Relevances of the Möbius energy to harmonic maps

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Abstract

This study examines the Möbius energies for knots and links. These energies have a Möbius-invariant decomposition, that is, up to constant, they are decomposed into two parts which are invariant under Möbous transformations. The first decomposed energy is reminiscent of the fractional harmonic map, and the second part is just the energy of wave maps. Considering this, this study examines the relevances between the Möbius energy and harmonic maps and proposes a new formulation for the variational problem of Möbius energy.

1 Introduction

The Möboius energy for knots is defined by O'Hara ([10]) as one of the knot energies, and Freedman-He Wang ([4]) discovered its Möbius-invariant property. These studies were conducted during the first half of 1990's. The energy a knot is defined as

$$\mathcal{E}_{\mathrm{kt}}(\boldsymbol{f}) = \iint_{(\mathbb{R}/L\mathbb{Z})^2} \left(\frac{1}{\|\boldsymbol{f}(s_1) - \boldsymbol{f}(s_2)\|_{\mathbb{R}^n}^2} - \frac{1}{\mathscr{D}(\boldsymbol{f}(s_1), \boldsymbol{f}(s_2))^2} \right) ds_1 ds_2,$$

where $\boldsymbol{f} : \mathbb{R}/L\mathbb{Z} \to \mathbb{R}^n$ is the parametrization by the arc-length of a closed curve embedded in $\mathbb{R}^n$. Further, $\mathscr{D}$ is the intrinsic distance on the curve $\mathrm{Im}\boldsymbol{f}$, and the suffix "kt" means "knot".

In the recent decade, the understaning on energy has progressed. In 2012, Blatt ([1]) clarified the proper domain of the energy, that is, the energy of a knot is finite if and only if $\boldsymbol{f}$ is a bi-Lipschitz function that belongs to the Sobolev-Slobodeckii space $W^{\frac{3}{2},2}(\mathbb{R}/L\mathbb{Z})$ with respect to the arc-length parameter. This space is a closed linear subspace of Sobolev space $W^{1,2}(\mathbb{R}/L\mathbb{Z})$ with a finite Gagliardo semi-norm as follows:

$$\left(\iint_{(\mathbb{R}/L\mathbb{Z})^2} \frac{\|\boldsymbol{\tau}(s_1) - \boldsymbol{\tau}(s_2)\|_{\mathbb{R}^n}^2}{\mathscr{D}(\boldsymbol{f}(s_1), \boldsymbol{f}(s_2))^2} \, ds_1 ds_2 \right)^{\frac{1}{2}} < \infty,$$

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where $\boldsymbol{\tau} = \frac{d\boldsymbol{f}}{ds}$. Further, the present author found with Ishizeki that the energy can be decomposed into three Möbius-invariant parts:

$$\mathcal{E}_{\mathrm{kt}}(\boldsymbol{f}) = \mathcal{E}_{\mathrm{kt},1}(\boldsymbol{f}) + \mathcal{E}_{\mathrm{kt},2}(\boldsymbol{f}) + 4, \tag{1.1}$$

where $\mathcal{E}_{\mathrm{kt},1}(\boldsymbol{f})$, $\mathcal{E}_{\mathrm{kt},2}(\boldsymbol{f})$ are defined as

$$\mathcal{E}_{\mathrm{kt},1}(\boldsymbol{f}) = \iint_{(\mathbb{R}/L\mathbb{Z})^2} \frac{\|\Delta\boldsymbol{\tau}\|_{\mathbb{R}^n}^2}{2\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^2} ds_1 ds_2,$$
$$\mathcal{E}_{\mathrm{kt},2}(\boldsymbol{f}) = \iint_{(\mathbb{R}/L\mathbb{Z})^2} \frac{2}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^2} \left\langle \boldsymbol{\tau}(s_1) \wedge \frac{\Delta\boldsymbol{f}}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}}, \boldsymbol{\tau}(s_2) \wedge \frac{\Delta\boldsymbol{f}}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}} \right\rangle_{\bigwedge^2\mathbb{R}^n} ds_1 ds_2$$

in [5, 6, 7]. Here, Δ represents the difference in the functions:

$$\Delta\boldsymbol{f} = \boldsymbol{f}(s_1) - \boldsymbol{f}(s_2), \quad \Delta\boldsymbol{\tau} = \boldsymbol{\tau}(s_1) - \boldsymbol{\tau}(s_2).$$

Geometrically, the first decomposed energy measures the *bending* of the curve, and the second performs the *twisting*.

