位相群における小部分集合の作用素
–SMALL SUBSET OPERATORS OF TOPOLOGICAL GROUPS–

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If X is a topological space, we shall use the symbol $cl_X(Y)$ to denote the closure of the subset $Y \subseteq X$ in X.

If $(G, \cdot)$ is a group and $X, Y \subseteq G$, then we let

$$X \cdot Y = \{x \cdot y : x \in X, y \in Y\}. \tag{1}$$

All topological groups appearing in this manuscript are assumed Hausdorff.

1. Introduction

Definition 1.1. We say that a functional T is an operator (for a class $\mathcal{C}$ of topological groups) if the evaluation $T_G : \mathcal{P}(G) \to \mathcal{P}(G)$ is a well-defined function on the power set $\mathcal{P}(G)$ for $G \in \mathcal{C}$. When $\mathcal{C}$ is unambiguous, we may simply call T_G *an operator of topological groups*.

For our purposes, the word "operator" for a functional T from Definition 1.1 does not immediately carry the meaning of neither "interior" nor "closure" operator in the categorical sense; in fact, it shall actually be incompatible with the categorical notion as we do not restrict our use of operators to sub-objects from the category **TopGrp** (in which case it would be subgroups, see Definition 1.4). Instead, we are able to apply operators directly to individual *subsets* of a given topological group G. In addition, our purposes require us to assign different properties and behaviors to given operators T by assigning different adjectives to it based on either intrinsic or combinatorial properties. Our terminology closely follows that of [5], with one major difference: we remove certain restrictions on the universal quantifiers which appeared on the original properties of [5], this is done to make them available for singular topological groups G. Our reasoning is the following: certain operators may have strong combinatorial properties when considered for specific subclasses of topological groups (e.g, abelian groups, torsion groups, etc) which may not be present *at all* in a general setting. The following list includes the properties that we shall focus on in this manuscript.

Definition 1.2. Let G be a topological group and T be an operator. The operator T is said to be:

(i) *extensive for* G whenever $T_G(X)$ satisfies that $X \subseteq T_G(X)$.
(ii) *contractive for* G whenever $T_G(X)$ satisfies that $T_G(X) \subseteq X$.
(iii) *generative for* G whenever $T_G(X)$ is a subgroup of G for every $X \subseteq G$.
(iv) *algebraically normal for* G whenever $T_G(X)$ is a normal subgroup of G for every $X \subseteq G$.
(v) *topologically open for* G whenever $T_G(X)$ is open for every $X \subseteq G$.
(vi) *topologically closed for* G whenever $T_G(X)$ is closed for every $X \subseteq G$.
(vii) *central for* G whenever the inclusion $e \in X$ implies that $e \in T_G(X)$ for every $X \subseteq G$.
(viii) *monotone for* G whenever the inclusions $X \subseteq Y \subseteq G$ imply that $T_G(X) \subseteq T_G(Y)$.

(ix) *lax-commutative for a homomorphism* $f : G \to H$ whenever $f(T_G(X)) \subseteq T_H(f(X))$ holds for each $X \subseteq G$.

(x) *forward quotient-uniform for* G whenever $T_H(q(X)) \subseteq q(T_G(X))$ holds for each continuous quotient map $q : G \to H$ and each set $e \in X \subseteq H$.

(xi) *backward quotient-uniform for* G whenever $q^{-1}(T_H(Y)) \subseteq T_G(q^{-1}(Y))$ holds for each continuous quotient map $q : G \to H$ and each set $e \in Y \subseteq H$.

In the context of topological groups, we are able to consider operators that are built based purely on topological properties, algebraic properties or a combination of both. The following examples show some of these selection patterns.

Example 1.3. (1) The trivial central operator T for which $T_G(X) = \{e_G\}$ holds for every topological group G and every set $X \subseteq G$ is universally generative, universally normal, universally closed, universally central, universally monotone and lax-commutative for all homomorphisms.

