function $\mathsf{H} : \mathbb{P}^n(\mathbb{Q}) \to \mathbb{R}_{\geq 0}$ is a real-valued function on the set of rational points given by

$$\mathsf{H}(x_0 : \cdots : x_n) = \max\{|x_0|, \cdots, |x_n|\},$$

where $(x_0 : \cdots : x_n) \in \mathbb{P}^n(\mathbb{Q})$ with $x_i \in \mathbb{Z}$ and $\gcd(x_i) = 1$. Note that for any $T > 0$, the set

$$\{p \in \mathbb{P}^n(\mathbb{Q}) \mid \mathsf{H}(p) \leq T\},$$

is a finite set.

In general for a given number field F, a projective variety X defined over F, and $\mathcal{L} = (L, \{\| \cdot \|_v\})$ be an adelically metrized Cartier divisor on X, we associate a height function

$$\mathsf{H}_{\mathcal{L}} : X(F) \to \mathbb{R}_{\geq 0},$$

to a triple $(F, X, \mathcal{L})$. Moreover when L is ample, the set

$$\{x \in X(F) \mid \mathsf{H}_{\mathcal{L}}(x) \leq T\}$$

is a finite set. This is called as Northcott property.

Thus for any subset $Q \subset X(F)$, we define the counting function as

$$N(Q, \mathcal{L}, T) := \#\{x \in Q \mid \mathsf{H}_{\mathcal{L}}(x) \leq T\}.$$

Manin's conjecture predicts the asymptotic formula for $N(Q, \mathcal{L}, T)$ as $T \to \infty$ for an appropriate choice of Q. For more details of height functions we recommend to consult [CLT10] and [HS00].

2.2. **Geometric invariants in Manin's conjecture.** The asymptotic formula in Manin's conjecture is expressed in terms of certain birational invariants. In this section we introduce them. We assume that our ground field F is a field of characteristic 0.

Definition 2.2. Let X be a smooth projective variety defined over F and L be a big and nef $\mathbb{Q}$-Cartier divisor on X. (If some readers are not familiar with these terminology, then they can simply assume that L is ample.) Then the Fujita invariant or the a-invariant is defined as

$$a(X, L) := \min\{t \in \mathbb{R} \mid tL + K_X \in \overline{\mathrm{Eff}}^1(X)\},$$

where K_X is the canonical divisor on X and $\overline{\mathrm{Eff}}^1(X)$ is the cone of pseudo-effective divisors on X. By [BDPP13], having $a(X, L) > 0$ is equivalent to say that X is geometrically uniruled. When L is not big, we formally set $a(X, L) = \infty$.

When X is singular, we take a smooth resolution $\beta : \widetilde{X} \to X$ and define the Fujita invariant by

$$a(X, L) = a(\widetilde{X}, \beta^* L).$$

This is well-defined due to birational invariance of $a(X, L)$. ([HTT15, Proposition 2.7])

Fujita invariants have been studied by Takao Fujita under the name of Kodaira energy. ([Fuj92, Fuj96, Fuj97]) This has been quite useful in the context of classification theory of polarized projective varieties. Next we define one more invariant from birational geometry:

Definition 2.3. Let X be a geometrically uniruled smooth projective variety defined over F and L be a big and nef $\mathbb{Q}$-Cartier divisor on X. We define the b-invariant as

$$b(F, X, L) = \text{codimension of the minimal supported face of } \overline{\mathrm{Eff}}^1(X) \text{ containing } a(X, L)L + K_X.$$

When X is singular, we take a smooth resolution $\beta : \widetilde{X} \to X$ and define the b-invariant by

$$b(F, X, L) := b(F, \widetilde{X}, \beta^* L).$$

Again this does not depend on a choice of resolutions by [HTT15, Proposition 2.10].

These Fujita invariants and b-invariants play roles in the study of cylinders. (See [CPPZ20] for interested readers.) For Fano varieties, it is easy to compute these invariants:

Example 2.4. Let X be a smooth projective variety defined over F. We say X is Fano if $-K_X$ is ample. For a smooth hypersurface of degree d in $\mathbb{P}^n$, it is Fano if and only if $d \leq n$.

Let X be a Fano variety defined over F and $L = -K_X$. Then we have

$$a(X, L) = 1, \quad b(F, X, L) = \rho(X),$$

where $\rho(X)$ is the Picard number of X.