The squared Gagliardo semi-norm is a quite similar to the first decomposed energy $\mathcal{E}_{\mathrm{kt}}$. The squared Gagliardo semi-norm is an energy of fractional harmonic map from a circle to $\mathbb{S}^{n-1}$. Blatt-Reiter-Schikorra discussed the regularity of a critical point by using a similar argument to that for fractional harmonic maps ([3]). Here, we discuss similarity of the second energy to an energy of harmonic maps from a torus to $\mathbb{S}^{n-1}$. A key finding was presented in a recent study [9], where the Möbius energy $\mathcal{E}_{\mathrm{lk}}$ for 2-component links was considered. Let $(\mathrm{Im}\boldsymbol{f}_1, \mathrm{Im}\boldsymbol{f}_2)$ be a 2-component link, that of, a pair of two curves embedded in $\mathbb{R}^n$ without intersection. Further, let s_i be the arc-length parameter of $\boldsymbol{f}_i$, and let L_i be its total length. The energy is defined by

$$\mathcal{E}_{\mathrm{lk}}(\boldsymbol{f}) = \iint_{(\mathbb{R}/L_1\mathbb{Z})\times(\mathbb{R}/L_2\mathbb{Z})} \frac{1}{\|\boldsymbol{f}_1(s_1) - \boldsymbol{f}_2(s_2)\|_{\mathbb{R}^n}^2} ds_1 ds_2.$$

The suffix "lk" means "link". Based on the Möbius-invariant property of the cross ratio, the energy is also Möbius-invariant. This energy also has a Möbius-invariant decomposition corresponding to the (1.1). In the case, the second energy can be expressed by the Gauss map, which is just the wave map energy. In the case of knots, the second energy can be expressed by the *self*-Gauss map. Considering this, this studt proposes a new formulation to the variational problem for the Möbius energy.

2 Möbius-invariant decomposition and the parallelogram law

It is inefficient to deal with two energies $\mathcal{E}_{\mathrm{kt}}$ and $\mathcal{E}_{\mathrm{lk}}$ separately. In fact, $\mathcal{E}_{\mathrm{lk}}$ can be considered as a regularization of $\mathcal{E}_{\mathrm{kt}}$, as in [9, § 2]. Hence, we have unified the two. Let $\mathrm{Im}\boldsymbol{f}_i$ be a closed curve embedded in $\mathbb{R}^n$ of length L_i, and let s_i be the arc-length parameter. Set

$$\mathcal{E}_0(\boldsymbol{f}_1,\boldsymbol{f}_2)=\iint_{(\mathbb{R}/L_1\mathbb{Z})\times(\mathbb{R}/L_2\mathbb{Z})}\left(\frac{1}{\|\boldsymbol{f}_1(s_1)-\boldsymbol{f}_2(s_2)\|_{\mathbb{R}^n}^2}\right.\\ \left.-\frac{\partial^2}{\partial s_1\partial s_2}\log\|\boldsymbol{f}_1(s_1)-\boldsymbol{f}_2(s_2)\|_{\mathbb{R}^n}\right)ds_1ds_2.$$

It obviously holds that

$$\mathcal{E}_0(\boldsymbol{f}_1,\boldsymbol{f}_2)=\begin{cases}\mathcal{E}_{\mathrm{kt}}(\boldsymbol{f}_1)-4 & \text{when } \boldsymbol{f}_1\equiv\boldsymbol{f}_2,\\ \mathcal{E}_{\mathrm{lk}}(\boldsymbol{f}_1,\boldsymbol{f}_2) & \text{when } \mathrm{Im}\boldsymbol{f}_1\cap\mathrm{Im}\boldsymbol{f}_2=\emptyset.\end{cases}$$

We can simultaneously consider both energies $\mathcal{E}_{\mathrm{kt}}$ and $\mathcal{E}_{\mathrm{lk}}$ by addressing with $\mathcal{E}_0$. First, we state that $\mathcal{E}_0$ has a Möbius invariant decomposition corresponding to (1.1). We introduce a map $\boldsymbol{g}$ defined as

$$\boldsymbol{g}(s_1,s_2)=\frac{\boldsymbol{f}_1(s_1)-\boldsymbol{f}_2(s_2)}{\|\boldsymbol{f}_1(s_1)-\boldsymbol{f}_2(s_2)\|_{\mathbb{R}^n}}.$$

