The Borel hierarchy and the complete metrizability of spaces of metrics

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This article is a résumé of the paper [13] with some additional remarks. In [13], the Borel hierarchy and the complete metrizability of spaces of metrics are investigated, which are important issues for recognizing topological types of function spaces in infinite-dimensional topology. Given a metrizable space X, denote by $C(X^2)$ the space of continuous bounded real-valued functions on X^2 with the sup-metric. Let $PM(X)$ be the subspace consisting of continuous bounded pseudometrics on X, and $AM(X)$ be the subspace consisting of bounded admissible metrics on X. Recall that $C(X^2)$ and $PM(X)$ are complete metric spaces, where $PM(X)$ is closed in $C(X^2)$. By virtue of the result in [8], we can prove that $AM(X)$ is always a Baire space. It is well-known that completely metrizable spaces are Baire (the Baire category theorem, cf. [11, Theorem 8.4]), but the inverse does not hold. It is interesting to ask the following:

Problem. When is $AM(X)$ completely metrizable?

Due to the study of Y. Ishiki [7], if X is separable and locally compact, then $AM(X)$ is a G_δ set in $C(X^2)$, and hence it is completely metrizable. Moreover, the author [12] improved this result: the complete metrizability of $AM(X)$ follows from the σ-compactness of X. Now we shall consider the inverse of it.

Denote by $\mathfrak{A}_n$ (respectively, $\mathfrak{M}_n$) the additive (respectively, multipulicative) absolute Borel class with respect to a natural number n, refer to [5, Problems 4.5.7 and 4.5.8] and [15, Section 5.11]. Note that $\mathfrak{A}_0 = \{\emptyset\}$ and $\mathfrak{M}_0$ is consisting of compact metrizable spaces. Furthermore,

- $\mathfrak{M}_1$ is the class of completely metrizable spaces,
- $\mathfrak{A}_1$ is the class of σ-locally compact metrizable spaces (σ-compact metrizable spaces in the separable case), see [16].

The author [13] showed that the Borel hierarchy of $AM(X)$ is restricted by the one of X.

Theorem. *Let X be a metrizable space and $n \geq 1$ be a natural number. If $AM(X) \in \mathfrak{A}_n$ (respectively, $\mathfrak{M}_n$), then $X \in \mathfrak{M}_n$ (respectively, $\mathfrak{A}_n$).*

Applying the above theorem and [12, Proposition 3], we can give an equivalent condition on the complete metrizability of $AM(X)$.

Corollary. *Suppose that X is a separable metrizable space. Then $X \in \mathfrak{A}_1$ if and only if $AM(X) \in \mathfrak{M}_1$.*

Two key ideas play significant roles in the proof of Theorem. The following example shows us the one of them.

Example. Let $Q = [0,1] \cap \mathbb{Q}$ and $P = [0,1] \cap \mathbb{P}$ with the usual metric, where $\mathbb{Q}$ is the set of rationals and $\mathbb{P}$ is the one of irrationals. The space $AM(X)$ on the topological sum $X = P \oplus (0,1]$ is not completely metrizable. Indeed, define $i : Q \to AM(X)$ by for all $q \in Q$,

$$i(q)(x,y) = \begin{cases} |x-y| & \text{if } (x,y) \in P^2 \text{ or } (0,1]^2, \\ |x-q| + y & \text{if } x \in P \text{ and } y \in (0,1], \\ x + |y-q| & \text{if } x \in (0,1] \text{ and } y \in P. \end{cases}$$

Then it is a closed embedding from $Q \notin \mathfrak{M}_1$. Consequently, $AM(X) \notin \mathfrak{M}_1$.

Combining the efforts of C. Bessaga [3], T. Banakh [1], O. Pikhurko [14] and M. Zarichnyi [17] (see also [4, Theorem 2] and [2, Theorem 1.2], and moreover [9] in the unbounded case), we can obtain the following lemma, which is the another key ingredient in the proof of Theorem.

Lemma. *For a metrizable space Y and a closed subset $A \subset Y$, there exists a continuous function $e : AM(A) \to AM(Y)$ such that $e(d)|_{A^2} = d$ for any $d \in AM(A)$.*

Using this lemma, we can prove the following proposition, which is a generalization of Example.