(2) The closure operator cl for which $\mathrm{cl}_G(X)$ denotes the topological closure of X in G is universally extensive, universally closed, universally central, universally monotone and lax-commutative for all continuous homomorphisms.

(3) The algebraic generation operator $\langle\cdot\rangle$ for which $\langle X\rangle_G$ denotes the smallest subgroup of G containing X is universally extensive, universally central, universally monotone, universally generative, lax-commutative for all homomorphisms, universally forward quotient uniform and universally backwards quotient-uniform.

(4) The cyclic operator Cyc for which $\mathrm{Cyc}_G(X)$ denotes the set $\{x \in X : \langle x\rangle \subseteq X\}$ is universally contractive, universally central, universally monotone and universally backwards quotient-uniform. In addition, it is lax-commutative for any homomorphism between topological groups.

(5) The heart operator N for which $\mathrm{N}_G(X)$ denotes the union of all normal subgroups of G contained in X is universally generative, universally normal, universally central, universally monotone and universally backward quotient-uniform. It is lax-commutative for any homomorphism between topological groups which has super-normal image (see [5]).

Let us now do a quick comparison between the operators we consider here, versus the traditional categorical notion of a closure operator:

Definition 1.4 (Categorical closure operators of topological groups)**.** Recall that **TopGrp** denotes the category of all Hausdorff topological groups as the **TopGrp**–objects, together with continuous homomorphisms as the **TopGrp**–maps between the **TopGrp**–objects. Let G and H denote topological groups. H is said to be a *subobject of* G whenever H is a (topological) subgroup of G; categorically this is denoted as $H \leq_{\mathbf{TopGrp}} G$ (but we shall omit the sub-index). An operator C in the category **TopGrp** is a functional which assigns to each topological group G and each subgroup H of G a unique subgroup of G denoted by $C_G(H)$. The operator C is said to be:

(i) *extensive* whenever $H \leq C_G(H)$ holds for each (topological) subgroup H of a topological group G.

(ii) *monotone* whenever $K \leq H$ implies that $C_G(K) \leq C_G(H)$ holds for a pair of (topological) subgroups K, H of G.

(iii) *continuous* whenever $f(C_G(K)) \leq C_H(f(K))$ holds for any continuous homomorphism $f : G \to H$ between topological groups, and any $K \leq G$.

If an operator C in the category **TopGrp** satisfies conditions (i)–(iii), then it is said to be a *closure operator of topological groups.*

We note that conditions (i)—(iii) of Definition 1.4 have their clear counterparts in our Definition 1.2 in the form of *extensivity* (i), *monotonicity* (viii) and *lax-commutativity* (ix). However, the properties for the operators we consider in this manuscript are *not* restricted to (topological) subgroups (i.e, subojects in the category **TopGrp**). A complete summary of the theory of closure operators for **TopGrp** can be found in [3]; and a general reference book of categorical closure operators with applications beyond topology and algebra (including discrete mathematics) can be found in [6].

2. COMPOSITION OF OPERATORS

Definition 2.1. Let T and S be operators of topological groups. We define $S \circ T$ to be the topological group operator satisfying the equality

$$(S \circ T)_G(X) = S_G(T_G(X)) \tag{2}$$

for each topological group G and each subset $X \subseteq G$. The operator $S \circ T$ shall be referred to as the *composition* of T and S.

Proposition 2.2. *Let T and S be operators of topological groups. Let G be a topological group.*

(i) If S is normal for G, then $S \circ T$ is normal for G.
(ii) If S is closed for G, then $S \circ T$ is closed for G.
(iii) Let N be the heart operator. If T is closed for G, then $\mathrm{N} \circ T$ is closed for G.
(iv) Let N be the heart operator. If T is generative for G, then $\mathrm{N} \circ T$ is normal for G.
(v) If T and S are monotone for G, then $S \circ T$ is monotone for G.
(vi) Assume T and S are lax-commutative for a homomorphism $f : G \to H$ between topological groups G and H. If S is monotone for H, then $S \circ T$ is lax-commutative for f.
(vii) Assume T and S are forward quotient-uniform for G. If S is monotone for G and T is central for G, then $S \circ T$ is forward quotient-uniform for G.
(viii) Assume T and S are backward quotient-uniform for G. If S is monotone for G and T is central for G, then $S \circ T$ is backward quotient-uniform for G.