Clearly $\boldsymbol{g}$ maps $(\mathbb{R}/L_1\mathbb{Z})\times(\mathbb{R}/L_2\mathbb{Z})$ into $\mathbb{S}^{n-1}$ in the links. This is called the *Gauss map* in link theory. In the case of knot, that is, $\boldsymbol{f}_1\equiv\boldsymbol{f}_2$, this is defined on $\{(s_1,s_2)\in\mathbb{R}/L_1\mathbb{Z})^2|\,s_1\neq s_2\}$, and call it the *self-Gauss map*. As $\|\boldsymbol{g}\|_{\mathbb{R}^n}$ is always 1 in its domain, $\boldsymbol{g}$ is perpendicular to $\frac{\partial\boldsymbol{g}}{\partial s_i}$. It holds that

$$\boldsymbol{\tau}_1=\frac{d\boldsymbol{f}_1}{ds_1}=\frac{\partial}{\partial s_1}\left(\|\boldsymbol{f}_1-\boldsymbol{f}_2\|\boldsymbol{g}\right)=\left(\frac{\partial}{\partial s_1}\|\boldsymbol{f}_1-\boldsymbol{f}_2\|\right)\boldsymbol{g}+\|\boldsymbol{f}_1-\boldsymbol{f}_2\|\frac{\partial\boldsymbol{g}}{\partial s_1},$$

which is the orthogonal decomposition of $\boldsymbol{\tau}_1$ into $\boldsymbol{g}$ and $\frac{\partial\boldsymbol{g}}{\partial s_i}$ directions. Set

$$\boldsymbol{\tau}_1^*=\left(\frac{\partial}{\partial s_1}\|\boldsymbol{f}_1-\boldsymbol{f}_2\|\right)\boldsymbol{g}-\|\boldsymbol{f}_1-\boldsymbol{f}_2\|\frac{\partial\boldsymbol{g}}{\partial s_1}.$$

This is the unit vector reflecting $\boldsymbol{\tau}_1$ with respect to $\boldsymbol{g}$ direction. This is shown in Figure 1. Similarly, we have

$$\boldsymbol{\tau}_2=-\left(\frac{\partial}{\partial s_2}\|\boldsymbol{f}_1-\boldsymbol{f}_2\|\right)\boldsymbol{g}-\|\boldsymbol{f}_1-\boldsymbol{f}_2\|\frac{\partial\boldsymbol{g}}{\partial s_2},$$

and set

$$\boldsymbol{\tau}_2^*=-\left(\frac{\partial}{\partial s_2}\|\boldsymbol{f}_1-\boldsymbol{f}_2\|\right)\boldsymbol{g}+\|\boldsymbol{f}_1-\boldsymbol{f}_2\|\frac{\partial\boldsymbol{g}}{\partial s_2}.$$

The (self-)Gauss map provides a new interpretation of the decomposition. We set

$$\begin{aligned}\mathscr{M}_0(\boldsymbol{f}_1,\boldsymbol{f}_2)(s_1,s_2)&=\frac{\|\boldsymbol{\tau}_1(s_1)-\boldsymbol{\tau}_2^*(s_1,s_2)\|_{\mathbb{R}^n}^2}{2\|\boldsymbol{f}_1(s_1)-\boldsymbol{f}_2(s_2)\|_{\mathbb{R}^n}^2}=\frac{\|\boldsymbol{\tau}_1^*(s_1,s_2)-\boldsymbol{\tau}_2(s_2)\|_{\mathbb{R}^n}^2}{2\|\boldsymbol{f}_1(s_1)-\boldsymbol{f}_2(s_2)\|_{\mathbb{R}^n}^2},\\ \mathscr{M}_1(\boldsymbol{f}_1,\boldsymbol{f}_2)(s_1,s_2)&=\frac{\|\boldsymbol{\tau}_1(s_1)-\boldsymbol{\tau}_2(s_2)\|_{\mathbb{R}^n}^2}{2\|\boldsymbol{f}_1(s_1)-\boldsymbol{f}_2(s_2)\|_{\mathbb{R}^n}^2}=\frac{\|\boldsymbol{\tau}_1^*(s_1,s_2)-\boldsymbol{\tau}_2^*(s_1,s_2)\|_{\mathbb{R}^n}^2}{2\|\boldsymbol{f}_1(s_1)-\boldsymbol{f}_2(s_2)\|_{\mathbb{R}^n}^2},\\ \mathscr{M}_2(\boldsymbol{f}_1,\boldsymbol{f}_2)(s_1,s_2)&=\frac{(\boldsymbol{\tau}_1(s_1)-\boldsymbol{\tau}_1^*(s_1,s_2))\cdot(\boldsymbol{\tau}_2(s_2)-\boldsymbol{\tau}_2^*(s_1,s_2))}{2\|\boldsymbol{f}_1(s_1)-\boldsymbol{f}_2(s_2)\|_{\mathbb{R}^n}^2}.\end{aligned}$$

