PROCEEDINGS IN RIMS SYMPOSIUM "IMAGINARY POWERS OF (k,a)-GENERALIZED HARMONIC OSCILLATOR"

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We will give a generalization of a classical result obtained by Stempak and Torrea [18] on the imaginary powers $\mathcal{L}^{-i\sigma}$ of classical harmonic oscillator $\mathcal{L} =: -\triangle + \|x\|^2$ in the context of (k,a)-generalized Fourier analysis developed by S. Ben Saïd, T. Kobayashi and B. Ørsted [4] for $a=2$ and 1, respectively. In [18], the imaginary power $\mathcal{L}^{-i\sigma}(f)$ of $\mathcal{L}$ was defined as

$$\mathcal{L}^{-i\sigma}(f)(x) = \frac{1}{\Gamma(i\sigma)} \int_0^\infty e^{-t\mathcal{L}}(f)(x)\, t^{i\sigma-1} dt$$

according to its spectrum and the following integral formula

$$\lambda^{-i\sigma} = \frac{1}{\Gamma(i\sigma)} \int_0^\infty e^{-t\lambda} t^{i\sigma-1} dt, \quad \lambda > 0.$$

The result by K. Stempak and J. L. Torrea [18] shows that the operator $\mathcal{L}^{-i\sigma}$ is bounded on $L^p(\mathbb{R}^N)$ $(1<p<\infty)$ and weakly bounded on $L^1(\mathbb{R}^N)$ according to the method of classical Calderón–Zygmund theory. We will define and investigate the imaginary powers $(-\triangle_{k,a})^{-i\sigma}$ $(\sigma \in \mathbb{R})$ of the (k,a)-generalized harmonic oscillator $-\triangle_{k,a} = -\|x\|^{2-a}\triangle_k + \|x\|^a$ for $a=2$ and 1, where $\triangle_k$ is the Dunkl Laplacian, and prove the L^p-boundedness $(1<p<\infty)$ and weak L^1-boundedness of such operators by developing the Calderón–Zygmund theory adapted to (k,a)-generalized Fourier analysis. The classical result obtained by Stempak and Torrea in [18] corresponds to the case when $k \equiv 0$ and $a=2$ for the operator $(-\triangle_{k,a})^{-i\sigma}$. The special case when the finite reflection group G is isomorphic to $\mathbb{Z}_2^N$ and $a=2$ for such result was given by A. Nowak and K. Stempak in [14].

In the following sections we firstly collect some basic facts in the (k,a)-generalized Fourier analysis developed by S. Ben Saïd, T. Kobayashi and B. Ørsted [4], especially for the two particular cases for $a=2$ and 1. And then we develop the Calderón–Zygmund theory adapted to the (k,a)-generalized setting for $a=2$ and 1 respectively, in order to prove the L^p-boundedness $(1<p<\infty)$ and weak L^1-boundedness of $(-\triangle_{k,a})^{-i\sigma}$.

1. An introduction on (k,a)-generalized Fourier analysis

1.1. **Dunkl theory.**

Dunkl theory is a far-reaching generalization of classical Fourier analysis related to root system initiated by Dunkl [7], in which finite reflection groups G play the role of orthogonal group $O(N)$. The framework of Dunkl theory is as follows: Given a root system R in the Euclidean space $\mathbb{R}^N$, denote by σ_α the reflection in the hyperplane orthogonal to α and G the finite subgroup of $O(N)$ generated by the reflections σ_α associated to the root system. We assume the root system R is normailized without loss of generality. Define a multiplicity function $k: R \to \mathbb{C}$ such that k is G-invariant, that is, $k(\alpha) = k(\beta)$ if σ_α and σ_β are conjugate. The *Dunkl operators* T_j, $1 \le j \le N$, which were introduced in [7], are defined by the following deformations by difference operators of directional derivatives ∂_j:

$$T_j f(x) = \partial_j f(x) + \sum_{\alpha \in R^+} k(\alpha)\alpha_j \frac{f(x) - f(\sigma_\alpha(x))}{\langle \alpha,\ x\rangle},$$

where $\langle \cdot, \cdot \rangle$ denotes the standard Euclidean inner product and R^+ is any fixed positive subsystem of R. They commute pairwise and are skew-symmetric with respect to the G-invariant measure $dm_k(x) = h_k(x)dx$, where $h_k(x) = \prod_{\alpha \in R} | \langle \alpha,\ x\rangle |^{k(\alpha)}$. The Dunkl Laplacian $\triangle_k = \sum_{j=1}^N T_j^2$ has the following explicit expression,

$$\triangle_k f(x) = \triangle f(x) + 2 \sum_{\alpha \in R^+} k(\alpha) \left(\frac{\langle \nabla f, \alpha \rangle}{\langle \alpha, x \rangle} - \frac{f(x) - f(\sigma_\alpha(x))}{\langle \alpha, x \rangle^2} \right).$$

where $\triangle$ stands for the classical Euclidean Laplacian. The eigenfunction of $\triangle_k$ for fixed y is the integral kernel of the generalized Fourier transform called Dunkl transform. It takes the place of the exponential function $e^{-i\langle x, y\rangle}$ in classical Fourier transform.