2.3. **Thin sets.** Next we define the notion of thin sets which plays a central role in Manin's conjecture:

Definition 2.5. Let F be a field of characteristic 0 and let X be a variety defined over F. Suppose that $V \subset X$ is a proper closed subset. Then $V(F) \subset X(F)$ is called as a type I thin set. Next let $f : Y \to X$ be a dominant and generically finite morphism of degree ≥ 2 from a variety Y. Then $f(Y(F)) \subset X(F)$ is called as a type II thin set. A thin set is any subset of a finite union of type I and II thin sets on X.

Note that this notion is not meaningful when F is algebraically closed because any subset of $X(F)$ is a thin set. However, the following theorem shows that this notion makes sense over number fields:

Theorem 2.6 (Hilbert)**.** *Let F be a number field. Then $\mathbb{P}^n(F)$ is not thin.*

By the above theorem, for any F-rational variety X, the set $X(F)$ is not thin. In general, for any geometrically rationally connected variety X, it is expected that $X(F)$ is not thin as soon as there is a F-rational point. Indeed this follows from Colliot-Thélène's conjecture on Brauer-Manin obstructions for weak approximation.

2.4. **Manin's conjecture.** In this section, we introduce Manin's conjecture. The following conjecture has been formulated through a series of work [FMT89, BM90, Pey95, BT98, Pey03]:

Conjecture 2.7 (Batyrev–Manin–Peyre–Tschinkel)**.** Let F be a number field and X be a geometrically rationally connected smooth projective variety defined over F. Let $\mathcal{L}$ be an adelically metrized big and nef $\mathbb{Q}$-divisor on X. Suppose that $X(F)$ is not thin. Then there exists a thin set $Z \subset X(F)$ such that we have

$$N(X(F) \setminus Z, \mathcal{L}, T) \sim c(F, \mathcal{L}, Z)T^{a(X,L)}(\log T)^{b(F,X,L)-1}$$

as $T \to \infty$ where $c(F, \mathcal{L}, Z)$ is Peyre's constant introduced by Peyre and Batyrev–Tschinkel in [Pey95, BT98].

The set Z in this conjecture is called as an exceptional set. A reason why we remove the contribution from an exceptional set is that it is possible that rational points are accumulating along subvarieties. For example, let S be a smooth cubic surface defined over F with $L = -K_S$. Then we have $a(S, L) = 1$ and $b(F, S, L) = \rho(S)$. So if Manin's conjecture is true, then the magnitude of the asymptotic formula will be $T(\log T)^{\rho(S)-1}$. On the other hand, it is well-known that over any algebraically closed field, S contains 27 lines. If a line $\ell \subset S$ is defined over F, then we have $a(\ell, L|_\ell) = 2$ and $b(F, \ell, L|_\ell) = 1$. So the asymptotic formula for ℓ is given by T^2. Thus if we do not remove the contribution from ℓ, then the asymptotic formula for $N(S(F), \mathcal{L}, T)$ would be T^2 which does not reflect the global geometry of S.

Originally it was expected that the exceptional set Z can be chosen to be contained in a proper closed subset ([BM90]), however, there is a counter example for the closed set version of Manin's conjecture ([BT96]). In the beginning of 21st century, Peyre first suggested that the exceptional set Z should be a thin set ([Pey03]). Recently there are some examples of (weak) Fano varieties which Manin's conjecture has been verified after removing Zariski dense exceptional set. ([LR19, BHB20])

2.5. **The main theorem.** In a joint work [LST22] with Brian Lehmann and Akash Kumar Sengupta, the author geometrically defined a conjectural exceptional set and proved that it is indeed a thin set. This result is a consequence of a series work on exceptional sets in Manin's conjecture ([HTT15, LTT18, HJ17, LT17, LT19, Sen21]). Here is the main theorem of [LST22]:

Theorem 2.8 ([LST22, Theorem 1.4], Lehmann-Sengupta-Tanimoto, 2022)**.** *Let F be a field of characteristic 0. Let X be a smooth projective geometrically uniruled variety defined over F and L be a big and nef $\mathbb{Q}$-divisor on X. As $f : Y \to X$ runs over all generically finite morphism to the image from a smooth projective variety Y such that*

$$(a(X, L), b(F, X, L)) < (a(Y, f^*L), b(F, Y, f^*L)),$$

in the lexicographic order, the union

$$\bigcup_{f:Y\to X} f(Y(F)) \subset X(F),$$

is a thin set.