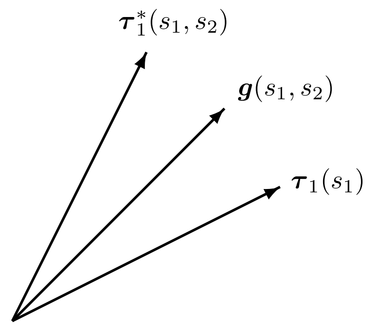

Figure 1: $\boldsymbol{\tau}_1$ and $\boldsymbol{\tau}_1^*$

The following, which is generalization of (1.1), was proven in [9].

Theorem 2.1 *Each of* $\mathscr{M}_i$ *is absolutely integrable on* $(\mathbb{R}/L_1\mathbb{Z}) \times (\mathbb{R}/L_2\mathbb{Z})$. *Thus, the following holds. Set*

$$\mathcal{E}_i(\boldsymbol{f}_1, \boldsymbol{f}_2) = \iint_{(\mathbb{R}/L_1\mathbb{Z})\times(\mathbb{R}/L_2\mathbb{Z})} \mathscr{M}_i(\boldsymbol{f}_1, \boldsymbol{f}_2)\, ds_1 ds_2.$$

Then the decomposition

$$\mathcal{E}_0(\boldsymbol{f}_1, \boldsymbol{f}_2) = \mathcal{E}_1(\boldsymbol{f}_1, \boldsymbol{f}_2) + \mathcal{E}_2(\boldsymbol{f}_1, \boldsymbol{f}_2)$$

holds. Further, all $\mathcal{E}_i$ *values are invariant under Möbius transformations.*

The decomposition follows from

$$\|\boldsymbol{a} - \boldsymbol{d}\|^2 + \|\boldsymbol{b} - \boldsymbol{c}\|^2 = \|\boldsymbol{a} - \boldsymbol{c}\|^2 + \|\boldsymbol{b} - \boldsymbol{d}\|^2 + 2(\boldsymbol{a} - \boldsymbol{b}) \cdot (\boldsymbol{c} - \boldsymbol{d}). \tag{2.1}$$

We insert

$$\boldsymbol{a} = \frac{\boldsymbol{\tau}(s_1)}{2\|\boldsymbol{f}(s_1) - \boldsymbol{f}(s_2)\|_{\mathbb{R}^n}}, \quad \boldsymbol{b} = \frac{\boldsymbol{\tau}^*(s_1, s_2)}{2\|\boldsymbol{f}(s_1) - \boldsymbol{f}(s_2)\|_{\mathbb{R}^n}},$$
$$\boldsymbol{c} = \frac{\boldsymbol{\tau}(s_2)}{2\|\boldsymbol{f}(s_1) - \boldsymbol{f}(s_2)\|_{\mathbb{R}^n}}, \quad \boldsymbol{d} = \frac{\boldsymbol{\tau}^*(s_1, s_2)}{2\|\boldsymbol{f}(s_1) - \boldsymbol{f}(s_2)\|_{\mathbb{R}^n}}$$

into (2.1). Consequently, we obtain

$$\mathscr{M}_0(\boldsymbol{f}) = \mathscr{M}_1(\boldsymbol{f}) + \mathscr{M}_2(\boldsymbol{f}).$$

