Global existence and large-time behavior of solutions to a 3D-model associated with grain boundary motion

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1 Introduction

In this paper, we consider a three dimensional model for grain boundary motion with time-dependent external forces as follows:

System (*P*)

$$\begin{cases} \partial_t \eta - \Delta \eta + g(\eta) + \alpha'(\eta)|\nabla \mathbf{u}| = f_0 & \text{in } Q := (0,\infty)\times\Omega, \\ \nabla\eta\cdot\mathbf{n}_\Gamma = 0 & \text{on } \Sigma := (0,\infty)\times\partial\Omega, \\ \eta(0,x) = \eta_0(x) & \text{for } x\in\Omega. \end{cases}$$

$$\begin{cases} \partial_t \mathbf{u} = \pi_{\mathbf{u}}\left(\operatorname{div}\left(\alpha(\eta)\dfrac{\nabla\mathbf{u}}{|\nabla\mathbf{u}|} + \kappa^2\nabla\mathbf{u}\right) + \mathbf{f}\right) & \text{in } Q, \\ \left(\alpha(\eta)\frac{\nabla\mathbf{u}}{|\nabla\mathbf{u}|} + \kappa^2\nabla\mathbf{u}\right)\mathbf{n}_\Gamma = \mathbf{0} & \text{on } \Sigma := (0,\infty)\times\partial\Omega, \\ \mathbf{u}(0,x) = \mathbf{u}_0(x) & \text{for } x\in\Omega. \end{cases}$$

Here $N\in\mathbb{N}$, $1<M\in\mathbb{N}$, $\kappa>0$ are fixed constants, $\Omega\subset\mathbb{R}^N$ is a bounded domain with Lipschitz boundary $\partial\Omega$, $\mathbf{n}_\Gamma$ is the unit normal vector on $\partial\Omega$. $[\eta,\mathbf{u}]\in\mathbb{R}\times\mathbb{R}^M$ is a pair of unknown functions to (P). g, α, f_0 are $\mathbb{R}$-valued given functions, $\mathbf{f}$ is an $\mathbb{R}^M$-valued given function. $\pi_{\mathbf{u}}$ is the projection onto the tangent space at $\mathbf{u}\in\mathbb{S}^{M-1}$, i.e.,

$$\pi_{\mathbf{u}}\mathbf{w} := \mathbf{w} - (\mathbf{w}\cdot\mathbf{u})\mathbf{u}, \quad \text{for} \quad \mathbf{w}\in\mathbb{R}^M.$$

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The second equation in (P) is formally derived by the constrained L^2-gradient descent flow:

$$\frac{\partial \mathbf{u}}{\partial t} \in -\pi_{\mathbf{u}}\left(\frac{\delta \mathcal{F}}{\delta \mathbf{u}}(\eta, \mathbf{u}) + \mathbf{f}\right)$$

of the following energy functional $\mathcal{F}$:

Free-energy:

$$\begin{aligned} &[\eta, \mathbf{u}] \in L^2(\Omega) \times L^2(\Omega; \mathbb{R}^M) \mapsto \mathcal{F}(\eta, \mathbf{u}) \\ &:= \begin{cases} \frac{1}{2}\int_\Omega |\nabla \eta|^2\,dx + \int_\Omega G(\eta)\,dx + \int_\Omega \alpha(\eta)|\nabla \mathbf{u}|\,dx + \frac{\kappa^2}{2}\int_\Omega |\nabla \mathbf{u}|^2\,dx, \\ \qquad \text{if } \ \eta \in H^1(\Omega) \text{ and } \mathbf{u} \in H^1(\Omega; \mathbb{R}^M), \\ +\infty, \quad \text{otherwise;} \end{cases} \end{aligned} \tag{1.1}$$

under the range constrained condition $\mathbf{u} \in \mathbb{S}^{M-1}$. Here G is a primitive function of g.

The unknown functions η and $\mathbf{u}$ describe the orientation order and the orientation in a polycrystal, respectively. If $M = 4$, $\mathbf{u}$ is an element of the 3-dimensional sphere $\mathbb{S}^3$ corresponding to a 3-dimensional rotation using the quaternion representation.

