Global existence of weak solutions to a singular nonlocal phase field system with inertial term

Shunsuke Kurima
Department of Mathematics, Tokyo University of Science

1. Introduction

This paper deals with the phase field system

$$\begin{cases}(\alpha(\theta))_t + \varphi_t - \Delta\theta = f & \text{in } \Omega \times (0,T), \\ \lambda\varphi_{tt} + \varphi_t + A\varphi + \beta(\varphi) + \pi(\varphi) = \theta & \text{in } \Omega \times (0,T).\end{cases} \tag{E1}$$

Here $\Omega \subset \mathbb{R}^3$ is a bounded domain, β is a monotone function (e.g., $\beta(\varphi) = \varphi^3$), π is a continuous function (e.g., $\pi(\varphi) = -\varphi$), f is a given function. In the case that $\alpha(\theta) = \theta$, $\lambda = 0$, $A\varphi = -\Delta\varphi$, (E1) is the classical phase field model proposed by Caginalp [**3**] (see e.g., [**2, 4, 7, 10, 19**]). In the case that $\alpha(\theta) = \ln\theta$, the first equation in (E1) gives account of an entropy balance. In the case that $\alpha(\theta) = \ln\theta$, $\lambda = 0$, $A\varphi = -\Delta\varphi$, (E1) has been studied (see e.g., [**5, 9**]). However, there are rapid phase transformation processes in nonequilibrium dynamics for which the inertial term φ_{tt} must be taken into account (see e.g., [**11**] and references therein). In the case that $\alpha(\theta) = \theta$, $\lambda = 1$, $A\varphi = -\Delta\varphi$, (E1) is the parabolic-hyperbolic phase field system (see e.g., [**15, 23, 24**]). In the case that $\alpha(\theta) = \theta$, $\lambda = 1$, $A\varphi = a\varphi - J * \varphi$, where $a(x) := \int_\Omega J(x-y)\,dy$, $(J * \varphi)(x) := \int_\Omega J(x-y)\varphi(y)\,dy$ for $x \in \Omega$, J is an interaction kernel such that $J(x) = J(-x)$, (E1) is the nonlocal phase field system with inertial term (see e.g., [**12, 16**]).

The phase field system proposed by Penrose–Fife [**21**]

$$\begin{cases}\theta_t + \varphi_t - \Delta\left(-\frac{1}{\theta}\right) = f & \text{in } \Omega \times (0,T), \\ \varphi_t - \Delta\varphi + \beta(\varphi) + \pi(\varphi) = -\frac{1}{\theta} & \text{in } \Omega \times (0,T)\end{cases} \tag{E2}$$

has been studied (see e.g., [**8, 13**]). Also, the parabolic-hyperbolic Penrose–Fife phase field system

$$\begin{cases}\theta_t + \varphi_t - \Delta\left(-\frac{1}{\theta}\right) = f & \text{in } \Omega \times (0,T), \\ \varphi_{tt} + \varphi_t - \Delta\varphi + \beta(\varphi) + \pi(\varphi) = -\frac{1}{\theta} & \text{in } \Omega \times (0,T)\end{cases} \tag{E2'}$$

has been studied (see e.g., [**6**]).

In this paper, we will derive existence for a singular nonlocal phase field system with inertial term. More precisely, in this paper, we will prove existence of weak solutions to the nonlocal phase field system with inertial term related to the entropy balance

$$\begin{cases} (\ln\theta)_t + \varphi_t - \Delta\theta = f & \text{in } \Omega\times(0,T), \\ \varphi_{tt} + \varphi_t + a\varphi - J*\varphi + \beta(\varphi) + \pi(\varphi) = \theta & \text{in } \Omega\times(0,T), \\ \partial_\nu\theta = 0 & \text{on } \partial\Omega\times(0,T), \\ (\ln\theta)(0) = \ln\theta_0,\ \varphi(0)=\varphi_0,\ \varphi_t(0) = v_0 & \text{in } \Omega, \end{cases} \tag{P}$$

where $\Omega \subset \mathbb{R}^d$ $(d = 1,2,3)$ is a bounded domain with smooth boundary $\partial\Omega$, ∂_ν denotes differentiation with respect to the outward normal of $\partial\Omega$. Moreover, we assume the four conditions:

(C1) $J(-x) = J(x)$ for all $x \in \mathbb{R}^d$ and $\sup_{x\in\Omega}\int_\Omega |J(x-y)|\,dy < +\infty$.