The relation (2.1) is a variant of

$$\|\boldsymbol{x} + \boldsymbol{y}\|^2 - \|\boldsymbol{x} - \boldsymbol{y}\|^2 = 4\boldsymbol{x} \cdot \boldsymbol{y}, \tag{2.2}$$

which is equivalent to the *parallelogram law*. Using $\boldsymbol{g}$ and its derivatives, the energy densities can be written as

$$\begin{aligned}
\mathscr{M}_0(\boldsymbol{f}_1,\boldsymbol{f}_2) &= \frac{(\boldsymbol{\tau}_1-\boldsymbol{\tau}_2\cdot\boldsymbol{g})^2}{2\|\boldsymbol{f}_1-\boldsymbol{f}_2\|_{\mathbb{R}^n}^2}+\frac{1}{2}\left\|\frac{\partial\boldsymbol{g}}{\partial s_1}-\frac{\partial\boldsymbol{g}}{\partial s_2}\right\|_{\mathbb{R}^n}^2,\\
\mathscr{M}_1(\boldsymbol{f}_1,\boldsymbol{f}_2) &= \frac{(\boldsymbol{\tau}_1-\boldsymbol{\tau}_2\cdot\boldsymbol{g})^2}{2\|\boldsymbol{f}_1-\boldsymbol{f}_2\|_{\mathbb{R}^n}^2}+\frac{1}{2}\left\|\frac{\partial\boldsymbol{g}}{\partial s_1}+\frac{\partial\boldsymbol{g}}{\partial s_2}\right\|_{\mathbb{R}^n}^2,\\
\mathscr{M}_2(\boldsymbol{f}_1,\boldsymbol{f}_2) &= -2\frac{\partial\boldsymbol{g}}{\partial s_1}\cdot\frac{\partial\boldsymbol{g}}{\partial s_2}.
\end{aligned}$$

Indeed, these follows from

$$\frac{\boldsymbol{\tau}_1}{\|\boldsymbol{f}_1-\boldsymbol{f}_2\|_{\mathbb{R}^n}}=\frac{(\boldsymbol{\tau}_1\cdot\boldsymbol{g})\boldsymbol{g}}{\|\boldsymbol{f}_1-\boldsymbol{f}_2\|_{\mathbb{R}^n}}+\frac{\partial\boldsymbol{g}}{\partial s_1},\quad \frac{\boldsymbol{\tau}_2^*}{\|\boldsymbol{f}_1-\boldsymbol{f}_2\|_{\mathbb{R}^n}}=\frac{(\boldsymbol{\tau}_2\cdot\boldsymbol{g})\boldsymbol{g}}{\|\boldsymbol{f}_1-\boldsymbol{f}_2\|_{\mathbb{R}^n}}+\frac{\partial\boldsymbol{g}}{\partial s_2}$$

and the fact that $\boldsymbol{g}$ is perpendicular to its derivatives. This can also be referred to as the parallelogram law as follows. The law gives us

$$\begin{aligned}
\frac{1}{2}\left\|\frac{(\boldsymbol{\tau}_1-\boldsymbol{\tau}_2)\cdot\boldsymbol{g}}{\|\boldsymbol{f}_1-\boldsymbol{f}_2\|_{\mathbb{R}^n}}\boldsymbol{g}+\boldsymbol{o}\right\|_{\mathbb{R}^n}^2 &= \frac{1}{2}\left\|\frac{(\boldsymbol{\tau}_1-\boldsymbol{\tau}_2)\cdot\boldsymbol{g}}{\|\boldsymbol{f}_1-\boldsymbol{f}_2\|_{\mathbb{R}^n}}\boldsymbol{g}-\boldsymbol{o}\right\|_{\mathbb{R}^n}^2+0,\\
\frac{1}{2}\left\|\frac{\partial\boldsymbol{g}}{\partial s_1}-\frac{\partial\boldsymbol{g}}{\partial s_2}\right\|_{\mathbb{R}^n}^2 &= \frac{1}{2}\left\|\frac{\partial\boldsymbol{g}}{\partial s_1}+\frac{\partial\boldsymbol{g}}{\partial s_2}\right\|_{\mathbb{R}^n}^2-2\frac{\partial\boldsymbol{g}}{\partial s_1}\cdot\frac{\partial\boldsymbol{g}}{\partial s_2}.
\end{aligned}$$

The sum of the left-hand rides is $\mathscr{M}_0(\boldsymbol{f}_1,\boldsymbol{f}_2)$, and that of the first terms of the right-hand side is $\mathscr{M}_1(\boldsymbol{f}_1,\boldsymbol{f}_2)$. Simarly the sum of the second terms is $\mathscr{M}_2(\boldsymbol{f}_1,\boldsymbol{f}_2)$.