In [12], the authors proposed a model for grain boundary motion in the three dimensional case as an extension of the two dimensional Kobayashi-Warren-Carter model [13]. In the model, there are two unknowns: $0 \le \eta \le 1$ and $\mathbf{u}$ representing, respectively, the orientation order (0 being the value for a totally disordered phase and 1 the one for a totally ordered phase) and the orientation, i.e., an element in the space of rotations in $\mathbb{R}^3$: $SO(3)$. Considering a quaternion representation for rotations, and after identifying quaternions with elements in the unit hypersphere $\mathbb{S}^3$ in $\mathbb{R}^4$, we formulated (P) as the constrained L^2-gradient descent flow of the energy functional $\mathcal{F}$ under $\mathbf{u} \in \mathbb{S}^{M-1}$ in [16]. The flow is constrained to the fact that the orientation $\mathbf{u}$ needs to be in the unit hypersphere. In fact, the projection onto the tangent space at $\mathbf{u} \in \mathbb{S}^{M-1}$ in the equation for $\mathbf{u}$ ensures that $|\mathbf{u}(t)| = 1$ if $|\mathbf{u}_0| = 1$.

The problem (P) without forcing terms, Moll–Shirakawa–W. [16] proved the existence of solutions $[\eta, \mathbf{u}]$ on any finite interval $(0, T)$, satisfying $0 \le \eta \le 1$, $|\mathbf{u}| = 1$ a.e. in $\Omega \times (0, T)$. In addition, Moll–Shirakawa–W. [17] considered an external time-dependent forcing $[f_0, \mathbf{f}]$ in the system, for both unknown functions $[\eta, \mathbf{u}]$ and formulated the system as the L^2-gradient descent flow of the free energy with forcing. We note that, due to the presence of an external forcing in the equation for η, the condition $0 \le \eta \le 1$ as in [16, Section 3] cannot be anymore guaranteed. Then, we allow that $\eta \in \mathbb{R}$ and we interpret its values as $\eta \ge 1$ means that the phase is totally ordered and $\eta \le 0$ means a totally disordered phase.

The purpose of this paper is to introduce the results in Moll–Shirakawa–W. [16, 17]. In particular, we provide outline of the proof of global existence of solutions to system (P) and an invariance principle for the solutions. In Section 2, we prepare some mathematical tools. In Section 3, we explain a derivation of the model. In Section 4, we introduce assumptions and the Main results. In Section 5, we prepare an approximating system for (P). We approximate the constraint $|\mathbf{u}| = 1$ via a Ginzburg-Landau type penalization on the free energy (with a parameter δ). Then, we smooth out the singularity of the Euclidean norm $|\cdot|$ with the approximation $\sqrt{\varepsilon^2 + |\cdot|^2}$. Finally, in order to obtain continuity of the solutions for proving an invariance principle, we also add an N-laplacian term to the energy with parameter ν. Existence and uniqueness of solutions to the approximating system is obtained via the use of nonlinear semigroup theory. In Section 6, together with the results in Section 5, we are able to pass to the limit in the weak formulation of the system first with $\delta \to 0^+$, thus recovering the constraint $|\mathbf{u}| = 1$. For the limit with $\nu \to 0$ and $\varepsilon \to 0$, we only show the results. Meanwhile, we obtain an invariance principle for some particular cases of initial data $\mathbf{u}_0$ and forcings $\mathbf{f}$.

2 Preliminaries

2.1 Multi-vectors

In this subsection, we recall some definitions and basic properties about multi-vectors that we need in our analysis. We refer to e.g. [11, Chapter 1] and [8, Chapter 1] for details.

Let $m \in \mathbb{N}$. The spaces $\Lambda_0(\mathbb{R}^m)$ and $\Lambda_1(\mathbb{R}^m)$ are defined as

$$\Lambda_0(\mathbb{R}^m) := \mathbb{R} \quad \text{and} \quad \Lambda_1(\mathbb{R}^m) := \mathbb{R}^m,$$

respectively. For any integer $2 \le k \le m$, the k-th exterior power of $\mathbb{R}^m$, denoted by $\Lambda_k(\mathbb{R}^m)$, is defined as a set spanned by generators; i.e elements of the form

$$\mathbf{u}_1 \wedge \cdots \wedge \mathbf{u}_k, \quad \mathbf{u}_i \in \mathbb{R}^m, \ i = 1, \ldots, k.$$