Proof. Let G be an arbitrary topological group. (i),(ii) and (v) are verified easily.

(iii) Let X be a subset of G. By definition, $\mathrm{N}_G(T_G(X)) \subseteq T_G(X)$. Since $T_G(X)$ is closed, the inclusion $cl_G(\mathrm{N}_G(T_G(X))) \subseteq T_G(X)$ holds. The closure of a normal subgroup of G is a normal subgroup of G, implying that $\mathrm{N}_G(T_G(X)) = cl_G(\mathrm{N}_G(T_G(X)))$ by maximality of $\mathrm{N}_G(T_G(X))$ as a normal subgroup of $T_G(X)$. This proves that $\mathrm{N}_G(T_G(X))$ is closed.

(vi) Assume $X \subseteq G$ is arbitrary. Since S is lax-commutative for f, we have that

$$f(S_G \circ T_G(X)) = f(S_G(T_G(X))) \subseteq S_H(f(T_G(X))). \tag{3}$$

Since T is lax-commutative for f, we have $f(T_G(X)) \subseteq T_H(f(X))$. Since S is monotone for H, we have

$$S_H(f(T_G(X))) \subseteq S_H(T_H(f(X))) = S_H \circ T_H(f(X)). \tag{4}$$

By (3) and (4) we have that

$$f(S_G \circ T_G(X)) \subseteq S_H \circ T_H(f(X)).$$

This proves that $S \circ T$ is lax-commutative for f.

(vii) Let $q : G \to H$ be a quotient mapping from G to a topological group H. Let $X \subseteq G$ satisfy $e \in G$. Since T is forward quotient-uniform and central for G, we have

$$e_H \in T_H(q(X)) \subseteq q(T_G(X)).$$

Since S is monotone for G, the above implies that

$$S_H(T_H(q(X))) \subseteq S_H(q(T_G(X))). \tag{5}$$

On the other hand, since S is forward quotient-uniform for G and $e_H \in q(T_G(X))$ we have

$$S_H(q(T_G(X))) \subseteq q(S_G(T_G(X))). \tag{6}$$

Combining (5) and (6) we get

$$S_H \circ T_H(q(X)) \subseteq q(S_G \circ T_G(X)). \tag{7}$$

Since G, H and the quotient map $q : G \to H$ were arbitrary, this proves that $S \circ T$ is forward quotient-uniform for G.

(viii) Let $q : G \to H$ be a quotient mapping from G to a topological group H. Let $Y \subseteq H$ satisfy $e \in Y$. Since T is central, the inclusion $e \in Y$ implies that $e \in T_H(Y)$. Since S is quotient uniform for G and $e \in T_H(Y)$, we now have

$$q^{-1}(S_H \circ T_H(Y)) = q^{-1}(S_H(T_H(Y))) \subseteq S_G(q^{-1}(T_H(Y))). \tag{8}$$

Since T is backward quotient-uniform for G, the inclusion $q^{-1}(T_H(Y)) \subseteq T_G(q^{-1}(Y))$ holds. Since S is monotone for G, this implies that

$$S_G(q^{-1}(T_H(Y))) \subseteq S_G(T_G(q^{-1}(Y))) = S_G \circ T_G(q^{-1}(Y)). \tag{9}$$

By (8) and (9) we have

$$q^{-1}(S_H \circ T_H(Y)) \subseteq S_G \circ T_G(q^{-1}(Y)).$$

Since G, H and the quotient map $q : G \to H$ were arbitrary, this proves that $S \circ T$ is backward quotient-uniform for G.