We set the angles φ_0 and φ_1 as

$$\begin{aligned}
\boldsymbol{\tau}_1(s_1)\cdot\boldsymbol{\tau}_2^*(s_1,s_2)=\boldsymbol{\tau}_1^*(s_1,s_2)\cdot\boldsymbol{\tau}_2(s_2)=\cos\varphi_0(s_1,s_2),\\
\boldsymbol{\tau}_1(s_1)\cdot\boldsymbol{\tau}_2(s_2)=\boldsymbol{\tau}_1^*(s_1,s_2)\cdot\boldsymbol{\tau}_2^*(s_1,s_2)=\cos\varphi_1(s_1,s_2).
\end{aligned}$$

Then, the following can be easily observed:

Corollary 2.1 *It holds that*

$$\begin{aligned}
\mathcal{E}_0(\boldsymbol{f}_1,\boldsymbol{f}_2) &= \iint_{(\mathbb{R}/L_1\mathbb{Z})\times(\mathbb{R}/L_2\mathbb{Z})}\frac{1-\cos\varphi_0(s_1,s_2)}{\|\boldsymbol{f}_1(s_1)-\boldsymbol{f}_2(s_2)\|_{\mathbb{R}^n}^2}ds_1ds_2,\\
\mathcal{E}_1(\boldsymbol{f}_1,\boldsymbol{f}_2) &= \iint_{(\mathbb{R}/L_1\mathbb{Z})\times(\mathbb{R}/L_2\mathbb{Z})}\frac{1-\cos\varphi_1(s_1,s_2)}{\|\boldsymbol{f}_1(s_1)-\boldsymbol{f}_2(s_2)\|_{\mathbb{R}^n}^2}ds_1ds_2,\\
\mathcal{E}_2(\boldsymbol{f}_1,\boldsymbol{f}_2) &= \iint_{(\mathbb{R}/L_1\mathbb{Z})\times(\mathbb{R}/L_2\mathbb{Z})}\frac{\cos\varphi_1(s_1,s_2)-\cos\varphi_0(s_1,s_2)}{\|\boldsymbol{f}_1(s_1)-\boldsymbol{f}_2(s_2)\|_{\mathbb{R}^n}^2}ds_1ds_2
\end{aligned}$$

The statement for $\mathcal{E}_0$ is known as *Doyle-Schuramm's cosine formula*. Here, φ_0 is the comformal angle, which is defined as follows. Let C_{12} be the oriented circle tangent to Im$\boldsymbol{f}$ at $\boldsymbol{f}(s_1)$ and passing through $\boldsymbol{f}(s_2)$, and let C_{21} be the oriented circle tangent to Im$\boldsymbol{f}$ at $\boldsymbol{f}(s_2)$ and passing through $\boldsymbol{f}(s_1)$. The orientation of C_{ij} is that of the tangent vector $\boldsymbol{\tau}(s_j)$ at $\boldsymbol{f}(s_j)$. The angle between C_{12} and C_{12} at $\boldsymbol{f}(s_1)$ (also at $\boldsymbol{f}(s_2)$) is thus obtained as φ_0. This is shown in Figure 2.

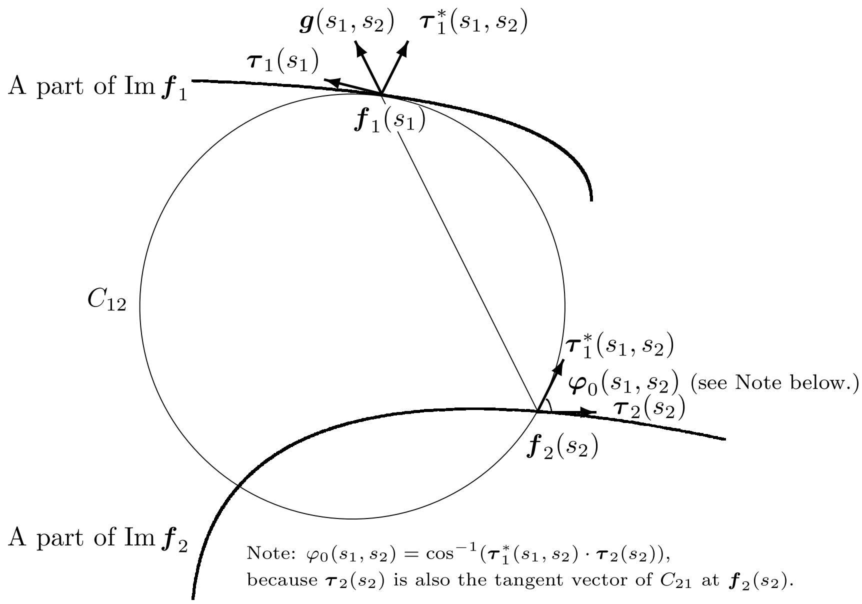

Figure 2: The conformal angle φ_0

3 New formulation of the variational problem

The most important type of study is a variational stufy; however, the variational formula is complex (Ishizeki-Nagasawa [6]). Thus, in this section, we propose the following new formulation for this problem. As discussed in the previous section, the second decomposed energy is expressed as:

$$\mathcal{E}_2(\boldsymbol{f}) = \iint_{(\mathbb{R}/L\mathbb{Z})^2} \frac{(\boldsymbol{\tau}_1 - \boldsymbol{\tau}_1^*)\cdot(\boldsymbol{\tau}_2 - \boldsymbol{\tau}_2^*)}{2\|\boldsymbol{f}_1 - \boldsymbol{f}_2\|_{\mathbb{R}^n}^2}\, ds_1 ds_2, \tag{3.1}$$

and

$$\mathcal{E}_2(\boldsymbol{f}) = -2\iint_{(\mathbb{R}/L\mathbb{Z})^2} \frac{\partial \boldsymbol{g}}{\partial s_1}\cdot\frac{\partial \boldsymbol{g}}{\partial s_2}\, ds_1 ds_2 \tag{3.2}$$

using of $\boldsymbol{g}$. The map $\boldsymbol{g}$ is used to define of $\boldsymbol{\tau}_i^*$; however, it does not explicitly appear in (3.1). Hence, we refer to (3.1) as the *indirect* expression. This expression yields the cosine formula. However, $\boldsymbol{g}$ appears explicitly in (3.2). Therefore, it is called the *direct* expression. This indicates the relationship between the second energy and wave maps.

Here, we consider the Möbius energy for knots. As it is invariant under dilation, the total length is assumed to be 1.

As mentioned in the Introduction, the first energy $\mathcal{E}_1$ is an analog of the fractional harmonic map $\boldsymbol{v} : \mathbb{T} \to \mathbb{S}^{n-1}$. Furthe, $\boldsymbol{g}$ is a map from $\mathbb{T}^2 = (\mathbb{R}/\mathbb{Z})^2$ to $\mathbb{S}^{n-1}$. Now, we define the energy E_2 for these maps as

$$E_2(\boldsymbol{w}) = -2\iint_{\mathbb{T}^2} \frac{\partial \boldsymbol{w}}{\partial s_1}\cdot\frac{\partial \boldsymbol{w}}{\partial s_2}\, ds_1 ds_2.$$

This represents the energy of wave map $\boldsymbol{w} : \mathbb{T}^2_{\pm} \to \mathbb{S}^{n-1}$. Here, $\mathbb{T}^2_{\pm}$ is $\mathbb{T}^2$ equipped with the Lorenz metric.

We set the energy E_1 by

$$E_1(\boldsymbol{v}) = \frac{1}{2} \iint_{\mathbb{T}^2} \frac{\|\boldsymbol{v}(s_1) - \boldsymbol{v}(s_2)\|^2_{\mathbb{R}^n}}{\left\| \int_{s_2}^{s_1} \boldsymbol{v}(s)\, ds \right\|^2_{\mathbb{R}^n}} \, ds_1 ds_2.$$

Then, by the decomposition of $\mathcal{E}_0$,

$$\mathcal{E}_0(\boldsymbol{f}) = E_1(\boldsymbol{\tau}) + E_2(\boldsymbol{g}).$$

However, $\boldsymbol{\tau}$ and $\boldsymbol{g}$ is not independent of each other; thus, the relationship is expressed as

$$\boldsymbol{g}(s_1, s_2) = \frac{\boldsymbol{f}(s_1) - \boldsymbol{f}(s_2)}{\|\boldsymbol{f}(s_1) - \boldsymbol{f}(s_2)\|_{\mathbb{R}^n}} = \frac{\int_{s_2}^{s_1} \boldsymbol{\tau}(s)\, ds}{\left\| \int_{s_2}^{s_1} \boldsymbol{\tau}(s)\, ds \right\|_{\mathbb{R}^n}}.$$