Generators are subject to the following rules:

$$(a\mathbf{v} + b\mathbf{w}) \wedge \mathbf{u}_2 \wedge \cdots \wedge \mathbf{u}_k = a(\mathbf{v} \wedge \mathbf{u}_2 \wedge \cdots \wedge \mathbf{u}_k) + b(\mathbf{w} \wedge \mathbf{u}_2 \wedge \cdots \wedge \mathbf{u}_k);$$

$$\mathbf{u}_1 \wedge \cdots \wedge \mathbf{u}_k \text{ changes sign if two entries are transposed;}$$

$$\left.\begin{array}{l} \text{for any basis } \{\mathbf{e}_1, \ldots, \mathbf{e}_m\} \text{ of } \mathbb{R}^m \text{, the set} \\ \qquad \left\{ \ \mathbf{e}_\alpha := \mathbf{e}_{\alpha_1} \wedge \cdots \wedge \mathbf{e}_{\alpha_k} \ \middle| \ \alpha = [\alpha_1, \ldots, \alpha_k] \in I(k,m) \ \right\} \\ \text{forms the basis of } \Lambda_k(\mathbb{R}^m) \text{, where} \\ \ I(k,m) := \left\{ \ \alpha = [\alpha_1, \ldots, \alpha_k] \in \mathbb{Z}^k \ \middle| \ 1 \le \alpha_1 < \cdots < \alpha_k \le m \ \right\}. \end{array}\right\} \tag{2.1}$$

The elements of $\Lambda_k(\mathbb{R}^m)$ are called multi-vectors (or k-vectors), and $\Lambda_k(\mathbb{R}^m)$ is a vector space of dimension $\binom{m}{k}$. Given $k, \ell \in \{0, \ldots, m\}$ with $k + \ell \le m$, there exists a unique bilinear map $(\boldsymbol{\lambda}, \boldsymbol{\mu}) \to \boldsymbol{\lambda} \wedge \boldsymbol{\mu}$ from $\Lambda_k(\mathbb{R}^m) \times \Lambda_\ell(\mathbb{R}^m)$ to $\Lambda_{k+\ell}(\mathbb{R}^m)$, whose effect on generators is

$$(\mathbf{u}_1 \wedge \mathbf{u}_2 \wedge \cdots \wedge \mathbf{u}_k) \wedge (\mathbf{v}_1 \wedge \mathbf{v}_2 \wedge \cdots \wedge \mathbf{v}_\ell) = \mathbf{u}_1 \wedge \mathbf{u}_2 \wedge \cdots \wedge \mathbf{u}_k \wedge \mathbf{v}_1 \wedge \mathbf{v}_2 \wedge \cdots \wedge \mathbf{v}_\ell.$$

There is an isomorphism between $\Lambda_k(\mathbb{R}^m)$ and $\Lambda_{m-k}(\mathbb{R}^m)$, called the *Hodge-star operator*:

$$*(\,\cdot\,) : \Lambda_k(\mathbb{R}^m) \ni \boldsymbol{\lambda} \longrightarrow *\boldsymbol{\lambda} \in \lambda_{m-k}(\mathbb{R}^m),$$

which is defined on the basis as

$$\begin{gathered} *(\mathbf{e}_{\alpha_1}\wedge\cdots\wedge\mathbf{e}_{\alpha_k}) := \mathbf{e}_{\alpha_{k+1}}\wedge\cdots\wedge\mathbf{e}_{\alpha_m}, \\ \text{for all permutations } \{\alpha_1,\ldots,\alpha_m\} \text{ of } \{1,\ldots,m\}, \text{ having positive signature.} \end{gathered} \tag{2.2}$$

In particular, in what follows we will systematically identify $\Lambda_{m-1}(\mathbb{R}^m)$ with $\mathbb{R}^m$ and $\Lambda_m(\mathbb{R}^m)$ with $\mathbb{R}$. We will use the following well known formulas (see e.g. [8, (1.64)] and [8, Table 1.2]) :

$$*(*\boldsymbol{\lambda}) = (-1)^{k(m-k)}\boldsymbol{\lambda} \quad \text{for all} \quad \boldsymbol{\lambda}\in\Lambda_k(\mathbb{R}^m), \tag{2.3}$$

and

$$\begin{gathered} \mathbf{a}\wedge*(\mathbf{b}\wedge\mathbf{c}) = (\mathbf{a}\cdot\mathbf{c})*\mathbf{b}-(\mathbf{a}\cdot\mathbf{b})*\mathbf{c} \;\text{ in } \mathbb{R}^m \;(=\Lambda_{m-1}(\mathbb{R}^m)), \\ \text{for all } \;\mathbf{a},\mathbf{b},\mathbf{c}\in\mathbb{R}^m \;(=\Lambda_1(\mathbb{R}^m)). \end{gathered} \tag{2.4}$$