The operators ∂_j and T_j are intertwined by a Laplace-type operator (see [8])

$$V_k f(x) = \int_{\mathbb{R}^N} f(y) d\mu_x(y) \tag{1.1}$$

associated to a family of probability measures $\left\{\mu_x | \ x \in \mathbb{R}^N\right\}$ with compact support (see [16]), that is, $T_j \circ V_k = V_k \circ \partial_j$. Specifically, the support of μ_x is contained in the convex hull $co(G \cdot x)$, where $G \cdot x = \{g \cdot x | \ g \in G\}$ is the orbit of x. And if $k > 0$, then $G.x \subseteq supp\mu_x$ (see [10]). For any Borel set B and any $r > 0$, $g \in G$, the probability measures satisfy

$$\mu_{rx}(B) = \mu_x\left(r^{-1}B\right),\ \mu_{gx}(B) = \mu_x\left(g^{-1}B\right).$$

Spherical h-harmonics

The study of Dunkl theory originates from a generalization of spherical harmonics, with the Dunkl weight measure $dm_k(x) = h_k(x)dx$ playing the role of Lebesgue measure dx in the classical theory of spherical harmonics. Let P_m be the space of homogeneous polynomials on $\mathbb{R}^N$ of degree m. The so called Dunkl Laplacian $\triangle_k$ was constructed in such a way that $P_m \cap ker\triangle_k$ are orthogonal to each other for $m = 0, 1, \cdots$ with respect to Dunkl weight measure m_k. It Denote $\mathcal{H}_k^m\left(\mathbb{R}^N\right) := P_m \cap ker\triangle_k$ to be the space of h-harmonic polynomials of degree m. Then the elements in the restriction $\mathcal{H}_k^m\left(\mathbb{R}^N\right)|_{\mathbb{S}^{N-1}}$ of $\mathcal{H}_k^m\left(\mathbb{R}^N\right)$ to the unit sphere $\mathbb{S}^{N-1}$ were called spherical h-harmonics. The spaces $\mathcal{H}_k^m\left(\mathbb{R}^N\right)|_{\mathbb{S}^{N-1}},\ m = 0, 1, \cdots$ are finite dimensional and there is the spherical harmonics decomposition

$$L^2\left(\mathbb{S}^{N-1}, h_k(x')\, d\sigma(x')\right) = \sum_{m \in \mathbb{N}}^{\oplus} \mathcal{H}_k^m\left(\mathbb{R}^N\right)|_{\mathbb{S}^{N-1}}, \tag{1.2}$$

where $d\sigma$ denotes the spherical measure. For each fixed $m \in \mathbb{N}$, denote by $d(m) = \dim\left(\mathcal{H}_k^m\left(\mathbb{R}^N\right)|_{\mathbb{S}^{N-1}}\right)$. Let $\{Y_i^m:\ i = 1, 2, \cdots, d(m)\}$ be an orthonormal basis of $\mathcal{H}_k^m\left(\mathbb{R}^N\right)|_{\mathbb{S}^{N-1}}$. They are the eigenvectors of the generalized Laplace–Beltrami operator $\triangle_k|_{\mathbb{S}^{N-1}}$.

1.2. (k, a)**-generalized Fourier analysis.**

In [4], S. Ben Saïd, T. Kobayashi and B. Ørsted gave a further far-reaching generalization of Dunkl theory by introducing a parameter $a > 0$ arisen from the "interpolation" of the two $sl(2, \mathbb{R})$ actions on the Weil representation of the metaplectic group $Mp(N, \mathbb{R})$ and the minimal unitary representation of the conformal group $O(N + 1, 2)$. They gave an a-deformation of the Dunkl harmonic oscillator $\triangle_k - \|x\|^2$ as follows

$$\triangle_{k,a} := \|x\|^{2-a} \triangle_k - \|x\|^a.$$

The operator is an essentially self-adjoint operator on $L^2\left(\mathbb{R}^N, \vartheta_{k,a}\left(x\right)dx\right)$ with only negative discrete spectrum, where $\vartheta_{k,a}\left(x\right) = \|x\|^{a-2} h_k(x)$. They then proved the existence of a (k,a)-generalized holomorphic semigroup $\mathcal{I}_{k,a}\left(z\right) := \exp\left(\frac{z}{a}\triangle_{k,a}\right)$, $\Re z \geq 0$ with infinitesimal generator $\frac{1}{a}\triangle_{k,a}$. This holomorphic semigroup recedes to the Hermite semigroup studied by Howe [11] when $k \equiv 0$ and $a = 2$; to the Laguerre semigroup studied by Kobayashi and Mano [12, 13] when $k \equiv 0$ and $a = 1$; to the Dunkl Hermite semigroup studied by Rösler [15] when $k \geq 0$, $a = 2$ and $z = 2t$, $t > 0$.

From the spherical harmonic decomposition (1.2) of $L^2\left(\mathbb{S}^{N-1}, h_k\left(x'\right) d\sigma(x')\right)$, there is a unitary isomorphism (see [4, (3.25)])

$$\sum_{m\in\mathbb{N}}^{\oplus} \left(\mathcal{H}_k^m\left(\mathbb{R}^N\right)|_{\mathbb{S}^{N-1}}\right) \otimes L^2\left(\mathbb{R}_+, r^{\,2\langle k\rangle+N+a-3}dr\right) \xrightarrow{\sim} L^2\left(\mathbb{R}^N, \vartheta_{k,a}\left(x\right)dx\right).$$

In [4] the authors constructed an orthonormal basis $\left\{ \Phi_{l,m,j}^{(a)} \,\middle|\, l \in \mathbb{N},\ m \in \mathbb{N},\ j = 1, 2, \cdots, d(m) \right\}$ of $L^2(\mathbb{R}^N,$ $\vartheta_{k,1}\left(x\right)dx)$. They are eigenfunctions for the (k,a)-generalized harmonic oscillator $-\Delta_{k,a} = -\|x\|^{2-a}\triangle_k + \|x\|^a$, i.e.,

$$-\Delta_{k,a}\Phi_{l,m,j}^{(a)}\left(x\right) = \left(2l + \lambda_{k,a,m} + 1\right)\Phi_{l,m,j}^{(a)}\left(x\right), \tag{1.3}$$

where $\lambda_{k,a,m} := \frac{2m+2\langle k\rangle+N-2}{a}$ and $\langle k\rangle := \sum_{\alpha\in R^+} k(\alpha)$. The (k,a)-generalized Laguerre holomorphic semigroup $e^{\frac{z}{a}\triangle_{k,a}}$ $(\Re z \geq 0)$ on $L^2\left(\mathbb{R}^N, \vartheta_{k,a}\left(x\right)dx\right)$ has its spectral decomposition