□

3. Coset saturations

In this section we introduce the most important (unary) operation in this theory of operators of topological groups. This operation was originally introduced by Dikranjan and Shakhmatov in [5] for a specific operator **S** (that we shall introduce in the final Section 4), in this manuscript we now formally define this unary operation for *any* operator of topological groups.

Definition 3.1. Let T be an operator of topological groups. For each ordinal α let us define an α-th *coset saturation* $T^{(\alpha)}$ of T as follows. For each topological group G and every $X \subseteq G$ we let

$$T_G^{(0)}(X) = \{e_G\}. \tag{10}$$

If $\alpha > 0$ and $T_G^{(\beta)}$ is defined for each $\beta < \alpha$ we define

$$T_G^{(\alpha)}(X) = T_G(X \cdot \bigcup_{\beta<\alpha} T_G^{(\beta)}(X)). \tag{11}$$

The following lemma summarizes property preservation under coset saturations.

Lemma 3.2. *Let G be a topological group and T be an operator of topological groups.*

(i) If T is central for G, then each $T^{(\alpha)}$ is also central for G.

(ii) If T is monotone for G, then each $T^{(\alpha)}$ is also monotone for G.
(iii) If T is monotone and extensive for G, then $T^{(\alpha)}$ is extensive for G for each $\alpha > 0$.
(iv) If T is monotone for G and lax-commutative for a homomorphism $f : G \to H$ between topological groups, then each $T^{(\alpha)}$ is also lax-commutative for f.

Proof. Let G be an arbitrary topological group.

Properties (i)–(iii) are easily seen to hold.

(iv). By Remark 1.3 the operator $T_G^{(0)}$ is lax-commutative for all homomorphisms. Assume that for each $\beta < \alpha$ the operator $T_G^{(\beta)}$ is lax-commutative for f. Let $X \subseteq G$ be arbitrary. For each $\beta < \alpha$ we have $f(T_G^{(\beta)}(X)) \subseteq T_H^{(\beta)}(f(X))$ by lax-commutativity of $T_G^{(\beta)}$ for f. Since f is a homomorphism, the previous α many inclusions imply that

$$f(X \cdot \bigcup_{\beta<\alpha} T_G^{(\beta)}(X)) = f(X) \cdot \bigcup_{\beta<\alpha} f(T_G^{(\beta)}(X)) \subseteq f(X) \cdot \bigcup_{\beta<\alpha} T_H^{(\beta)}(f(X)). \tag{12}$$

Since T is monotone for G, (12) and (11) (in this order) imply that

$$T_H(f(X \cdot \bigcup_{\beta<\alpha} T_G^{(\beta)}(X))) \subseteq T_H(f(X) \cdot \bigcup_{\beta<\alpha} T_H^{(\beta)}(f(X))) = T_H^{(\alpha)}(f(X)). \tag{13}$$

On the other hand, since T is lax-commutative for f, we can see that

$$f(T_G^{(\alpha)}(X)) = f(T_G(X \cdot \bigcup_{\beta<\alpha} T_G^{(\beta)}(X))) \subseteq T_H(f(X \cdot \bigcup_{\beta<\alpha} T_G^{(\beta)}(X))). \tag{14}$$

Combining (13) and (14) we obtain

$$f(T_G^{(\alpha)}(X)) \subseteq T_H^{(\alpha)}(f(X)).$$

Since the set $X \subseteq G$ was arbitrary, this proves that $T^{(\alpha)}$ is lax-commutative for f. □

4. The operator fullness criterion

Definition 4.1. Let G be a topological group and T be an operator. We say that

(i) G has local T-concentration of degree α of if $\{A \subseteq G : T_G(A) = G\}$ contains an open neighbourhood basis of the identity of G.
(ii) G is locally T-trivial whenever $\{A \subseteq G : T_G(A) = \{e_G\}\}$ contains an open neighbourhood basis of the identity of G.