We next introduce the inner product on $\Lambda_k(\mathbb{R}^m)$. Given two generators $\boldsymbol{\lambda} = \boldsymbol{\lambda}_1\wedge\ldots\wedge\boldsymbol{\lambda}_k$, $\boldsymbol{\mu} = \boldsymbol{\mu}_1\wedge\ldots\wedge\boldsymbol{\mu}_k$, we define

$$\langle\boldsymbol{\lambda},\boldsymbol{\mu}\rangle_k := \det\left(\langle\boldsymbol{\lambda}_i,\boldsymbol{\mu}_j\rangle_{i,j=1}^k\right). \tag{2.5}$$

The inner product on $\Lambda_k(\mathbb{R}^m)$ is an extension by linearity of this definition. Then, we easily see that

$$\langle\boldsymbol{\lambda},\boldsymbol{\mu}\rangle_k = \boldsymbol{\lambda}\wedge*\boldsymbol{\mu}. \tag{2.6}$$

Moreover,

$$|\boldsymbol{\lambda}|_k := \langle\boldsymbol{\lambda},\boldsymbol{\lambda}\rangle_k^{\frac12} = \left(\sum_{\alpha\in I(k,m)}|\lambda_\alpha|^2\right)^{\frac12}, \quad \text{where} \quad \boldsymbol{\lambda} = \sum_{\alpha\in I(k,m)}\lambda_\alpha\mathbf{e}_\alpha. \tag{2.7}$$

2.2 Vector valued functions

Let X be a Banach space with dual X' and let $V\subset\mathbb{R}^d$ be a bounded open set endowed with the Lebesgue measure $\mathcal{L}^d$. A function $u : V\to X$ is called *simple* if there exist $x_1,\ldots,x_n\in X$ and $V_1,\ldots,V_n$ $\mathcal{L}^m$-measurable subsets of V such that $u = \sum_{i=1}^n x_i\chi_{V_i}$. The function u is called *strongly measurable* if there exists a sequence of simple functions $\{u_n\}$ such that $\|u_n(x)-u(x)\|_X\to 0$ as $n\to+\infty$ for almost all $x\in V$. If $1\le p<\infty$, then $L^p(V;X)$ stands for the space of (equivalence classes of) strongly measurable functions $u : V\to X$ with

$$\|u\|_{L^p(V;X)} := \left(\int_V\|u(x)\|_X^p\,dx\right)^{\frac1p} < \infty.$$

Endowed with this norm, $L^p(V;X)$ is is a Banach space. For $p=\infty$, the symbol $L^\infty(V;X)$ stands for the space of (equivalence classes of) strongly measurable functions $u : V\to X$ such that

$$\|u\|_{L^\infty(V;X)} := \operatorname{esssup}\{\|u(x)\|_X : \; x\in V\} < \infty.$$

If $V=(0,T)$ with $0<T\le\infty$, we write $L^p(0,T;X) = L^p((0,T);X)$. For $1\le p<\infty$, $L^{p'}(0,T;X')$ ($\frac1p+\frac1{p'}=1$) is isometric to a subspace of $(L^p(0,T;X))'$, with equality if and only if X' has the Radon-Nikodým property (see for instance [9]).

We consider the vector space $\mathcal{D}(V;X) := C_0^\infty(V;X)$, endowed with the topology for which a sequence $\varphi_n\to 0$ as $n\to+\infty$ if there exists $K\subset V$ compact such that

$\mathrm{supp}(\varphi_n) \subset K$ for any $n \in \mathbb{N}$ and $D^\alpha \varphi_n \to 0$ uniformly on K as $n \to +\infty$ for all multi-index α. We denote by $\mathcal{D}'(V;X)$ the space of distributions on V with values in X; that is, the set of all linear continuous maps $T : \mathcal{D}(V;X) \to \mathbb{R}$. As is well known, $L^p(V;X) \subset \mathcal{D}'(V;X)$ through the standard continuous injection. Given $T \in \mathcal{D}'(V;X)$, the distributional derivative of T is defined by

$$\langle D_i T, \varphi\rangle := -\langle T, \partial_i \varphi\rangle \quad \text{for any } \varphi \in \mathcal{D}(V;X) \text{ and any } i \in \{1,\dots,d\}. \tag{2.8}$$