$$e^{\frac{z}{a}\triangle_{k,a}}\left(f\right)\left(x\right) = \sum_{l,m,j} e^{-z(2l+\lambda_{k,a,m}+1)}\left\langle f, \Phi_{l,m,j}^{(a)}\right\rangle_{k,a}\Phi_{l,m,j}^{(a)}\left(x\right),\ f \in L^2\left(\mathbb{R}^N, \vartheta_{k,a}\left(x\right)dx\right), \tag{1.4}$$

where $\langle f, g\rangle_{k,a} = \int_{\mathbb{R}^N} f(x)g(x)\vartheta_{k,a}(x)dx$. It is a Hilbert–Schmidt operator for $\Re z > 0$ and a unitary operator on $\Re z = 0$ (see [4, Theorem 3.39]). By Schwartz kernel theorem, the operator $e^{\frac{z}{a}\triangle_{k,a}}$ $(\Re z \geq 0)$ has the following integral representation (see [4, (4.56)])

$$e^{\frac{z}{a}\triangle_{k,a}}\left(f\right)\left(x\right) = c_{k,a}\int_{\mathbb{R}^N} f\left(y\right)\Lambda_{k,a}\left(x, y; z\right)\vartheta_{k,a}\left(y\right)dy, \tag{1.5}$$

where $c_{k,a} = \left(\int_{\mathbb{R}^N}\exp\left(-\frac{1}{a}\|x\|^a\right)\vartheta_{k,a}\left(x\right)dx\right)^{-1}$.

The (k,a)-generalized Fourier transform is then defined as the boundary value $z = \frac{\pi i}{2}$ of the semigroup, i.e.,

$$F_{k,a} = e^{i\pi\left(\frac{2\langle k\rangle+N+a-2}{2a}\right)}\mathcal{I}_{k,a}\left(\frac{\pi i}{2}\right).$$

It has the following integral representation on $L^2\left(\mathbb{R}^N, \vartheta_{k,a}\left(x\right)dx\right)$ (see [4, (5.8)])

$$F_{k,a}f\left(\xi\right) = c_{k,a}\int_{\mathbb{R}^N} f\left(y\right)B_{k,a}\left(\xi, y\right)\vartheta_{k,a}\left(y\right)dy,\ \ \xi \in \mathbb{R}^N,$$

where $c_{k,a}$ is a constant and $B_{k,a}\left(x, y\right) = e^{i\frac{\pi}{2}\left(\frac{2\langle k\rangle+N+a-2}{2a}\right)}\Lambda_{k,a}\left(x, y; i\frac{\pi}{2}\right)$ is a symmetric kernel which is the eigenfunction of $-\|x\|^{2-a}\Delta_k$ for fixed y (see [4, Theorem 5.7]), i.e.,

$$\|x\|^{2-a}\Delta_k^x B_{k,a}(x, y) = -\|y\|^a B_{k,a}(x, y).$$

It recedes to the Dunkl transform when $a = 2$ for $f \in \left(L^1 \cap L^2\right)\left(\mathbb{R}^N, \vartheta_{k,a}\left(x\right)dx\right)$. In [4, 5], it was shown that the integral kernel $B_{k,a}\left(x, y\right)$ satisfies the condition

$$\left|B_{k,a}\left(x, y\right)\right| \leq \left|B_{k,a}\left(0, y\right)\right| \leq 1 \tag{1.6}$$

if $a=2/n,\ n\in\mathbb{N}$ assuming that $2\langle k\rangle+N+a-3\geq 0$. In such cases one can define the (k,a)-generalized translation operator on $L^2\left(\mathbb{R}^N,\vartheta_{k,a}(x)\,dx\right)$ by

$$F_{k,a}\left(\tau_y f\right)(\xi):=B_{k,a}\left((-1)^n y,\xi\right)F_{k,a}\left(f\right)(\xi),\ \ \xi\in\mathbb{R}^N.$$

And it can be written via an integral for $f\in\mathcal{L}_k^1\left(\mathbb{R}^N\right)$,

$$\tau_y f\left(x\right)=c_{k,a}\int_{\mathbb{R}^N}B_{k,a}\left((-1)^n x,\xi\right)B_{k,a}\left((-1)^n y,\xi\right)F_{k,a}\left(f\right)(\xi)\,\vartheta_{k,a}\left(\xi\right)d\xi,$$

where $\mathcal{L}_k^1\left(\mathbb{R}^N\right):=\left\{f\in L^1\left(\mathbb{R}^N,\vartheta_{k,a}(x)\,dx\right):\ F_{k,a}\left(f\right)\in L^1\left(\mathbb{R}^N,\vartheta_{k,a}(x)\,dx\right)\right\}$, combining the inversion formula of the (k,a)-generalized Fourier transform for $a=\frac{2}{n},\ n\in\mathbb{N}$ (see [4, Theorem 5.3]). We can observe that $\tau_y f(x)=\tau_x f(y)$.

The two special cases for $a=2$ (the Dunkl case) and $a=1$ are of particular interest because for the two cases the structures will be richer. And in particular, we have the formulae for radial functions of the generalized translations for the two special cases. For the case of $a=2$, the generalized translation corresponds to the classical translation operator $f\mapsto f(x+\cdot)$; and for $a=1$, it corresponds to the classical translation $f\mapsto f(x-\cdot)$. We will discuss more on the properties of the generalized translations for the two cases.