(C2) $\beta : \mathbb{R}\to\mathbb{R}$ is a single-valued maximal monotone function such that there exists a lower semicontinuous convex function $\widehat{\beta} : \mathbb{R} \to [0,+\infty)$ satisfying that $\widehat{\beta}(0) = 0$ and $\beta = \partial\widehat{\beta}$, where $\partial\widehat{\beta}$ is the subdifferential of $\widehat{\beta}$. Moreover, $\beta : \mathbb{R}\to\mathbb{R}$ is local Lipschitz continuous.

(C3) $\pi : \mathbb{R}\to\mathbb{R}$ is a Lipschitz continuous function.

(C4) $f \in L^2(\Omega\times(0,T)) \cap L^1(0,T;L^\infty(\Omega))$, $\theta_0 \in L^2(\Omega)$, $\ln\theta_0 \in L^2(\Omega)$, $\varphi_0, v_0 \in L^\infty(\Omega)$.

We define weak solutions of (P) as follows.

Definition 1.1. For $T > 0$, (θ,φ) is called a *weak solution* of (P) on $[0,T]$ if (θ,φ) satisfies

$$\theta \in L^2(0,T;H^1(\Omega)),\ \ln\theta \in H^1(0,T;(H^1(\Omega))^*) \cap L^\infty(0,T;L^2(\Omega)),$$
$$\varphi \in W^{2,2}(0,T;L^2(\Omega)) \cap W^{1,\infty}(0,T;L^\infty(\Omega)),$$

$$\langle(\ln\theta)_t, w\rangle_{(H^1(\Omega))^*,H^1(\Omega)} + (\varphi_t, w)_{L^2(\Omega)} + \int_\Omega \nabla\theta\cdot\nabla w = (f,w)_{L^2(\Omega)} \quad \text{a.e. in } (0,T) \ \text{ for all } w \in H^1(\Omega),$$

$$\varphi_{tt} + \varphi_t + a\varphi - J*\varphi + \beta(\varphi) + \pi(\varphi) = \theta \quad \text{a.e. in } \Omega\times(0,T),$$

$$(\ln\theta)(0) = \ln\theta_0,\ \varphi(0) = \varphi_0,\ \varphi_t(0) = v_0 \quad \text{a.e. in } \Omega.$$

The following theorem is concerned with existence of weak solutions to (P).

Theorem 1.1 (K. [**17**])**.** *Assume that* (C1)-(C4) *hold. Then for all* $T > 0$ *there exists a weak solution* (θ,φ) *of* (P) *on* $[0,T]$.

2. $L^2(0,T;L^2(\Omega))$-estimate for $\beta(\varphi)$

In the case that $\lambda = 1$, to obtain the $L^2(0,T;L^2(\Omega))$-estimate for $\beta(\varphi)$ is more difficult compared to the case that $\lambda = 0$. In the case that $\alpha(\theta) = \theta$, $\lambda = 1$, $A\varphi = -\Delta\varphi$, assuming that $|\beta''(r)| \le c(1+|r|)$ for all $r \in \mathbb{R}$, where $c > 0$ is some constant, we can derive the $L^\infty(0,T;L^2(\Omega))$-estimate for $\beta(\varphi)$ by establishing the $L^\infty(0,T;H^1(\Omega))$-estimate for φ and by the continuity of the embedding $H^1(\Omega) \hookrightarrow L^6(\Omega)$. On the other hand, in the case that $\alpha(\theta) = \theta$, $\lambda = 1$, $A\varphi = a\varphi - J * \varphi$, since the regularity of φ is lower compared to the case that $A\varphi = -\Delta\varphi$, it seems to be difficult to obtain the $L^2(0,T;L^2(\Omega))$-estimate for $\beta(\varphi)$ in the same way as in the case that $\alpha(\theta) = \theta$, $\lambda = 1$, $A\varphi = -\Delta\varphi$. In the case that $\alpha(\theta) = \theta$, $\lambda = 1$, $A\varphi = a\varphi - J * \varphi$, assuming that $\varphi_0, v_0 \in L^\infty(\Omega)$, establishing the $L^2(0,T;H^2(\Omega))$-estimate for θ, using the continuity of the embedding $H^2(\Omega) \hookrightarrow L^\infty(\Omega)$, we can derive the $L^\infty(\Omega \times (0,T))$-estimate for φ and then we can obtain the $L^\infty(\Omega \times (0,T))$-estimate for $\beta(\varphi)$ by the continuity of β. However, in the case that $\lambda = 1$, $\alpha(\theta) = \ln\theta$, $A\varphi = a\varphi - J * \varphi$, since the regularity of θ is lower compared to the case that $\alpha(\theta) = \theta$, it seems to be difficult to establish the $L^2(0,T;H^2(\Omega))$-estimate for θ in the same way as in the case that $\alpha(\theta) = \theta$, $\lambda = 1$, $A\varphi = a\varphi - J * \varphi$.