General notations for matrices. Let $d, m \in \mathbb{N}$. If $A = [a_k^\ell] = \left[a_k^\ell\right]_{\substack{1\le \ell\le m\\ 1\le k\le d}} \in \mathbb{R}^{md}$ is an $m \times d$ matrix, we write

$$\begin{cases} \mathbf{a}^\ell = [a_1^\ell, \dots a_d^\ell] \in \mathbb{R}^d & \text{for } \ell = 1,\dots,m,\\ \mathbf{a}_k = {}^{\mathrm{t}}[a_k^1, \dots, a_k^m] \in \mathbb{R}^m & \text{for } k = 1,\dots,d. \end{cases}$$

If $B = [b_k^\ell] = \left[b_k^\ell\right]_{\substack{1\le \ell\le m\\ 1\le k\le d}} \in \mathbb{R}^{md}$ is also an $m \times d$ matrix, we let

$$A : B = \sum_{\ell=1}^m \sum_{k=1}^d a_k^\ell b_k^\ell \quad \text{and} \quad |A| = (A:A)^{\frac{1}{2}} = \left(\sum_{\ell=1}^m \sum_{k=1}^d (a_k^\ell)^2\right)^{\frac{1}{2}}.$$

Given $A = \left[\mathbf{a}_1, \dots, \mathbf{a}_m\right] \in \mathbb{R}^{d\times m}$ with $\mathbf{a}_i \in \mathbb{R}^d$, $i = 1,\dots,m$, and $\mathbf{b} \in \mathbb{R}^d$, we let

$$\begin{aligned} A \wedge \mathbf{b} &:= (\mathbf{a}_1 \wedge \mathbf{b}, \dots, \mathbf{a}_m \wedge \mathbf{b}),\\ *(A \wedge \mathbf{b}) &:= (*(\mathbf{a}_1 \wedge \mathbf{b}), \dots, *(\mathbf{a}_m \wedge \mathbf{b})). \end{aligned}$$

2.3 Multi-vector fields

Let $d, m \in \mathbb{N}$. Let $V \subset \mathbb{R}^d$ be a bounded open set. A multi-vector distribution in U is a linear continuous map $\boldsymbol{\lambda} \in \mathcal{D}'(U;\Lambda_k(\mathbb{R}^m))$ (see §2.2). It may be expressed in terms of the basis (2.1) as

$$\boldsymbol{\lambda} = \sum_{\alpha\in I(k,m)} \lambda_\alpha \mathbf{e}_\alpha, \quad \text{with } \lambda_\alpha \in \mathcal{D}'(V;\mathbb{R}^m) \text{ for any } \alpha \in I(k,m).$$

Then, according to (2.8),

$$D_i\boldsymbol{\lambda} = \sum_{\alpha\in I(k,m)} D_i\lambda_\alpha \mathbf{e}_\alpha \quad \text{for any } i \in \{1,\dots,d\}. \tag{2.9}$$

From (2.9), the following two identities are easily seen to hold for $k, \ell \in \mathbb{N}$ and $i \in \{1,\dots,d\}$:

$$D_i(\boldsymbol{\lambda} \wedge \boldsymbol{\eta}) = D_i\boldsymbol{\lambda} \wedge \boldsymbol{\eta} + \boldsymbol{\lambda} \wedge D_i\boldsymbol{\eta} \tag{2.10}$$

for any $\boldsymbol{\lambda} \in L^2(V;\Lambda_k(\mathbb{R}^m))$ such that $D_i\boldsymbol{\lambda} \in L^2(V;\Lambda_k(\mathbb{R}^m))$ and any $\boldsymbol{\eta} \in L^2(V;\Lambda_\ell(\mathbb{R}^m))$ such that $D_i\boldsymbol{\eta} \in L^2(V;\Lambda_\ell(\mathbb{R}^m))$;

$$*(D_i\boldsymbol{\lambda}) = D_i(*\boldsymbol{\lambda}) \qquad \text{for any } \boldsymbol{\lambda} \in \mathcal{D}'(U;\Lambda_k(\mathbb{R}^m)). \tag{2.11}$$

For any $k \in \mathbb{N}$, $[\Lambda_k(\mathbb{R}^m)]^m$ is a Banach space with the norm

$$\|\mathcal{A}\|_{[\Lambda_k(\mathbb{R}^m)]^m} := \left(\sum_{i=1}^m |\mathcal{A}_i|_k^2\right)^{\frac{1}{2}}, \quad \text{for } \mathcal{A} = (\mathcal{A}_1, \dots, \mathcal{A}_m)$$

with $|\cdot|_k$ given by (2.7).