Proposition. *If a metrizable space X contains a closed topological copy of $\mathbb{P}$, then $AM(X)$ is not completely metrizable.*

Sketch of Proof. Take any closed embedding $h : \mathbb{P} \to X$, and put $A_1 = h([0,1/3] \cap \mathbb{P})$ and $A_2 = h([2/3,1] \cap \mathbb{P})$ with the natural metric induced by $\mathbb{P}$. Let $Q' = [2/3,1] \cap \mathbb{Q}$ and define an embedding $i : Q' \to AM(A_1 \oplus A_2)$ by the same way as Example. Due to the above lemma, we can find an extensor $e : AM(A_1 \oplus A_2) \to AM(X)$. Then the composition $e \circ i$ is a closed embedding from $Q' \notin \mathfrak{M}_1$. Therefore $AM(X) \notin \mathfrak{M}_1$. □

Theorem will be shown as follows:

Sketch of Proof. We only discuss the case that $AM(X) \in \mathfrak{M}_n$. Supposing that $X \notin \mathfrak{A}_n$, we can find a bounded complete metric space Y so that X is not an $\mathfrak{A}_n$ subset of Y. Set $B = X \setminus \mathrm{int}_Y X$ and $Z = \mathrm{cl}_Y B \setminus X$, where "$\mathrm{int}_Y$" is the interior operator and "cl_Y" is the closure operator on Y respectively, so B is not an $\mathfrak{A}_n$ set in Y and $Z \notin \mathfrak{M}_n$. Take two open subsets U and U' of $\mathrm{cl}_Y B$ so that $Z \cap \mathrm{cl}_Y U \notin \mathfrak{M}_n$, $Z \cap U' \neq \emptyset$ and $\mathrm{cl}_Y U \cap \mathrm{cl}_Y U' = \emptyset$. Fix any $a \in Z \cap U'$ and choose $\{a_n\} \subset X \cap U'$ converging to a. Let $A = \{a_n\}$, $B' = B \cap \mathrm{cl}_Y U$ and $Z' = Z \cap \mathrm{cl}_Y U$. Define an embedding $i : Z' \to AM(A \oplus B')$ by the same technique as Example and take an extending map $e : AM(A \oplus B') \to AM(X)$ as in Lemma, so $AM(X)$ admits a closed embedding $e \circ i$ from $Z' \notin \mathfrak{M}_n$. It follows that $AM(X) \notin \mathfrak{M}_n$. □

Remark 1. We can consider that there exists certain "complementary" relation between X and $AM(X)$. In Example, the subspaces $Q \notin \mathfrak{M}_1$ and $P \notin \mathfrak{A}_1$ are complementary to each other in $[0,1]$. The embedding i constructs a metric on $X = P \bigoplus (0,1]$ by connecting $q \in [0,1] \backslash P$ and $0 \in [0,1] \backslash (0,1]$. Similarly, we can define an embedding $j : P \to PM(X) \backslash AM(X)$ by for any $p \in P$,

$$i(p)(x,y) = \begin{cases} |x-y| & \text{if } (x,y) \in P^2 \text{ or } (0,1]^2, \\ |x-p|+y & \text{if } x \in P \text{ and } y \in (0,1], \\ x+|y-p| & \text{if } x \in (0,1] \text{ and } y \in P. \end{cases}$$

Then $i(Q)$ and $j(P)$ are complementary to each other in some kind of "boundary" between $AM(X)$ and $PM(X) \setminus AM(X)$. We can observe the similar relations between $Q' \notin \mathfrak{M}_1$ and $A_2 \notin \mathfrak{A}_1$ in Proposition, and between $Z' \notin \mathfrak{M}_n$ and $B' \notin \mathfrak{A}_n$ in Theorem, and they induces certain complementary relation between X and $AM(X)$ in the Borel hierarchy.

Remark 2. In [12], the space $AM(X)$ is homeomorphic to Hilbert space ℓ_2 if X is compact, and homeomorphic to the Banach space ℓ_∞ of bounded real-valued functions on natural numbers with the sup-norm if X is σ-compact but not compact. It is known that ℓ_∞ is isometrically universal for separable metric spaces (the Fréchet embedding theorem, cf. [6, p. 101]), see also [10, Lemma 2.8] in the non-separable case. Observe that the embeddings i in Example, Proposition, and Theorem are in fact isometric. Remark that the extension e in Lemma can be chosen as an isometric operator. Hence the composition $e \circ i$ is also isometric, and more $AM(X)$ admits an isometric embedding from some metric spaces. Using these techniques, Y. Ishiki and the author [10] have investigated the isometric universality of spaces of metrics.

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