If we have $f : Y \to X$ satisfying the inequality in the statement and the asymptotic formula for Manin's conjecture, then we need to remove the contribution of $f(Y(F))$ from the counting function. Thus to define the exceptional set, it is natural to take the union of all such contributions and we ask if this is indeed a thin set. Our main theorem answers this question affirmatively. Also in [LST22], we consider the case of $(a(X, L), b(F, X, L)) = (a(Y, f^*L), b(F, Y, f^*L))$, and the key point here is the notion of face contracting morphisms.

The main ingredients of the proof of Theorem 2.8 are the following:

- the minimal model program ([BCHM10]);
- boundedness of singular Fano varieties (BAB conjecture, [Bir19, Bir21]);
- Hilbert's irreducibility theorem, and;
- universal families of accumulating maps up to twists ([LST22]).

In particular the proofs utilize the boundedness of Fano varieties with canonical singularities. This has been proved by Birkar recently, and he received a Fields medal for this contribution. In some sense, what we did is to translate a finite statement in algebraic geometry to a finite statement in arithmetic geometry.

3. Geometric Manin's conjecture

There is a version of Manin's conjecture over global function fields such as $\mathbb{F}_q(t)$, and this leads to Geometric Manin's conjecture which is a version of Manin's conjecture over function fields of complex curves. This has been introduced by Brian Lehmann and the author, and in this section, we discuss this conjecture.

3.1. **Moduli spaces of curves.** We work over the field $\mathbb{C}$ of complex numbers. Let B be a smooth projective curve of genus $g(B)$. Let $K(B)$ be the function field of B. Geometric Manin's conjecture concerns $K(B)$-rational points on a smooth Fano variety X_η defined over $K(B)$. By valuative criterion, this is equivalent to consider sections of any fixed integral model $\mathcal{X} \to B$ of X_η. To this end we introduce the notion of integral models for Fano varieties over $K(B)$:

Definition 3.1. A morphism $\pi : \mathcal{X} \to B$ from a projective variety is called as a Fano fibration if the following properties hold:

- $\mathcal{X}$ is a smooth projective variety;
- π is flat with connected fibers, and;
- the generic fiber X_η is a smooth Fano variety defined over $K(B)$.

Let $\pi : \mathcal{X} \to B$ be a Fano fibration over B. Then we define the space of sections $\mathrm{Sec}(\mathcal{X}/B)$ as the Zariski open set of the Hilbert scheme parametrizing sections of π. The scheme $\mathrm{Sec}(\mathcal{X}/B)$ consists of countably many irreducible components. Let α be an algebraic class of sections, and we define $\mathrm{Sec}(\mathcal{X}/B, \alpha)$ to be the space of sections of class α. This is a quasi-projective scheme of finite type over $\mathbb{C}$.

Let $M \subset \mathrm{Sec}(\mathcal{X}/B)$ be an irreducible component of the space of sections. Then we have

$$\dim M \geq -K_{\mathcal{X}/B}.C + (\dim \mathcal{X} - 1)(1 - g(B)),$$

where $-K_{\mathcal{X}/B}$ be the relative anticanonical divisor and $C \in M$. The right hand side is called as the expected dimension of M. We also have an upper bound:

$$\dim M \leq -K_{\mathcal{X}/B}.C + (\dim \mathcal{X} - 1)(1 - g(B)) + h^1(C, T_{\mathcal{X}/B}|_C),$$

where $T_{\mathcal{X}/B}$ is the relative tangent bundle of $\mathcal{X}/B$ and $h^1(C, T_{\mathcal{X}/B}|_C)$ is the dimension of the first cohomology of the restricted relative tangent bundle. In particular when $h^1(C, T_{\mathcal{X}/B}|_C) = 0$, the expected dimension and the actual dimension coincide.

3.2. **Geometric Manin's conjecture.** In the lecture held at Berlin in 1988 ([Bat88]), Batyrev introduced a heuristic argument for Manin's conjecture over global function fields. His heuristic was relied on some assumptions on properties of the space of sections, and Geometric Manin's conjecture is a refinement of these assumptions. This has been first introduced by Brian Lehmann and the author in [LT19]:

Conjecture 3.2 (Geometric Manin's conjecture, (Batyrev, Lehmann–Tanimoto))**.** Let B be a smooth projective curve defined over $\mathbb{C}$ and let $\pi : \mathcal{X} \to B$ be a Fano fibration. Then the following statements hold:

(1) pathological components of $\mathrm{Sec}(\mathcal{X}/B)$ are controlled by Fujita invariants;

(2) for a sufficiently positive algebraic class α of sections, there exists a unique non-pathological component in $\mathrm{Sec}(\mathcal{X}/B, \alpha)$ which should be counted in Manin's conjecture. We call this unique component as Manin component;
(3) Manin components exhibit homological/motivic stability.