3 Derivation of the model

In [12] the authors generalized the existing two dimensional model by Kobayashi et al [13, 14] to the case of 3D-crystals. In essence, it consists in the L^2-gradient descent flow of the following energy functional:

$$[\eta, \theta] \in [H^1(\Omega)]^2 \mapsto \frac{1}{2}\int_\Omega |\nabla \eta|^2\, dx + \int_\Omega G(\eta)\, dx + \int_\Omega \alpha(\eta)|\nabla \theta|\, dx + \frac{\kappa^2}{2}\int_\Omega |\nabla \theta|^2\, dx \in [0, \infty].$$

The unknowns $\eta = \eta(t, x)$ and $\theta = \theta(t, x)$ represent, respectively, the "orientation order" and the "orientation angle" in a polycrystal. $\eta = 1$ corresponds to a completely ordered state while $\eta = 0$ corresponds to the state where no meaningful value of mean orientation exists. α is a nonnegative function corresponding to the spatial mobility of grain boundaries while G is a single well potential ensuring that only the ordered state $\eta = 1$ is stable.

In order to generalize the model, one needs to consider orientations in 3D and misorientations, since the term $|\nabla\theta|$ represents the misorientation on a short scale. In 3D case, orientations are elements of $SO(3)$ which is the special orthogonal group in $\mathbb{R}^3$. In [12], the term $|\nabla\theta|$ is substituted by the corresponding Euclidean norm in $\mathbb{R}^9$; i.e. $||\nabla P||_{\mathbb{R}^9} := \left(\sum_{i,j=1}^3 |\nabla p_{i,j}|^2\right)^{\frac{1}{2}}$ for $P = [p_{i,j}]_{i,j}$. Then, one has to compute the gradient descent flow for the constrained energy under $P \in SO(3)$, thus ensuring that the solutions for the orientation variable still belong to $SO(3)$.

In [19, 20], instead, a quaternion representation is used for $SO(3)$. Since quaternions can be identified as elements in the unit sphere $\mathbb{S}^3$ in $\mathbb{R}^4$, the authors replaced the term $|\nabla\theta|$ by the Euclidean norm of the gradient of the quaternion: i.e. $|\nabla \mathbf{q}| := \left(\sum_{i=0}^3 |\nabla q^i|^2\right)^{\frac{1}{2}}$, for $\mathbf{q} = (q^0, q^1, q^2, q^3) \in \mathbb{S}^3$.

We take the point of view of [19, 20] and we consider the energy functional (1.1) constrained to functions with values in the unit sphere $\mathbb{S}^{M-1}$ of $\mathbb{R}^M$ with $1 < M \in \mathbb{N}$. The system (P) has the projection onto the tangent space at $\mathbf{u} \in \mathbb{S}^{M-1}$. Through the following formal calculation, we get the representation of the projection:

$$\begin{aligned} &-\pi_{\mathbf{u}}\left(\frac{\delta}{\delta \mathbf{u}}\int_\Omega \alpha(\eta)|\nabla \mathbf{u}|dx + \mathbf{f}\right) \\ &= \operatorname{div}\left(\alpha(\eta)\frac{\nabla \mathbf{u}}{|\nabla \mathbf{u}|}\right) + \mathbf{f} - \left(\operatorname{div}\left(\alpha(\eta)\frac{\nabla \mathbf{u}}{|\nabla \mathbf{u}|}\right)\cdot \mathbf{u}\right)\mathbf{u} - (\mathbf{f}\cdot\mathbf{u})\mathbf{u} \\ &= \operatorname{div}\left(\alpha(\eta)\frac{\nabla \mathbf{u}}{|\nabla \mathbf{u}|}\right) + \mathbf{f} + \alpha(\eta)|\nabla \mathbf{u}|\mathbf{u} - (\mathbf{f}\cdot\mathbf{u})\mathbf{u}. \end{aligned}$$

Here, we computed that

$$\operatorname{div}\left(\alpha(\eta)\frac{\nabla \mathbf{u}}{|\nabla \mathbf{u}|}\right)\cdot \mathbf{u} = \operatorname{div}\left(\alpha(\eta)\frac{{}^t\nabla \mathbf{u}}{|\nabla \mathbf{u}|}\mathbf{u}\right) - \alpha(\eta)\frac{\nabla \mathbf{u}}{|\nabla \mathbf{u}|} : \nabla \mathbf{u}.$$

Moreover, we note that $({}^t\nabla \mathbf{u})\mathbf{u} = 0$ by $\mathbf{u} \in \mathbb{S}^{M-1}$, and $\nabla \mathbf{u} : \nabla \mathbf{u} = |\nabla \mathbf{u}|^2$.