The case of $a=2$

For the case of $a=2$ (the Dunkl case), the radial formula of Dunkl translations was found by Rösler in [17]

$$\tau_x f(-y)=V_k\left(\widetilde{f}(\sqrt{\|x\|^2+\|y\|^2-2\langle y,\ \cdot\rangle})\right)(x),$$

where $f(x)=\widetilde{f}(\|x\|)\in\mathcal{S}_{rad}(\mathbb{R}^N)$. This formula was first proved by Rösler for radial Schwartz functions, and was then extended to all continuous radial functions by F. Dai and H. Wang [9]. This shows that Dunkl translation is positive on radial functions because the intertwining operator V_k is positive, i.e., if $f\in L^2(m_k)$ is **radial** and $f\geq 0$, then $\tau_x f\geq 0$. For a nonnegative radial function f in $L^1(\mathbb{R}^N,h_k(x)dx)$, we have

$$\int_{\mathbb{R}^N}\tau_x f(y)h_k(y)dy=\int_{\mathbb{R}^N}f(y)h_k(y)dy,\ x\in\mathbb{R}^N.$$

The following is a theorem in analogy to the property of classical translation that if $\operatorname{supp} f=B(0,r)$, then $\operatorname{supp} f(x-\cdot)=B(x,r)$, where $B(x,r)=\{y\in\mathbb{R}^N:\|x-y\|\leq r\}$.

Theorem 1.1. *([19]) Assume $k>0$. Let $f=f_0\left(\|\cdot\|\right)$ be a nonnegative radial function on $L^2\left(\mathbb{R}^N,h_k\left(x\right)dx\right)$, $suppf=B(0,r)$, then*

$$supp\tau_x f=\bigcup_{g\in G}B(gx,r),$$

where $B\left(x,r\right):=\left\{y\in\mathbb{R}^N:\|x-y\|\leq r\right\}$.

The case of $a=1$

The radial formula of $(k,1)$-generalized translations was found by S. B. Saïd and L. Deleaval in [3] for $2\langle k\rangle+N-2>0$,

$$\begin{aligned}\tau_y f(x)=&\frac{\Gamma\left(\frac{N-1}{2}+\langle k\rangle\right)}{\sqrt{\pi}\Gamma\left(\frac{N-2}{2}+\langle k\rangle\right)}\times\\&V_k\left(\int_{-1}^{1}f_0\left(\|x\|+\|y\|-\sqrt{2\left(\|x\|\,\|y\|+\langle\cdot,y\rangle\right)}u\right)\left(1-u^2\right)^{\frac{N}{2}+\langle k\rangle-2}du\right)(x).\end{aligned}\tag{1.7}$$

where $f(x) = \widetilde{f}(\|x\|)$ is a radial function in $\mathcal{L}_k^1(\mathbb{R}^N)$. This formula can also be extended to all continuous radial functions analogously. The $(k,1)$-generalized translation is also positive on radial functions from the positivity of the intertwining operator. For a nonnegative radial function f in $L^1(\mathbb{R}^N, \vartheta_{k,1}(x)dx)$,

$$\int_{\mathbb{R}^N} \tau_x f(y)\vartheta_{k,1}(y)dy = \int_{\mathbb{R}^N} f(y)\vartheta_{k,1}(y)dy,\ x \in \mathbb{R}^N.$$

We also have a theorem parallel to the Theorem 1.1 for $a = 2$.

Theorem 1.2. *([20]) Assume $k > 0$ and $2\langle k\rangle + N - 2 > 0$. Let $f = f_0(\|\cdot\|)$ be a nonnegative radial function in $L^2(\mathbb{R}^N, \vartheta_{k,1}(x)\,dx)$, supp$f = B(0,r)$, then*

$$supp\tau_x f = \bigcup_{g\in G} B(gx, r),$$

where $B(x,r) := \left\{y \in \mathbb{R}^N : d(x,y) \le r\right\}$ *and* $d(x,y) = \sqrt{\|x\| + \|y\| - \sqrt{2(\|x\|\,\|y\| + \langle x,y\rangle)}}$.

It was shown in [20] that the function $d(x,y) = \sqrt{\|x\| + \|y\| - \sqrt{2(\|x\|\,\|y\| + \langle x,y\rangle)}}$ is a metric satisfying the three properites: 1) $d(x,y) > 0$ if and only if $x \neq y$; 2) $d(x,y) = d(y,x)$; 3) $d(x,y) \le d(x,z) + d(z,y)$. And it is not difficult to check $\mathbb{R}^N$ equipped with the metric is complete and that the closure of the open ball with respect to the metric is the closed ball.

2. Calderón–Zygmund theory

We review the classical Calderón–Zygmund theory on general homogeneous space first. Let (X,d) be a metric space. Denote $B(x,r)$ to be the ball $B(x,r) := \{y \in X : d(x,y) \le r\}$ for $x \in X$. If there exists a doubling measure m, i.e., there exists a measure m such that for some absolute constant C,

$$m(B(x,2r)) \le Cm(B(x,r)),\ \forall x \in \mathbb{R}^N,\ r > 0, \tag{2.1}$$

then (X,d) is a space of homogeneous type. The Calderón–Zygmund theory on a space of homogeneous type (X,d,m) says that for $f \in L^1(X,m) \cap L^2(X,m)$ and a function $K(x,y)$ on $L^2(X \times X, m \times m)$, define the operator S with the integral kernel $K(x,y)$ as

$$S(f)(x) = \int_X K(x,y) f(y)\,dm(y).$$

If S is bounded on $L^2(X,m)$ and $K(x,y)$ satisfies the Hörmander type condition

$$\int_{d(x,y)>2d(y,y_0)} |K(x,\ y) - K(x,\ y_0)|\,dm(x) \le C,\ \ y, y_0 \in \mathbb{R}^N, \tag{2.2}$$

then the operator S is bounded on $L^p(X,m)$ $(1 < p \le 2)$ and weakly bounded on $L^1(X,m)$. We refer to [6, Chapter III] for this theory.