An idea to use homological stability in Batyrev's heuristic is due to Ellenberg and Venkatesh (see, e.g., [EVW16]). Geometric Manin's conjecture (1) has been established over complex numbers in the author's recent preprint which is a joint work with Brian Lehmann and Eric Riedl ([LRT23]), and we will explain results in this paper in the next section. Roughly speaking, Geometric Manin's conjecture (2) claims irreducibility of the moduli space $\mathrm{Sec}(\mathcal{X}/B, \alpha)$. This has been extensively studied by various mathematicians when $B = \mathbb{P}^1$:

- smooth Fano hypersurfaces ([HRS04] and [RY19]);
- rational homogeneous spaces ([Tho98] and [KP01]);
- toric varieties ([Bou16]);
- del Pezzo surfaces ([Tes09] and [BLRT23]);
- moduli spaces of vector bundles over curves ([Cas04] and [MTiB20]);
- smooth Fano threefolds ([CS09], [Cas04], [LT19], [LT21], [BLRT22], [ST22], and [BJ22]);
- del Pezzo fibrations ([LT24] and [LT22]), and;
- del Pezzo manifolds ([CS09] and [Oka22]).

Many results in this list utilized Bend and Break techniques found by Shigefumi Mori, and the proofs are based on induction on degree of curves. In particular, Lehmann and the author proposed Movable Bend and Break conjecture to prove Geometric Manin's conjecture (2), and this has been established in dimension ≤ 3 ([Tes09], [LT24], [LT22], and [BLRT22]). Regarding Geometric Manin's conjecture (3), there have been studies on Cohen–Jones–Segal conjecture ([CJS00]), and there are sporadic results on this topic, e.g., low degree affine hypersurfaces ([BS20]). Motivic Manin's conjecture has been extensively studied for equivariant compactifications of vector groups in [CLL16], [Bil23], and [Fai23].

3.3. **Main theorems.** In this section, we discuss the main results of [LRT23] which is joint work with Brian Lehmann and Eric Riedl. These results confirm Geometric Manin's conjecture (1) over complex numbers in full generality. To this end, we explain what we mean by "pathological components" in Geometric Manin's conejcture:

Definition 3.3. Let $\pi : \mathcal{X} \to B$ be a Fano fibration. A section $s : C \to \mathcal{X}$ of π is relatively free if $T_{\mathcal{X}/B}|_C$ is globally generated and $H^1(C, s^*T_{\mathcal{X}/B}) = 0$.

Let $M \subset \mathrm{Sec}(\mathcal{X}/B)$ be an irreducible component. If M parametrizes a relatively free section, then one can conclude that a general section parametrized by M is relatively free. Moreover, since $H^1(C, s^*T_{\mathcal{X}/B}) = 0$, the dimension of M is given by the expected dimension and a point on $\mathrm{Sec}(\mathcal{X}/B)$ corresponding to a relatively free section is a smooth point of the moduli space $\mathrm{Sec}(\mathcal{X}/B)$. In particular, such a component is generically reduced. In this way, any component generically parametrizing relatively free sections are easier to understand and the deformation theory works well for such a component in an expected way. On the other hand, components only parametrizing non-free sections are difficult to understand. A priori, we do not know what the dimension of such a component is and it could be generically non-reduced. So we mean these non-free components by pathological components. Geometric Manin's conjecture (1) provides a tool to access information regarding such components. Here is the main result from [LRT23]:

Theorem 3.4 ([LRT23, Theorem 1.3], Lehmann–Riedl–Tanimoto, 2023)**.** *Let $\pi : \mathcal{X} \to B$ be a Fano fibration. There is a constant $\xi = \xi(\pi)$ with the following properties. Let M be an irreducible component of $\mathrm{Sec}(\mathcal{X}/B)$ parametrizing a family of non-relatively free sections C which satisfy $-K_{\mathcal{X}/B} \cdot C \geq \xi$. Let $\mathcal{U}^\nu$ denote the normalization of the universal family over M and let $ev : \mathcal{U}^\nu \to \mathcal{X}$ denote the evaluation map. Then either:*