4 Assumptions and Main Theorems

We start with setting up the assumptions in the principal part of this paper. The assumptions also fix the notations in the system (P), and in its approximating problems.

(A0) $1 < N \in \mathbb{N}$, $1 < M \in \mathbb{N}$, and $\kappa > 0$.

(A1) $\Omega \subset \mathbb{R}^N$ is a bounded domain with Lipschitz boundary $\Gamma := \partial\Omega$ and unit outer normal $\mathbf{n}_\Gamma$. We use the following notation:

$$H := L^2(\Omega),\ \mathbb{X} := L^2(\Omega; \mathbb{R}^M),\ \mathfrak{X} := H \times \mathbb{X},$$

$$V := H^1(\Omega),\ \mathbb{W} := H^1(\Omega; \mathbb{R}^M),\ \mathfrak{W} := V \times \mathbb{W}.$$

(A2) $g \in C^1(\mathbb{R})$ is a fixed Lipschitz function such that g has a potential $0 \leq G \in C^2(\mathbb{R})$, i.e. $G'(s) = \frac{d}{ds}G(s) = g(s)$ on $\mathbb{R}$. Moreover, g satisfies that

$$\liminf_{s \to -\infty} g(s) = -\infty \quad \text{and} \quad \limsup_{s \to \infty} g(s) = \infty.$$

(A3) $0 < \alpha \in C^2(\mathbb{R})$ is such that

- $\alpha'(0) = 0$, $\quad \alpha'' \geqq 0$ on $\mathbb{R}$, and α and $\alpha\alpha'$ are Lipschitz continuous on $\mathbb{R}$.
- $\alpha^* := \inf \alpha(\mathbb{R}) > 0$.

(A4) For any $\varepsilon \geq 0$, $\gamma_\varepsilon : \mathbb{R}^{MN} \longrightarrow [0, \infty)$ is a continuous convex function, defined as

$$\gamma_\varepsilon : W = \left[w_k^\ell\right]_{\substack{1 \leq \ell \leq M \\ 1 \leq k \leq N}} \in \mathbb{R}^{MN} \mapsto \gamma_\varepsilon(W) := \sqrt{\varepsilon^2 + |W|^2} \in \mathbb{R}.$$

(A5) For any $\delta > 0$, $\Pi_\delta \in C^2(\mathbb{R}^M)$ is the following function:

$$\Pi_\delta : \mathbf{w} \in \mathbb{R}^M \mapsto \Pi_\delta(\mathbf{w}) := \frac{1}{4\delta}(|\mathbf{w}|^2 - 1)^2 \in \mathbb{R}.$$

We let $\varpi_\delta \in C^1(\mathbb{R}^M; \mathbb{R}^M)$ be the gradient of Π_δ, i.e.:

$$\varpi_\delta : \mathbf{w} \in \mathbb{R}^M \mapsto \varpi_\delta(\mathbf{w}) := \nabla \Pi_\delta(\mathbf{w}) = \frac{1}{\delta}\big(|\mathbf{w}|^2 - 1\big)\mathbf{w} \in \mathbb{R}^M.$$

(A6) $\mathfrak{f} := [f_0, \mathbf{f}] \in L^2_{\mathrm{loc}}([0, \infty); \mathfrak{X})$ is a fixed pair of forcings with $f_0 \in L^\infty(Q)$ and $\mathbf{f} = [f_1, \dots, f_M] \in L^2_{\mathrm{loc}}([0, \infty); \mathbb{X})$.

(A7) $U_0 := [\eta_0, \mathbf{u}_0] \in L^\infty(\Omega) \times \mathbb{X}$ is a fixed pair of initial data.