The closedness of $\boldsymbol{f}$ requires

$$\int_{\mathbb{T}} \boldsymbol{\tau}(s)\, ds = \boldsymbol{o}$$

We arrive at the following formulation: Find critical points

$$E(\boldsymbol{v}, \boldsymbol{w}) = E_1(\boldsymbol{v}) + E_2(\boldsymbol{w})$$

in the class

$$\left\{ (\boldsymbol{v}, \boldsymbol{w}) \;\middle|\; \boldsymbol{v} : \mathbb{T} \to \mathbb{S}^{n-1},\ \int_{\mathbb{T}} \boldsymbol{v}\, ds = \boldsymbol{o},\ \boldsymbol{w} : \mathbb{T}^2 \to \mathbb{S}^{n-1} \right\}$$

under the constraint

$$\boldsymbol{w}(s_1, s_2) = \frac{\int_{s_2}^{s_1} \boldsymbol{v}(s)\, ds}{\left\| \int_{s_2}^{s_1} \boldsymbol{v}(s)\, ds \right\|_{\mathbb{R}^n}}.$$

The expected advantages are the following.

- The variational formulae may be easier to handle than the original energy $\mathcal{E}_0$.
- There are certain contributions to topological constraints by use of the self-Gauss map.

The second advantage is as follows. We have a similar decomposition of the Möbius energy for 2-component-links $(\boldsymbol{f}_1, \boldsymbol{f}_2)$. In this case, the Gauss map is

$$\frac{\boldsymbol{f}_1(s_1) - \boldsymbol{f}_2(s_2)}{\|\boldsymbol{f}_1(s_1) - \boldsymbol{f}_2(s_2)\|_{\mathbb{R}^n}}$$

When $n = 3$, the mapping degree of $\boldsymbol{g}$ is the linking number

$$\mathrm{lk}(\boldsymbol{f}_1, \boldsymbol{f}_2) = \mathrm{degree}(\boldsymbol{g}) = \iint \det(\partial_{s_1}\boldsymbol{g}, \partial_{s_2}\boldsymbol{g}, \boldsymbol{g})\, ds_1 ds_2$$

Thus, the Gauss map includes the topological information of the link.

Simultaneously, there are several disadvantages.

- This is a variational problem with a non-local constraint.
- The question is, can we apply enormous results of harmonic maps to our problem ? In other words, even if $(\boldsymbol{v}, \boldsymbol{w})$ is a critical points, then neither $\boldsymbol{v}$ is a critical point of E_1, nor $\boldsymbol{w}$ is that of E_2.

However, the new formulation still provides many ingredient from harmonic map theory to the Möbius theory.

4 Related open problem

We have discussed the Möbius energy for knots and 2-component links. Consider the energy for m-component, where $m \geqq 3$. Freedman-He-Wang in [4] proposed

$$\widetilde{\mathcal{E}}^{(m)}(\boldsymbol{f}_1, \cdots, \boldsymbol{f}_m) = \sum_{1 \leqq i < j \leqq m} \mathcal{E}(\boldsymbol{f}_i, \boldsymbol{f}_j).$$

Because this is the sum energies of 2-component links, it is Möbius invariant and decomposable.

The energy

$$\mathcal{E}^{(m)}(\boldsymbol{f}_1, \cdots, \boldsymbol{f}_m) = \left\{ \int \cdots \int \frac{1}{\prod_{1 \leqq i < j \leqq m} \|\boldsymbol{f}_i(s_i) - \boldsymbol{f}_j(s_j)\|_{\mathbb{R}^n}^{2/(m-1)}}\, ds_1 \cdots ds_m \right\}^{\frac{2}{m}}$$

is also Möbius invariant. This fact follows from the Möbius-invariant property of the cross ratio. However, the decomposability is uncertain.

The Brascamp-Lieb inequality implies that:

$$\mathcal{E}^{(m)}(\boldsymbol{f}_1, \cdots, \boldsymbol{f}_m) \leqq \prod_{1 \leqq i < j \leqq m} \left\{ \mathcal{E}^{(2)}(\boldsymbol{f}_i, \boldsymbol{f}_j) \right\}^{\frac{2}{m(m-1)}} \leqq \widetilde{\mathcal{E}}^{(m)}(\boldsymbol{f}_1, \cdots, \boldsymbol{f}_m)$$

Hence, to study the decomposability, it is reduced to that of

$$\widetilde{\mathcal{E}}^{(m)}(\boldsymbol{f}_1, \cdots, \boldsymbol{f}_m) - \mathcal{E}^{(m)}(\boldsymbol{f}_1, \cdots, \boldsymbol{f}_m).$$

However, this seems to be an open problem.

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