Now we adapt the classical Calderón–Zygmund theory on general homogeneous space to the setting of (k,a)-generalized Fourier analysis for $a = 2$ and 1, respectively. Denote $dm_{k,a}(x) = \vartheta_{k,a}(x)\,dx$, $\vartheta_{k,a}(x) = \|x\|^{a-2} h_k(x)$, $2\langle k\rangle + N - a - 3 > 0$. We firstly claim that the metric spaces $(X, d, m_{k,a})$ for both $a = 2$ and $a = 1$ are of homogeneous type. Here for $a = 2$, $d(x,y)$ denotes the Euclidean metric, i.e., $d(x,y) = \|x - y\|$. And for $a = 1$, $d(x,y) = \sqrt{\|x\| + \|y\| - \sqrt{2(\|x\|\,\|y\| + \langle x,y\rangle)}}$. In particular, for $a = 2$, it was shown in [1] that $m_k(B(x,r)) \sim r^N \prod_{\alpha\in R} (|\langle x,\alpha\rangle| + r)^{k(\alpha)}$. And so, $m_{k,2}$ is a doubling measure. For $a = 1$, we have the following scaling property

$$m_{k,1}\left(B\left(tx, \sqrt{t}r\right)\right) = t^{2\langle k\rangle+N-1} m_{k,1}(B(x,r)),\ t > 0. \tag{2.3}$$

And $m_{k,1}(B(x,r)) = \int_{E(x_\omega,r)} \|z\|^{2\langle k\rangle+N-2} h_k(z)\,dz$ from polar coordinate transformation $z=\sqrt{\rho}\omega$, where $E(x_\omega,r)=\left\{z\in\mathbb{R}^N : \left\|z-\sqrt{\|x\|}\frac{x+\|x\|\omega}{\|x+\|x\|\omega\|}\right\|\le r\right\}$. It can be observed that $m_{k,1}(B(tx,r))$ is equivalent to a function which is non-decreasing as t grows. And then it can be easily shown that $m_{k,1}$ is also a doubling measure combining the scaling property (2.3). Now we are ready to give the Hörmander type condition adapted to (k,a)-generalized setting. However, the classical Calderón–Zygmund condition on homogeneous space is no longer valid in the setting of (k,a)-generalized Fourier analysis. And we define the distance between the two orbits $G.x$ and $G.y$,

$$d_G(x,y)=\min_{g\in G} d(gx,y).$$

Obviously, $d_G(x,y)\le d(x,y)$. The classical Hörmander type condition (2.2) on general homogeneous spaces is then modified in the following theorem in order to adapt the classical Calderón–Zygmund theory to the (k,a)-generalized setting. The proof for the case of $a=2$ (the Dunkl case) was shown in [2], and the similar proof also applies to the case of $a=1$.

Theorem 2.1. *([2, 20, 21]) For $2\langle k\rangle+N+a-3>0$, $a=2$ or 1, let K be a measurable function on $\mathbb{R}^N\times\mathbb{R}^N\backslash\left\{(x,g.x);x\in\mathbb{R}^N,\ g\in G\right\}$ such that the operator S is defined as*

$$S(f)(x)=\int_{\mathbb{R}^N} K(x,y)\,f(y)\,\vartheta_{k,a}(y)\,dy,\ G.x\cap \operatorname{supp} f=\varnothing$$

for any compactly supported function $f\in L^2\left(\mathbb{R}^N,\vartheta_{k,a}(x)\,dx\right)$. If S is bounded on $L^2\left(\mathbb{R}^N,\vartheta_{k,a}(x)\,dx\right)$ and K satisfies

$$\int_{d_G(x,y)>2d(y,y_0)} |K(x,\ y)-K(x,\ y_0)|\,\vartheta_{k,a}(x)\,dx\le C,\quad y,y_0\in\mathbb{R}^N,$$

then S extends to a bounded operator on $L^p\left(\mathbb{R}^N,\vartheta_{k,a}(x)\,dx\right)$ for $1<p\le 2$ and a weakly bounded operator on $L^1\left(\mathbb{R}^N,\vartheta_{k,a}(x)\,dx\right)$.

3. Imaginary powers of the (k,a)-generalized harmonic oscillator

From the spectral decomposition (1.3) we can define the imaginary powers $(-\triangle_{k,a})^{-i\sigma},\sigma\in\mathbb{R}$ for $f\in L^2\left(\mathbb{R}^N,\vartheta_{k,a}(x)\,dx\right)$ of the (k,a)-generalized harmonic oscillator $-\triangle_{k,a}$ naturally as

$$(-\triangle_{k,a})^{-i\sigma}(f)(x)=\sum_{l,m,j}(a(2l+\lambda_{k,a,m}+1))^{-i\sigma}\left\langle f,\Phi^{(a)}_{l,m,j}\right\rangle_{k,a}\Phi^{(a)}_{l,m,j}(x). \tag{3.1}$$

It is obviously a bounded operator on $L^2\left(\mathbb{R}^N,\vartheta_{k,a}(x)\,dx\right)$ from its spectrum.

For $a=2$ or 1, the reproducing kernel $\Lambda_{k,a}(x,y;z)$ of $e^{z\triangle_{k,a}}$ (see (1.5)) can be reformulated as follows

$$\Lambda_{k,a}(x,y;z)=\frac{\exp(-\frac{1}{a}(\|x\|^a+\|y\|^a)\tanh\frac{z}{2})}{\sinh(z)^{\frac{2\langle k\rangle+N+a-2}{a}}}\tau_y\left(e^{-\frac{1}{a\sinh z}\|\cdot\|^a}\right)((-1)^n x),\ a=\tfrac{2}{n},\ n=2 \text{ or } 1, \tag{3.2}$$

$$\begin{aligned}
&=\frac{\exp(-\frac{1}{a}(\|x\|^a+\|y\|^a)\tanh\frac{z}{2})}{\sinh(z)^{\frac{2\langle k\rangle+N+a-2}{a}}}\\
&\quad\times\begin{cases}\dfrac{\Gamma\left(\frac{N-1}{2}+\langle k\rangle\right)}{\sqrt{\pi}\Gamma\left(\frac{N-2}{2}+\langle k\rangle\right)}\times\\
V_k\left(\int_{-1}^1 e^{-\frac{1}{\sinh z}\left(\|x\|+\|y\|-\sqrt{2(\|x\|\|y\|+\langle\cdot,y\rangle)u}\right)}\left(1-u^2\right)^{\frac{N}{2}+\langle k\rangle-2}du\right)(x) & (a=1),\\
V_k\left(e^{-\frac{1}{2\sinh z}\left(\|x\|^2+\|y\|^2-2\langle\cdot,y\rangle\right)}\right)(x) & (a=2).\end{cases}
\end{aligned} \tag{3.3}$$