In this paper, it holds that

$$\frac{1}{2}|\varphi(x,t)|^2 = \frac{1}{2}|\varphi_0(x)|^2 + \int_0^t \varphi_t(x,s)\varphi(x,s)\,ds$$

and

$$\begin{aligned}
&\frac{1}{2}|\varphi_t(x,t)|^2 + \int_0^t |\varphi_t(x,s)|^2\,ds + \widehat{\beta}(\varphi(x,t)) \\
&= \int_0^t \theta(x,s)\varphi_t(x,s)\,ds + \frac{1}{2}|v_0(x)|^2 + \widehat{\beta}(\varphi_0(x)) \\
&\quad - \int_0^t (a(x)\varphi(x,s) - (J * \varphi(s))(x) + \pi(\varphi(x,s)))\varphi_t(x,s)\,ds.
\end{aligned}$$

Moreover, since $\theta \in D(\ln) = (0,+\infty)$, we see that

$$\int_0^t \theta(x,s)\varphi_t(x,s)\,ds \le \|\varphi_t\|_{L^\infty(\Omega\times(0,T))} \int_0^t \theta(x,s)\,ds.$$

Thus, deriving the $L^\infty(0,T;H^2(\Omega))$-estimate for $\int_0^t \theta(x,s)\,ds$ from the first equation in (P), using the continuity of the embedding $H^2(\Omega) \hookrightarrow L^\infty(\Omega)$, applying the Young inequality and the Gronwall lemma, we can establish the $L^\infty(\Omega \times (0,T))$-estimate for φ, whence we can obtain the $L^\infty(\Omega \times (0,T))$-estimate for $\beta(\varphi)$ by the continuity of β.

3. Approximation

Even if we consider the approximation

$$\begin{cases} (\ln_\varepsilon(\theta_\varepsilon))_t + (\varphi_\varepsilon)_t - \Delta\theta_\varepsilon = f & \text{in } \Omega\times(0,T), \\ (\varphi_\varepsilon)_{tt} + (\varphi_\varepsilon)_t + a\varphi_\varepsilon - J*\varphi_\varepsilon + \beta(\varphi_\varepsilon) + \pi(\varphi_\varepsilon) = \theta_\varepsilon & \text{in } \Omega\times(0,T), \\ \partial_\nu\theta_\varepsilon = 0 & \text{on } \partial\Omega\times(0,T), \\ (\ln_\varepsilon(\theta_\varepsilon))(0) = \ln_\varepsilon(\theta_0),\ \varphi_\varepsilon(0) = \varphi_0,\ (\varphi_\varepsilon)_t(0) = v_0 & \text{in } \Omega, \end{cases} \tag{P$_\varepsilon$}$$

we do not know whether we can derive a priori estimates for $(\mathrm{P})_\varepsilon$ or not. Here $\ln_\varepsilon$ is the Yosida approximation of ln on $\mathbb{R}$. Although we can obtain that

$$\frac{1}{2}|\varphi_\varepsilon(x,t)|^2 = \frac{1}{2}|\varphi_0(x)|^2 + \int_0^t (\varphi_\varepsilon)_t(x,s)\varphi_\varepsilon(x,s)\,ds$$

and

$$\begin{aligned} &\frac{1}{2}|(\varphi_\varepsilon)_t(x,t)|^2 + \int_0^t |(\varphi_\varepsilon)_t(x,s)|^2\,ds + \widehat{\beta}(\varphi_\varepsilon(x,t)) \\ &= \int_0^t \theta_\varepsilon(x,s)(\varphi_\varepsilon)_t(x,s)\,ds + \frac{1}{2}|v_0(x)|^2 + \widehat{\beta}(\varphi_0(x)) \\ &\quad - \int_0^t (a(x)\varphi_\varepsilon(x,s) - (J*\varphi_\varepsilon(s))(x) + \pi(\varphi_\varepsilon(x,s)))(\varphi_\varepsilon)_t(x,s)\,ds, \end{aligned}$$

since $\theta_\varepsilon > 0$ does not hold, we see that

$$\int_0^t \theta_\varepsilon(x,s)(\varphi_\varepsilon)_t(x,s)\,ds \not\leq \|(\varphi_\varepsilon)_t\|_{L^\infty(\Omega\times(0,T))}\int_0^t \theta_\varepsilon(x,s)\,ds,$$

whence we do not know whether the $L^\infty(\Omega\times(0,T))$-estimates for $\{\varphi_\varepsilon\}_\varepsilon$ and $\{\beta(\varphi_\varepsilon)\}_\varepsilon$ can be established or not.