(1) *ev is not dominant. Then the subvariety $\mathcal{Y}$ swept out by the sections parametrized by M satisfies*

$$a(\mathcal{Y}_\eta, -K_{\mathcal{X}/B}|_{\mathcal{Y}_\eta}) \geq a(\mathcal{X}_\eta, -K_{\mathcal{X}/B}|_{\mathcal{X}_\eta}).$$

(2) *ev is dominant. Letting $f : \mathcal{Y} \to \mathcal{X}$ denote the finite part of the Stein factorization of ev, we have*

$$a(\mathcal{Y}_\eta, -f^*K_{\mathcal{X}/B}|_{\mathcal{Y}_\eta}) = a(\mathcal{X}_\eta, -K_{\mathcal{X}/B}|_{\mathcal{X}_\eta}).$$

Furthermore, there is a dominant rational map $\phi : \mathcal{Y} \dashrightarrow \mathcal{Z}$ over B with connected fibers such that the dimension of $\mathcal{Z}$ is at least 2 and the following properties hold. Let C' denote a general section of $\mathcal{Y} \to B$ parametrized by M and let $\mathcal{W}' \subset \mathcal{Y}$ denote the unique irreducible component of the closure of $\phi^{-1}(\phi(C'))$ which maps dominantly to $\phi(C')$. There is a resolution $\psi : \mathcal{W} \to \mathcal{W}'$ such that the locus where ψ^{-1} is well-defined intersects C' and ψ has the following properties.

(a) *We have $a(\mathcal{W}_\eta, -\psi^* f^* K_{\mathcal{X}/B}|_{\mathcal{W}_\eta}) = a(\mathcal{X}_\eta, -K_{\mathcal{X}/B}|_{\mathcal{X}_\eta})$.*

(b) *The Iitaka dimension of*

$$K_{\mathcal{W}_\eta} - a(\mathcal{W}_\eta, -\psi^* f^* K_{\mathcal{X}/B}|_{\mathcal{W}_\eta})\psi^* f^* K_{\mathcal{X}/B}|_{\mathcal{W}_\eta}$$

is 0.

(c) *The general deformation of the strict transform of C' in $\mathcal{W}$ is relatively free in $\mathcal{W}$.*

(d) *There is a constant $T = T(\pi)$ depending only on π, but not M, such that the sublocus of M parametrizing deformations of the strict transform of C' in $\mathcal{W}$ has codimension at most T in M.*

When a generically finite B-morhpism $f : \mathcal{Y} \to \mathcal{X}$ to the image from a projective B-variety $\mathcal{Y}$ satisifes $a(\mathcal{Y}_\eta, -f^*K_{\mathcal{X}_\eta}) \geq a(\mathcal{X}_\eta, -K_{\mathcal{X}_\eta})$, we call such an f as an accumulating map. The above theorem shows that non-relatively free general sections of sufficiently large degree are coming from accumulating maps.

Let me explain in the case (2) of the above theorem, how we obtain a B-rational map ϕ assuming $\mathcal{Y} = \mathcal{X}$. In [LRT23], we prove Grauer-Mulich theorem which shows that if $T_{\mathcal{X}/B}$ is $[C]$-semi-stable, then its restriction $T_{\mathcal{X}/B}|_C$ is almost semi-stable. When a general section C is non-relatively free, one can show that $T_{\mathcal{X}/B}|_C$ admits a low slope quotient. Using our Grauer-Mulich theorem, this shows that $T_{\mathcal{X}/B}$ is $[C]$-unstable so that one can find a foliation of big slope in $T_{\mathcal{X}/B}$. It follows from [CP19] that this foliation induces a rational map $\phi : \mathcal{X} \dashrightarrow \mathcal{Z}$.

Example 3.5 (Cubic hypersurfaces)**.** Let us explain an application of Theorem 3.4. Let $\pi : \mathcal{X} \to B$ be a Fano fibration such that the generic fiber $\mathcal{X}_\eta$ is isomorphic to a smooth cubic hypersurface of dimension ≥ 5. Using adjunction theory, one can show that there is no accumulating map for $\mathcal{X}$. Thus Theorem 3.4 implies that any general section of sufficiently large degree on $\mathcal{X}$ is relatively free.