In what follow we put

$$K\left(x,y\right)=\int_{0}^{\infty}\Lambda_{k,a}\left(x,y;at\right)t^{i\sigma-1}dt. \tag{3.4}$$

It is then easy to verify that if $a=2$ or 1 and $2\left\langle k\right\rangle+N-a-3>0$, then the integral (3.4) converges absolutely for all $x,y\in\mathbb{R}^{N}$, $y\notin G.x$.

Based on the formula

$$\lambda^{-i\sigma}=\frac{1}{\Gamma\left(i\sigma\right)}\int_{0}^{\infty}e^{-t\lambda}t^{i\sigma-1}dt,\quad \lambda>0$$

and (3.1), (1.4), (1.5), we can write $\left(-\triangle_{k,a}\right)^{-i\sigma}$ in the following way (such definition goes back to [14] and [18])

$$\begin{aligned}\left(-\triangle_{k,a}\right)^{-i\sigma}\left(f\right)\left(x\right)&=\frac{1}{\Gamma\left(i\sigma\right)}\int_{0}^{\infty}e^{t\triangle_{k,a}}\left(f\right)\left(x\right)t^{i\sigma-1}dt\\&=\frac{c_{k,a}}{\Gamma\left(i\sigma\right)}\int_{0}^{\infty}t^{i\sigma-1}dt\int_{\mathbb{R}^{N}}f\left(y\right)\Lambda_{k,a}\left(x,y;t\right)\vartheta_{k,a}\left(y\right)dy.\end{aligned}$$

We can observe that this integral converges absolutely for all compactly supported functions $f\in L^{2}(\mathbb{R}^{N},\vartheta_{k,a}\left(x\right)dx)$ with $\operatorname{supp}f\cap G.x=\varnothing$. And for compactly supported functions $f\in L^{2}\left(\mathbb{R}^{N},\vartheta_{k,a}\left(x\right)dx\right)$, $G.x\cap\operatorname{supp}f=\varnothing$, $\left(-\triangle_{k,a}\right)^{-i\sigma}$ satisfies

$$\left(-\triangle_{k,a}\right)^{-i\sigma}\left(f\right)\left(x\right)=\frac{c_{k,a}}{\Gamma\left(i\sigma\right)}\int_{\mathbb{R}^{N}}K\left(x,y\right)f\left(y\right)\vartheta_{k,a}\left(y\right)dy$$

by changing the order of integration. We will show that the kernel $K\left(x,y\right)$ of $\left(-\triangle_{k,a}\right)^{-i\sigma}$ satisfies the condition in Theorem 2.1 to prove the following main theorem. And we will give the sketch of the proof of the theorem. For the detailed proof we refer to [20, 21].

Theorem 3.1. *([20, 21]) For $2\left\langle k\right\rangle+N+a-3>0$, $a=2$ or 1, the imaginary powers $\left(-\triangle_{k,a}\right)^{-i\sigma},\sigma\in\mathbb{R}$ are bounded on $L^{p}\left(\mathbb{R}^{N},\vartheta_{k,a}\left(x\right)dx\right)$, $1<p<\infty$, and weakly bounded on $L^{1}\left(\mathbb{R}^{N},\vartheta_{k,a}\left(x\right)dx\right)$.*

Sketch of Proof: Notice that $\left(-\triangle_{k,a}\right)^{-i\sigma}$ is symmetric on $L^{2}\left(\mathbb{R}^{N},\vartheta_{k,a}\left(x\right)dx\right)$. It suffices to show that the integral kernel $K\left(x,y\right)$ of $\left(-\triangle_{k,a}\right)^{-i\sigma}$ for $a=1$ or 2 satisfies the Calderón–Zygmund condition adapted to (k,a)-generalized Fourier analysis.

We write

$$\begin{aligned}K\left(x,y\right)&=\int_{0}^{1}\Lambda_{k,a}\left(x,y;at\right)t^{i\sigma-1}dt+\int_{1}^{\infty}\Lambda_{k,a}\left(x,y;at\right)t^{i\sigma-1}dt\\&=K^{(1)}\left(x,y\right)+K^{(2)}\left(x,y\right),\end{aligned}$$

It can be shown easily that $\int_{\mathbb{R}^{N}}\left|K^{(2)}\left(x,y\right)\right|\vartheta_{k,a}\left(x\right)dx\leq C$ for both $a=2$ and 1 and

$$\begin{aligned}&\int_{d_{G}(x,y)>2d(y,y_{0})}\left|K^{(2)}\left(x,y\right)-K^{(2)}(x,y_{0})\right|\vartheta_{k,a}\left(x\right)dx\\&\qquad\leq 2\int_{\mathbb{R}^{N}}\left|K^{(2)}\left(x,y\right)\right|\vartheta_{k,a}\left(x\right)dx\leq C.\end{aligned}$$

For $K^{(1)}\left(x,y\right)$,

$$\left|K^{(1)}\left(x,y\right)-K^{(1)}\left(x,y_{0}\right)\right|\leq\int_{0}^{1}\left|\Lambda_{k,a}\left(x,y;at\right)-\Lambda_{k,a}\left(x,y_{0};at\right)\right|\frac{1}{t}dt.$$

We will then show that $K^{(1)}(x,y)$ satisfies the adapted Calderón–Zygmund condition for $a=2$ and 1 respectively.