In this paper, to prove existence for (P), we employ the following time discretization scheme: find $(\theta_{n+1}, \varphi_{n+1})$ such that

$$\begin{cases} \frac{u_{n+1}-u_n}{h} + \frac{\varphi_{n+1}-\varphi_n}{h} - \Delta\theta_{n+1} = f_{n+1} & \text{in } \Omega, \\ \frac{v_{n+1}-v_n}{h} + v_{n+1} + a\varphi_n - J*\varphi_n + \beta(\varphi_{n+1}) + \pi(\varphi_{n+1}) = \theta_{n+1} & \text{in } \Omega, \\ v_{n+1} = \frac{\varphi_{n+1}-\varphi_n}{h} & \text{in } \Omega, \\ \partial_\nu\theta_{n+1} = 0 & \text{on } \partial\Omega \end{cases} \tag{P$_n$}$$

for $n = 0, ..., N-1$, where $h = \frac{T}{N}$, $N \in \mathbb{N}$,

$$u_j := h^{1/2}\theta_j + \ln\theta_j$$

for $j = 0, 1, ..., N$, and $f_k := \frac{1}{h}\int_{(k-1)h}^{kh} f(s)\,ds$ for $k = 1, ..., N$. Indeed, we can prove existence for $(\mathrm{P})_n$.

Theorem 3.1 (K. [**17**]). *Assume that* (C1)-(C4) *hold. Then there exists* $h_0 \in (0,1]$ *such that for all* $h \in (0,h_0)$ *there exists a unique solution of* $(\mathrm{P})_n$ *satisfying*

$$\theta_{n+1} \in H^2(\Omega),\ \theta_{n+1} \in D(\ln) = (0,+\infty)\ \textit{a.e. in}\ \Omega,\ \partial_\nu \theta_{n+1} = 0\ \textit{a.e. on}\ \partial\Omega,$$
$$\varphi_{n+1} \in L^\infty(\Omega) \quad \textit{for}\ n = 0, ..., N-1.$$

The problem $(\mathrm{P})_n$ can be written as

$$\begin{cases} h^{1/2}\theta_{n+1} + \ln\theta_{n+1} - h\Delta\theta_{n+1} \\ \qquad\qquad = -\varphi_{n+1} + hf_{n+1} + \varphi_n + h^{1/2}\theta_n + \ln\theta_n & \text{in } \Omega, \\ \partial_\nu\theta_{n+1} = 0 & \text{on } \partial\Omega, \\ \varphi_{n+1} + h\varphi_{n+1} + h^2\beta(\varphi_{n+1}) + h^2\pi(\varphi_{n+1}) \\ \qquad\qquad = h^2\theta_{n+1} + \varphi_n + hv_n + h\varphi_n - h^2a\varphi_n + h^2J*\varphi_n & \text{in } \Omega. \end{cases} \tag{P$'_n$}$$

Thus, to prove Theorem 3.1, it is enough to derive existence and uniqueness of solutions to $(\mathrm{P})'_n$ in the case that $n = 0$. Then the following two lemmas are necessary:

Lemma 3.2. *For all* $g \in L^2(\Omega)$ *and all* $h > 0$ *there exists a unique function* $\theta \in H^2(\Omega)$ *such that*

$$\theta \in D(\ln)\ \textit{a.e. in}\ \Omega,\ \ \partial_\nu\theta = 0\ \textit{a.e. on}\ \partial\Omega,\ \ h^{1/2}\theta + \ln\theta - h\Delta\theta = g \quad \textit{a.e. in}\ \Omega.$$