We explain one more theorem from [LRT23]: it shows that non-relatively free components are coming from a bounded family of accumulating maps:

Theorem 3.6 ([LRT23, Theorem 1.6], Lehmann–Riedl–Tanimoto, 2023)**.** *Let $\pi : \mathcal{X} \to B$ be a Fano fibration. Then we have*

(1) *There is a proper closed subset $\mathcal{V} \subsetneq \mathcal{X}$ such that if $M \subset \mathrm{Sec}(\mathcal{X}/B)$ is an irreducible component parametrizing a non-dominant family of sections then the sections parametrized by M are contained in $\mathcal{V}$.*

(2) *There are a proper closed subset $\mathcal{V} \subsetneq \mathcal{X}$ and a bounded family of smooth projective B-varieties $\mathcal{Y}$ equipped with B-morphisms $f : \mathcal{Y} \to \mathcal{X}$ satisfying the following properties:*
   - (a) *f is generically finite onto its image but not birational;*
   - (b) $a(\mathcal{Y}_\eta, -f^*K_{\mathcal{X}_\eta}|_{\mathcal{Y}_\eta}) \geq a(\mathcal{X}_\eta, -K_{\mathcal{X}_\eta})$;
   - (c) *if equality of Fujita invariants is achieved, then the Iitaka dimension of $K_{\mathcal{Y}_\eta} - a(\mathcal{Y}_\eta, -f^*K_{\mathcal{X}_\eta}|_{\mathcal{Y}_\eta})f^*K_{\mathcal{X}_\eta}|_{\mathcal{Y}_\eta}$ is zero;*

   If $M \subset \mathrm{Sec}(\mathcal{X}/B)$ is a component that generically parametrizes non-relatively free sections of sufficiently large degree, then a general section C parametrized by M satisfies either (i) $C \subset \mathcal{V}$ or (ii) $C = f(C')$ where $f : \mathcal{Y} \to \mathcal{X}$ is in our family, and C' is a relatively free section in $\mathcal{Y}$.

The main tools of the proof of this theorem are (i) the boundedness of singular Fano varieties (BAB conjecture) proved by Birkar in [Bir19] and [Bir21] as well as the construction of universal familes of accumulating maps up to twists in [LST22]. Furthermore, it relies on the space of twists constructed in [LRT23].

3.4. **An arithmetic application.** Finally we explain an arithmetic application of Theorem 3.6. Let F be a number field and B be a smooth projective curve defined over F. Let S be a finite set of places of F including all archimedean places, and we denote the ring of S-integers of F by $\mathfrak{o}_{F,S}$.

Let $\pi : X \to B$ be a Fano fibration defined over F and $\widetilde{\pi} : \mathcal{X} \to \mathcal{B}$ be an integral model of π over $\mathfrak{o}_{F,S}$. Let $V \subset X$ be a proper closed subset from Theorem 3.6 and let $\mathcal{V} \subset \mathcal{X}$ be the flat closure of V. Let v be a place not contained in S. We denote the reduction of π at v by $\pi_v : X_v \to B_v$ and this is defined over the residue field k_v.

We assume that this is a Fano fibration over k_v. Let V_v be the reduction of $\mathcal{V}$ at v. Let $\mathrm{Sec}(X_v/B_v, V_v)_{\leq d}$ be the Zariski open subset of $\mathrm{Sec}(X_v/B_v)$ parametrizing sections C of anticanonical degree $\leq d$ such that $C \not\subset V_v$. We consider the following counting function:

$$N(X_v \setminus V_v, -K_{X_v/B_v}, d) = \#\mathrm{Sec}(X_v/B_v, V_v)_{\leq d}(k_v).$$

Weak Manin's conjecture over $K(B_v)$ predicts that for any $\epsilon > 0$,

$$N(X_v \setminus V_v, -K_{X_v/B_v}, d) = o(q_v^{d(1+\epsilon)}),$$

as $d \to \infty$ where $q_v = \#k_v$. Our main theorem is the following:

Theorem 3.7 ([LRT23, Theorem 1.10], Lehmann–Riedl–Tanimoto, 2023)**.** *Assume that $d\epsilon > \dim X_\eta$. then we have*

$$\frac{N(X_v \setminus V_v, -K_{X_v/B_v}, d)}{q_v^{d(1+\epsilon)}} \to 0,$$

as $v \to \infty$.

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