For $a=2$, We need an estimate of the partial derivative of the integral kernel.

Lemma 3.2. *([21]) For $0<t<1$, we have*

$$\left|\frac{\partial \Lambda_{k,2}}{\partial y_i}(x,y;2t)\right| \le \frac{C}{t^{\langle k\rangle+\frac{N+1}{2}}}\tau_y\left(e^{-\frac{c}{t}\|\cdot\|^2}\right)(-x).$$

Then from the mean value theorem for integrals and the above lemma, we have

$$\begin{aligned}\left|K^{(1)}(x,y)-K^{(1)}(x,y_0)\right| &\le \|y-y_0\|\int_0^1 \frac{1}{t}dt\int_0^1\sum_{i=1}^N\left|\frac{\partial \Lambda_{k,2}}{\partial y_i}(x,y_\theta,;2t)\right|d\theta\\ &\le C\|y-y_0\|\int_0^1\frac{1}{t^{\langle k\rangle+\frac{N+3}{2}}}dt\int_0^1\tau_x\left(e^{-\frac{c}{t}\|\cdot\|^2}\right)(-y_\theta)d\theta,\end{aligned}$$

where $y_\theta=y_0+\theta(y-y_0)$. Notice that $\tau_x\left(e^{-\frac{c}{t}\|\cdot\|^2}\right)(-y_\theta)\le\tau_x\left(e^{-\frac{c}{4t}(\|\cdot\|+\|y-y_0\|)^2}\right)(-y_\theta)$ if $d_G(x,y)>2\|y-y_0\|$. Then from the properties of the Dunkl translation,

$$\begin{aligned}&\int_{d_G(x,y)>2\|y-y_0\|}\left|K^{(1)}(x,y)-K^{(1)}(x,y_0)\right|h_k(x)\,dx\\ &\le C\|y-y_0\|\int_0^1\frac{1}{t^{\langle k\rangle+\frac{N+3}{2}}}dt\int_0^1\left(\int_{\mathbb{R}^N}\tau_{y_\theta}\left(e^{-\frac{c}{4t}(\|\cdot\|+\|y-y_0\|)^2}\right)(-x)h_k(x)\,dx\right)d\theta\\ &=C\|y-y_0\|\int_0^1\frac{1}{t^{\langle k\rangle+\frac{N+3}{2}}}dt\int_{\mathbb{R}^N}e^{-\frac{c}{4t}(\|x\|+\|y-y_0\|)^2}h_k(x)\,dx\\ &\le C\|y-y_0\|\int_0^\infty\frac{1}{(r+\|y-y_0\|)^2}dr=C.\end{aligned}$$

For $a=1$, $2\langle k\rangle+N-2>0$, we need the following estimate of the difference quotient analogue, parallel to the estimate of the partial derivative in Lemma 3.2. As continuous rectifiable curves between two distinct points may fail with respect to the metric corresponding to $(k,1)$-generalized analysis, we can no longer make use of estimate of partial derivatives but will make use of an estimate of the difference quotient analogue instead. We refer to [20, 21] for the detailed proof of the following lemma.

Lemma 3.3. *([20, 21]) For $0<t<1$, $y\ne y_0$,*

$$\left|\frac{\Lambda_{k,1}(x,y;t)-\Lambda_{k,1}(x,y_0;t)}{d(y,y_0)}\right|\le\frac{C}{t^{2\langle k\rangle+N-\frac{1}{2}}}\left(\tau_{y_0}\left(e^{-\frac{c}{t}\|\cdot\|}\right)(x)+\tau_y\left(e^{-\frac{c}{t}\|\cdot\|}\right)(x)\right).$$

To prove the above lemma we need an enhancement of the triangle inequality of the metric $d(x,y)$.

Lemma 3.4. *([20, 21]) For $u\in[-1,1]$, $\eta\in co(G.x)$, and $x,y\in\mathbb{R}^N$,*

$$\left|\sqrt{\|x\|+\|y\|-\sqrt{2\left(\|x\|\,\|y\|+\langle\eta,y\rangle\right)u}}-\sqrt{\|x\|+\|z\|-\sqrt{2\left(\|x\|\,\|z\|+\langle\eta,z\rangle\right)u}}\right|\le d(y,z).$$

From Lemma 3.3,

$$\begin{aligned}\left|K^{(1)}(x,y)-K^{(1)}(x,y_0)\right| &\le\int_0^1\left|\Lambda_{k,1}(x,y;t)-\Lambda_{k,1}(x,y_0;t)\right|\frac{1}{t}dt\\ &\le Cd(y,y_0)\int_0^1\frac{1}{t^{2\langle k\rangle+N+\frac{1}{2}}}\left(\tau_{y_0}\left(e^{-\frac{c}{t}\|\cdot\|}\right)(x)+\tau_y\left(e^{-\frac{c}{t}\|\cdot\|}\right)(x)\right)dt.\end{aligned}$$

When $d_G(x,y) > 2d(y,y_0)$, it can be observed that

$$\tau_x\left(e^{-\frac{c}{t}\|\cdot\|}\right)(y_0) \le \tau_x\left(e^{-\frac{c}{4t}\left(\sqrt{\|\cdot\|}+d(y,y_0)\right)^2}\right)(y_0), \quad \tau_x\left(e^{-\frac{c}{t}\|\cdot\|}\right)(y) \le \tau_x\left(e^{-\frac{c}{4t}\left(\sqrt{\|\cdot\|}+d(y,y_0)\right)^2}\right)(y).$$