Proof. The operator $\mathcal{A} : D(\mathcal{A}) \subset L^2(\Omega) \to L^2(\Omega)$ defined as

$$\mathcal{A}\theta := -h\Delta\theta \quad \text{for } \theta \in D(\mathcal{A}) := \{\theta \in H^2(\Omega) \mid \partial_\nu\theta = 0 \ \text{ a.e. on } \partial\Omega\}$$

is maximal monotone. Also, since $\ln : D(\ln) \subset \mathbb{R} \to \mathbb{R}$ is maximal monotone, we can verify that the operator $\mathcal{B} : D(\mathcal{B}) \subset L^2(\Omega) \to L^2(\Omega)$ defined as

$$\mathcal{B}\theta := \ln\theta \quad \text{for } \theta \in D(\mathcal{B}) := \{\theta \in L^2(\Omega) \mid \theta \in D(\ln) \text{ a.e. in } \Omega,\ \ln\theta \in L^2(\Omega)\}$$

is maximal monotone. Moreover, we see that $1 \in D(\mathcal{A}) \cap D(\mathcal{B}) \neq \emptyset$ and $(\mathcal{A}\theta, \mathcal{B}_\tau\theta)_{L^2(\Omega)} \geq 0$ for all $\theta \in D(\mathcal{A})$ and all $\tau > 0$ by noting that $(-\Delta\theta, \ln_\tau\theta)_{L^2(\Omega)} \geq 0$ for all $\theta \in D(\mathcal{A})$ and all $\tau > 0$ (cf. Okazawa [**20**, Proof of Theorem 3 with $a = b = 0$]), where $\mathcal{B}_\tau$ is the Yosida approximation of $\mathcal{B}$ and $\ln_\tau$ is the Yosida approximation of ln on $\mathbb{R}$. Therefore we can conclude that the operator $\mathcal{A} + \mathcal{B} : D(\mathcal{A}) \cap D(\mathcal{B}) \subset L^2(\Omega) \to L^2(\Omega)$ is maximal monotone (cf. Barbu [**1**, Theoreme II.3.6]). □

Lemma 3.3. *For all* $g \in L^2(\Omega)$ *and all* $h \in (0, \min\{1, 1/\|\pi'\|_{L^\infty(\mathbb{R})}\})$ *there exists a unique function* $\varphi \in L^2(\Omega)$ *satisfying*

$$\varphi + h\varphi + h^2\beta(\varphi) + h^2\pi(\varphi) = g \quad \textit{a.e. in}\ \Omega.$$

Proof. We set the operator $\Phi : D(\Phi) \subset L^2(\Omega) \to L^2(\Omega)$ as

$$\Phi\varphi := h^2\beta(\varphi) \quad \text{for } \varphi \in D(\Phi) := \{\varphi \in L^2(\Omega) \mid \beta(\varphi) \in L^2(\Omega)\}.$$

Then this operator is maximal monotone. Also, we define the operator $\mathcal{L} : L^2(\Omega) \to L^2(\Omega)$ as

$$\mathcal{L}\varphi := h\varphi + h^2\pi(\varphi) \quad \text{for } \varphi \in L^2(\Omega).$$

Then this operator is Lipschitz continuous, monotone for all $h \in (0, \frac{1}{\|\pi'\|_{L^\infty(\mathbb{R})}})$. Therefore the operator $\Phi + \mathcal{L} : D(\Phi) \subset L^2(\Omega) \to L^2(\Omega)$ is maximal monotone (see e.g., [**22**, Lemma IV.2.1 (p.165)]). $\square$

We can show Theorem 3.1 by using Lemmas 3.2, 3.3, the Banach fixed-point theorem and by confirming that $\varphi_1 \in L^\infty(\Omega)$ (for details, see [**17**, Proof of Theorem 1.2]).

4. Outline of the proof of Theorem 1.1

In order to derive existence for (P) by passing to the limit in $(\mathrm{P})_n$ as $h \searrow 0$, we put

$$\widehat{u}_h(t) := u_n + \frac{u_{n+1} - u_n}{h}(t - nh),$$

$$\widehat{\varphi}_h(t) := \varphi_n + \frac{\varphi_{n+1} - \varphi_n}{h}(t - nh),$$

$$\widehat{v}_h(t) := v_n + \frac{v_{n+1} - v_n}{h}(t - nh)$$

for $t \in [nh, (n+1)h]$, $n = 0, ..., N-1$, and

$$\overline{u}_h(t) := u_{n+1},\ \overline{\theta}_h(t) := \theta_{n+1},\ \overline{\varphi}_h(t) := \varphi_{n+1},\ \underline{\varphi}_h(t) := \varphi_n,$$