Definition 4.2. We say that a topological group G is T-full for an operator T of topological groups if and only if the only locally T-trivial topological group quotient of G is the trivial group.

Our main result is a criterion for fully describing the class of groups with local T-concentration in terms of their topological group quotients (by the notion of T-fullness).

Theorem 4.3 ([15])**.** *Let T be a monotone, closed and normal operator for a topological group G. Assume T is lax-commutative for all topological group quotient maps of G. If T is backwards-quotient uniform for G then the following statements are equivalent:*

(i) G is T-full; and
(ii) G has local T-concentration of degree α for some ordinal α.

The full proof of this theorem is available in the upcoming [15].

Example 4.4 (The Dikranjan–Shakhmatov operator $\mathbf{S}$)**.** The operator $\mathbf{S}$ is defined as the composition

$$\mathbf{S} = \mathrm{N} \circ \mathrm{cl} \circ \langle \cdot \rangle \circ \mathrm{Cyc}. \tag{15}$$

Example 1.3 has the properties of each individual factor in the above composition, and the results from Section 2 allow us to deduce the properties preserved under the composition performed in (15): $\mathbf{S}$ is universally monotone, universally closed, universally normal, universally backwards-quotient uniform and lax-commutative for continuous homomorphisms between topological groups with super-normal image. All of these properties were computed manually in [5, Lemma 4.5, Lemma 4.9] (with backwards quotient-uniformity being unnamed).

While we have not defined the concept of super-normality in this manuscript (which is present in [5] and can be consulted in more detail in [4]), it suffices to note that any group is super-normal in itself. As a consequence, any surjective homomorphism between topological groups has super-normal image. This allows us to obtain the following result as a consequence of Theorem 4.3:

Corollary 4.5 ([14])**.** *TFAE for a topological group G:*

(i) G is S-full; and
(ii) G has local S-concentration of degree α for some ordinal α.

This result was previously obtained by the author in [14] by manually computing the coset iterations of the operator S. When the group G is Abelian, the above result becomes stronger:

Corollary 4.6 ([12, 13])**.** *TFAE for an Abelian topological group G:*

(i) G has local S-concentration of degree α for some ordinal α.
(ii) G admits no non-trivial continuous homomorphism to an NSS *group.*

The topological groups which have local S-concentration for an ordinal α are known as the groups satisfying the *small subgroup generating property for* α (denoted as $\mathtt{SSGP}(\alpha)$). The algebraic structure of the abelian $\mathtt{SSGP}(\alpha)$ groups (namely, the ones in Corollary 4.6) was fully described by Shakhmatov and the author in [10]. These properties were introduced by Dikranjan and Shakhmatov in [5] as a generalization of the small subgroup generating property of Gould [1, 7, 8]. The origin of this property is intimately linked to original techniques of Prodanov [9] and Dierolf and Warken [2] for constructing *minimally almost periodic groups* (:= groups for which any continuous homomorphism to a compact group is trivial). The common algorithm employed in their techniques was constructing groups which had no non-trivial homomorphism to an NSS group; precisely the property appearing in item (ii) of our Corollary 4.6. Examples of group topologies following the algorithm of Dierolf and Warken was produced by Shakhmatov and the author in [11] for infinitely generated free groups. Overall, we believe that the operator techniques we present here signify a humble start in a new type of degree theory in the realm of topological groups (as shown in Corollaries 4.5 and 4.6). Such a theory allows us to essentially model purely "functional" properties by calculating a specific type of "degree" or "dimension" that is to be attained by a given topological group G. In addition, this new approach allows us to go beyond the categorical setting, thereby allowing us the freedom to build new types of operators (and therefore, topological group decompositions) based on subsets of a topological group which may not necessarily be subgroups. We give a more complete introduction to this theory in the forthcoming paper [15] of the author.

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