Therefore, from the properties of the $(k,1)$-generalized translation,

$$\begin{aligned}
&\int_{d_G(x,y)>2d(y,y_0)} \left|K^{(1)}(x,y) - K^{(1)}(x,\ y_0)\right| \vartheta_{k,1}(x)\,dx \\
&\le Cd(y,y_0)\int_0^1 \frac{1}{t^{2\langle k\rangle+N+\frac{1}{2}}}\left(\int_{\mathbb{R}^N} \tau_{y_0}\left(e^{-\frac{c}{4t}\left(\sqrt{\|\cdot\|}+d(y,y_0)\right)^2}\right)(x)\vartheta_{k,1}(x)\,dx\right. \\
&\quad \left.+\int_{\mathbb{R}^N}\tau_y\left(e^{-\frac{c}{4t}\left(\sqrt{\|\cdot\|}+d(y,y_0)\right)^2}\right)(x)\vartheta_{k,1}(x)\,dx\right)dt \\
&= Cd(y,y_0)\int_0^1 \frac{1}{t^{2\langle k\rangle+N+\frac{1}{2}}}dt\int_{\mathbb{R}^N} 2e^{-\frac{c}{4t}\left(\sqrt{\|x\|}+d(y,y_0)\right)^2}\vartheta_{k,1}(x)\,dx \\
&\le Cd(y,y_0)\int_0^\infty r^{2\langle k\rangle+N-2}dr\int_0^1 \frac{2}{t^{2\langle k\rangle+N+\frac{1}{2}}}e^{-\frac{c}{4t}\left(\sqrt{r}+d(y,y_0)\right)^2}dt \\
&\le Cd(y,y_0)\int_0^\infty \frac{r^{2\langle k\rangle+N-2}}{\left(\sqrt{r}+d(y,y_0)\right)^{2\left(2\langle k\rangle+N-\frac{1}{2}\right)}}dr\int_0^\infty \frac{2}{u^{2\langle k\rangle+N+\frac{1}{2}}}e^{-\frac{c}{4u}}du \\
&\le Cd(y,y_0)\int_0^\infty \frac{1}{\left(\sqrt{r}+d(y,y_0)\right)^3}dr = C.
\end{aligned}$$

The proof of Theorem 3.1 is complete. □

References

[1] JP. Anker, J. Dziubański, A. Hejna, Harmonic functions, conjugate harmonic functions and the Hardy space H^1 in the rational Dunkl setting, Journal of Fourier Analysis and Applications, 25(5)(2019), 2356–2418.

[2] B. Amri, M. Sifi, Riesz transforms for the Dunkl transform, Ann. Math.Blaise Pascal 19(2012), no. 1, 247–262.

[3] S. Ben Saïd, L. Deleaval, Translation Operator and Maximal Function for the $(k,1)$-Generalized Fourier Transform, Journal of Functional Analysis, vol. 279, no. 8 (2020), 108706.

[4] S. Ben Saïd, T. Kobayashi and B. Ørsted, Laguerre semigroup and Dunkl operators, Compos. Math., 148(4)(2012), 1265–1336.

[5] D. Constales, H. De Bie and P. Lian, Explicit formulas for the Dunkl dihedral kernel and the (k,a)-generalized Fourier kernel, J. Math. Anal. Appl. 460(2) (2018) 900–926

[6] R. R. Coifman, G. Weiss, Analyse Hamonique Non-Commutative sur Certains Espaces Homogenes, Springer Berlin Heidelberg, 1971.

[7] C. F. Dunkl, Differential-difference operators associated to reflection groups, Trans. Amer. Math. Soc., 311, no. 1(1989), 167–183.

[8] C. F. Dunkl, Integral kernels with reflection group invariance, Canadian J Math43(1991), 1213–1227.

[9] F. Dai, H. Wang, A transference theorem for the Dunkl transform and its applications, J. Funct. Anal. 258.12(2010), 4052–4074.

[10] L. Gallardo, C. Rejeb, Support properties of the intertwining and the mean value operators in Dunkl theory. Proceedings of the American Mathematical Society 146.1(2017), 1.

[11] R. Howe, The oscillator semigroup. The mathematical heritage of Hermann Weyl (Durham, NC, 1987), 61–132, Proc. Sympos. Pure Math., 48, Amer. Math. Soc., Providence, RI, 1988.

[12] T. Kobayashi, G. Mano, The inversion formula and holomorphic extension of the minimal representation of the conformal group, Harmonic Analysis, Group Representations, Automorphic Forms and Invariant Theory: In honor of Roger Howe, (eds. J. S. Li, E. C. Tan, N. Wallach and C. B. Zhu), World Scientific(2007), 159–223.

[13] T. Kobayashi, G. Mano, The Schrödinger model for the minimal representation of the indefinite orthogonal group $O(p,q)$, Mem. Amer. Math. Soc. 213(1000), 2011.

[14] A. Nowak, K. Stempak, Imaginary Powers of the Dunkl Harmonic Oscillator, Symmetry Integrability and Geometry–Methods and Applications, vol. 5 (2009) p. 16.
[15] M. Rösler, Generalized Hermite polynomials and the heat equation for Dunkl operators, Comm. Math. Phys., 192(3)(1998), 519–542.
[16] M. Rösler, Positivity of Dunkl's intertwining operator. Duke Mathematical Journal, vol. 98, no. 3 (1999) pp. 445–463.
[17] M. Rösler, A positive radial product formula for the Dunkl kernel, Trans. Amer.Math. Soc. 355(2003), no. 6, 2413–2438.
[18] K. Stempak, J. L. Torrea. Higher Riesz Transforms and Imaginary Powers Associated to the Harmonic Oscillator, Acta Mathematica Hungarica, vol. 111, no. 1 (2006), 43–64.
[19] W. Teng, Dunkl translations, Dunkl–type BMO space and Riesz transforms for the Dunkl transform on L^∞, Funct Anal Its Appl 55, 304–315 (2021).
[20] W. Teng, Imaginary powers of $(k,1)$-generalized harmonic oscillator, Complex Anal. Oper. Theory 16, 89 (2022).
[21] W. Teng, The (k,a)-Generalized Fourier Analysis, doctoral thesis, Kwansei Gakuin University (2023).

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