$$\overline{v}_h(t) := v_{n+1},\ \overline{f}_h(t) := f_{n+1}$$

for $t \in (nh, (n+1)h]$, $n = 0, ..., N-1$, and we rewrite $(\mathrm{P})_n$ as

$$\begin{cases} (\widehat{u}_h)_t + (\widehat{\varphi}_h)_t - \Delta\overline{\theta}_h = \overline{f}_h & \text{in } \Omega \times (0,T), \\ (\widehat{v}_h)_t + \overline{v}_h + a\underline{\varphi}_h - J * \underline{\varphi}_h + \beta(\overline{\varphi}_h) + \pi(\overline{\varphi}_h) = \overline{\theta}_h & \text{in } \Omega \times (0,T), \\ \overline{v}_h = (\widehat{\varphi}_h)_t & \text{in } \Omega \times (0,T), \\ \overline{u}_h = h^{1/2}\overline{\theta}_h + \ln\overline{\theta}_h & \text{in } \Omega \times (0,T), \\ \partial_\nu\overline{\theta}_h = 0 & \text{on } \partial\Omega \times (0,T), \\ \widehat{u}_h(0) = h^{1/2}\theta_0 + \ln\theta_0,\ \widehat{\varphi}_h(0) = \varphi_0,\ \widehat{v}_h(0) = v_0 & \text{in } \Omega. \end{cases} \qquad (\mathrm{P})_h$$

Moreover, to obtain the $L^2(0,T;L^2(\Omega))$-estimate for $\{\beta(\overline{\varphi}_h)\}_h$, the following estimate is necessary (see [**17**, Lemma 4.5]):

Lemma 4.1. *There exists a constant $C > 0$ such that*

$$h \max_{1\le m\le N} \left\| \sum_{n=0}^{m-1} \theta_{n+1} \right\|_{H^2(\Omega)} \le C$$

for sufficiently small $h \in (0,1)$.

There exists a constant $C_1 > 0$ such that

$$\begin{aligned}
&\frac{1}{2}|\varphi_m(x)|^2 + \frac{1}{2}|v_m(x)|^2 + \widehat{\beta}(\varphi_m(x)) \\
&\le \frac{1}{2}\|\varphi_0\|^2_{L^\infty(\Omega)} + \frac{1}{2}\|v_0\|^2_{L^\infty(\Omega)} + \|\widehat{\beta}(\varphi_0)\|_{L^\infty(\Omega)} \\
&\quad + h\sum_{n=0}^{m-1} \theta_{n+1}(x)v_{n+1}(x) + C_1 h \sum_{n=0}^{m-1} \|\varphi_n\|^2_{L^\infty(\Omega)} \\
&\quad + \frac{\|\pi'\|^2_{L^\infty(\mathbb{R})}+1}{2} h \sum_{n=0}^{m-1} \|\varphi_{n+1}\|^2_{L^\infty(\Omega)} + 2h\sum_{n=0}^{m-1} \|v_{n+1}\|^2_{L^\infty(\Omega)} + \frac{|\pi(0)|^2}{2}T
\end{aligned}$$

for sufficiently small $h \in (0,1)$ and for a.a. $x \in \Omega$, $m = 1, ..., N$. On the other hand, since $\theta_j \in D(\ln) = (0,+\infty)$ a.e. in Ω for $j = 0,1,...,N$, it follows from Lemma 4.1 and the continuity of the embedding $H^2(\Omega) \hookrightarrow L^\infty(\Omega)$ that there exists a constant $C_2 > 0$ such that

$$\begin{aligned}
h\sum_{n=0}^{m-1} \theta_{n+1}(x)v_{n+1}(x) &\le h\left(\max_{1\le m\le N} \|v_m\|_{L^\infty(\Omega)}\right) \sum_{n=0}^{m-1} \theta_{n+1}(x) \\
&\le h\left(\max_{1\le m\le N} \|v_m\|_{L^\infty(\Omega)}\right) \left\| \sum_{n=0}^{m-1} \theta_{n+1} \right\|_{L^\infty(\Omega)} \\
&\le C_2 \max_{1\le m\le N} \|v_m\|_{L^\infty(\Omega)} = C_2\|\overline{v}_h\|_{L^\infty(\Omega\times(0,T))}
\end{aligned}$$

for sufficiently small $h \in (0,1)$ and for a.a. $x \in \Omega$, $m = 1, ..., N$. Hence there exists a constant $C_3 > 0$ such that

$$\begin{aligned}
&\|\varphi_m\|^2_{L^\infty(\Omega)} + \|v_m\|^2_{L^\infty(\Omega)} \\
&\le C_3 + C_3\|\overline{v}_h\|_{L^\infty(\Omega\times(0,T))} + C_3 h\sum_{j=0}^{m-1} \|\varphi_j\|^2_{L^\infty(\Omega)} + C_3 h \sum_{j=0}^{m-1} \|v_j\|^2_{L^\infty(\Omega)}
\end{aligned}$$

for sufficiently small $h \in (0,1)$ and $m = 1, ..., N$. Thus it follows from the discrete Gronwall lemma (see e.g., [**14**, Prop. 2.2.1]) that

$$\|\varphi_m\|^2_{L^\infty(\Omega)} + \|v_m\|^2_{L^\infty(\Omega)} \le C_4 + C_4\|\overline{v}_h\|_{L^\infty(\Omega\times(0,T))}$$

for sufficiently small $h \in (0,1)$ and $m = 1, ..., N$ with some $C_4 > 0$. Hence we have that

$$\|\overline{\varphi}_h\|^2_{L^\infty(\Omega\times(0,T))} + \|\overline{v}_h\|^2_{L^\infty(\Omega\times(0,T))} \le C_4 + C_4\|\overline{v}_h\|_{L^\infty(\Omega\times(0,T))}$$
$$\le C_4 + \frac{1}{2}\|\overline{v}_h\|^2_{L^\infty(\Omega\times(0,T))} + \frac{C_4^2}{2}.$$

Therefore we can establish the $L^\infty(\Omega \times (0,T))$-estimate for $\{\overline{\varphi}_h\}_h$ and then we can derive the $L^\infty(\Omega \times (0,T))$-estimate for $\{\beta(\overline{\varphi}_h)\}_h$ by the continuity of β.

To obtain that $\pi(\overline{\varphi}_h) \to \pi(\varphi)$ in $L^2(0,T;L^2(\Omega))$ as $h = h_j \searrow 0$, we need the strong convergence of $\{\widehat{\varphi}_h\}_h$ in $L^2(0,T;L^2(\Omega))$. However, we cannot establish the $L^p(0,T;H^1(\Omega))$-estimate ($1 \le p \le \infty$) for $\{\widehat{\varphi}_h\}_h$ and then we cannot apply the Aubin–Lions lemma for $\{\widehat{\varphi}_h\}_h$ and the compact embedding $H^1(\Omega) \hookrightarrow L^2(\Omega)$. Thus the following lemma are necessary (see [**17**, Lemma 5.1]):

Lemma 4.2. *There exists a constant $C > 0$ depending on the data such that*

$$\|\widehat{\varphi}_h - \widehat{\varphi}_\tau\|_{C([0,T];L^2(\Omega))} + \|\widehat{v}_h - \widehat{v}_\tau\|_{C([0,T];L^2(\Omega))} + \|\overline{v}_h - \overline{v}_\tau\|_{L^2(0,T;L^2(\Omega))}$$
$$\le C(h^{1/2} + \tau^{1/2}) + C\|\widehat{v}_h - \widehat{v}_\tau\|^{1/2}_{C([0,T];(H^1(\Omega))^*)}$$

for sufficiently small $h, \tau \in (0,1)$.

We can prove Theorem 1.1 by establishing some uniform estimates for $(\mathrm{P})_h$, by applying the Aubin–Lions lemma for $\{\widehat{v}_h\}_h$ and compact embedding $L^2(\Omega) \hookrightarrow (H^1(\Omega))^*$, by using Lemma 4.2, by passing to the limit in $(\mathrm{P})_h$ as $h = h_j \searrow 0$ (for details, see [**17**, Proof of Theorem 1.1]).

Remark 4.1. We can show that there exists a weak solution of the nonlocal Penrose–Fife type phase field system with inertial term

$$\begin{cases} \theta_t + \varphi_t - \Delta\left(-\frac{1}{\theta}\right) = f, \ \theta > 0 & \text{in } \Omega \times (0,T), \\ \varphi_{tt} + \varphi_t + a\varphi - J * \varphi + \beta(\varphi) + \pi(\varphi) = -\frac{1}{\theta} & \text{in } \Omega \times (0,T), \\ \partial_\nu\left(-\frac{1}{\theta}\right) + \left(-\frac{1}{\theta}\right) = g_\Gamma & \text{on } \partial\Omega \times (0,T), \\ \theta(0) = \theta_0 \ (> 0), \ \varphi(0) = \varphi_0, \ \varphi_t(0) = v_0 & \text{in } \Omega \end{cases}$$

(see K. [**18**]).

References

[1] V. Barbu, "Nonlinear Semigroups and Differential Equations in Banach spaces", Noordhoff International Publishing, Leyden, 1976.

[2] V. Barbu, P. Colli, G. Gilardi, G. Marinoschi, E. Rocca, *Sliding mode control for a nonlinear phase-field system*, SIAM J. Control Optim. **55** (2017), 2108–2133.

[3] G. Caginalp, *An analysis of a phase field model of a free boundary*, Arch. Rational Mech. Anal. **92** (1986), 205—245.

[4] G. Canevari, P. Colli, *Solvability and asymptotic analysis of a generalization of the Caginalp phase field system*, Commun. Pure Appl. Anal. **11** (2012), 1959–1982.

[5] P. Colli, M. Colturato, *Global existence for a singular phase field system related to a sliding mode control problem*, Nonlinear Anal. Real World Appl. **41** (2018), 128–151.

[6] P. Colli, M. Grasselli, A. Ito, *On a parabolic-hyperbolic Penrose-Fife phase-field system*, Electron. J. Differential Equations 2002, No. 100, 30 pp. (Erratum: Electron. J. Differential Equations 2002, No. 100, 32 pp.).

[7] P. Colli, S. Kurima, *Time discretization of a nonlinear phase-field system in general domains*, Commun. Pure Appl. Anal. **18** (2019) 3161–3179.

[8] P. Colli, Ph. Laurençot, *Weak solutions to the Penrose–Fife phase field model for a class of admissible heat flux laws*, Phys. D **111**, 311–334 (1998).

[9] M. Colturato, *Well-posedness and longtime behavior for a singular phase field system with perturbed phase dynamics*, Evol. Equ. Control Theory **7** (2018), 217–245.

[10] C.M. Elliott, S. Zheng, *Global existence and stability of solutions to the phase-field equations, in "Free Boundary Problems"*, Internat. Ser. Numer. Math. **95**, 46–58, Birkhäuser Verlag, Basel, (1990).

[11] P. Galenko, D. Jou, *Diffuse-interface model for rapid phase transformations in nonequilibrium systems*, Phys. Rev. E, **71** (2005), 046125.

[12] M. Grasselli, H. Petzeltová, G. Schimperna, *A nonlocal phase-field system with inertial term*, Quart. Appl. Math. **65** (2007), 451–469.

[13] A. Ito, N. Kenmochi, *Inertial set for a phase transition model of Penrose-Fife type*, Adv. Math. Sci. Appl. 10, 353–374 (2000) Correction, Adv. Math. Sci. Appl. **11**, 481 (2001).

[14] J.W. Jerome, "Approximations of Nonlinear Evolution Systems", Mathematics in Science and Engineering **164**, Academic Press Inc., Orlando, 1983.

[15] S. Kurima, *Time discretization of an initial value problem for a simultaneous abstract evolution equation applying to parabolic-hyperbolic phase-field systems*, ESAIM Math. Model. Numer. Anal. **54** (2020), 977–1002.

[16] S. Kurima, *Time discretization of a nonlocal phase-field system with inertial term*, Matematiche (Catania) **77** (2022), 47–66.

[17] S. Kurima, *Existence for a singular nonlocal phase field system with inertial term*, Acta Appl. Math., **178** (2022), Paper No. 10, 20 pp.

[18] S. Kurima, *Existence for a nonlocal Penrose–Fife type phase field system with inertial term*, Electron. J. Differential Equations (2023), Paper No. 40, 18 pp.

[19] T. Miyasita, *Global existence and exponential attractor of solutions of Fix-Caginalp equation*, Sci. Math. Jpn. **77** (2015), 339–355.

[20] N. Okazawa, *An application of the perturbation theorem for m-accretive operators*, Proc. Japan Acad. Ser. A Math. Sci. **59** (1983), 88–90.

[21] O. Penrose, P.C. Fife, *Thermodynamically consistent models of phase-field type for the kinetics of phase transitions*, Phys. D **43** (1990), 44–62.

[22] R. E. Showalter, “Monotone Operators in Banach Space and Nonlinear Partial Differential Equations”, Mathematical Surveys and Monographs, **49**, American Mathematical Society, Providence, RI, 1997.

[23] H. Wu, M. Grasselli, S. Zheng, *Convergence to equilibrium for a parabolic-hyperbolic phase-field system with Neumann boundary conditions*, Math. Models Methods Appl. Sci. **17** (2007), 125–153.

[24] H. Wu, M. Grasselli, S. Zheng, *Convergence to equilibrium for a parabolic-hyperbolic phase-field system with dynamical boundary condition*, J. Math. Anal. Appl. **329** (2007), 948-976.

Department of Mathematics
Tokyo University of Science
1-3 Kagurazaka, Shinjuku-ku, Tokyo 162-8601, JAPAN
E–mail address: shunsuke.kurima@gmail.com