Remark 4.1. From its definition, it immediately follows that $\gamma_\varepsilon : \mathbb{R}^{MN} \longrightarrow [0, \infty)$, $\varepsilon \geq 0$, are non-expansive over $\mathbb{R}^{MN}$. Also, if $\varepsilon > 0$, then $\gamma_\varepsilon \in C^\infty(\mathbb{R}^{MN})$, and if $\varepsilon = 0$, then the corresponding function γ_0 coincides with the (Euclidean) norm $\|\cdot\|_{\mathbb{R}^{MN}}$ on $\mathbb{R}^{MN}$. Additionally,

$$\partial\gamma_\varepsilon(W) = \begin{cases} \left\{ \dfrac{W}{\sqrt{\varepsilon^2 + |W|^2}} \right\} \ \big(= [\nabla\gamma_\varepsilon](W)\big), & \text{if } \varepsilon > 0, \\ \mathrm{Sgn}^{M,N}(W), & \text{if } \varepsilon = 0, \end{cases} \quad \text{for all } W = \left[w_k^\ell\right]_{\substack{1 \leq \ell \leq M \\ 1 \leq k \leq N}} \in \mathbb{R}^{MN},$$

where $\mathrm{Sgn}^{M,N}:\mathbb{R}^{MN}\longrightarrow 2^{\mathbb{R}^{MN}}$ is the *sign function* on $\mathbb{R}^{MN}$, i.e.

$$\mathrm{Sgn}^{M,N}(W):=\begin{cases}\left\{\dfrac{W}{|W|}\right\}, & \text{if } W\neq 0,\\ \left\{\ \tilde{W}\ \middle|\ \tilde{W}:\tilde{W}\leq 1\ \right\}, & \text{if } W=0,\end{cases}\quad \text{for all } W=\left[w_k^\ell\right]_{\substack{1\leq \ell\leq M\\1\leq k\leq N}}\in\mathbb{R}^{MN}.$$

Next, we define the notion of solutions to our system (P).

Definition 4.1. A pair of functions $U:=[\eta,\mathbf{u}]\in L^2_{\mathrm{loc}}([0,\infty);\mathfrak{X})$ is called a solution to the system (P), if

$$\begin{cases}U=[\eta,\mathbf{u}]\in W^{1,2}_{\mathrm{loc}}([0,\infty);\mathfrak{X})\cap L^\infty_{\mathrm{loc}}(0,\infty;\mathfrak{W}),\\ \eta\in L^\infty(Q) \text{ and } \mathbf{u}\in\mathbb{S}^{M-1},\ \text{a.e. in } Q,\end{cases}\tag{4.1}$$

$$\begin{gathered}\left(\partial_t\eta(t)+g(\eta(t))+\alpha'(\eta(t))|\nabla\mathbf{u}(t)|,\varphi\right)_H+\left(\nabla\eta(t),\nabla\varphi\right)_H=\left(f_0(t),\varphi\right)_H,\\ \text{for any } \varphi\in V,\ \text{a.e. } t>0,\ \text{ subject to } \eta(0)=\eta_0 \text{ in } H;\end{gathered}\tag{4.2}$$

and there exist functions $\mathcal{B}\in L^\infty(Q;\mathbb{R}^{MN})$ and $\mu\in L^1_{\mathrm{loc}}([0,\infty);L^1(\Omega))$, such that

$$\begin{cases}\mathcal{B}\in\mathrm{Sgn}^{M,N}(\nabla\mathbf{u}) \text{ in } \mathbb{R}^{MN},\\ \mu:=(\alpha(\eta)\mathcal{B}+\kappa^2\nabla\mathbf{u}):\nabla\mathbf{u}=\alpha(\eta)|\nabla\mathbf{u}|+\kappa^2|\nabla\mathbf{u}|^2,\end{cases}\quad\text{a.e. in } Q,\tag{4.3a}$$

and

$$\begin{aligned}\int_\Omega\partial_t\mathbf{u}(t)\cdot\boldsymbol{\psi}\,dx+\int_\Omega(\alpha(\eta(t))\mathcal{B}(t)+\kappa^2\nabla\mathbf{u}(t)):\nabla\boldsymbol{\psi}\,dx=\int_\Omega\mu(t)\mathbf{u}(t)\cdot\boldsymbol{\psi}\,dx\\ +\int_\Omega\left(\mathbf{f}(t)-(\mathbf{f}\cdot\mathbf{u})(t)\mathbf{u}(t)\right)\cdot\boldsymbol{\psi}\,dx\ \text{ for any } \boldsymbol{\psi}\in C^1(\overline{\Omega};\mathbb{R}^M),\\ \text{a.e. } t>0,\ \text{ subject to } \mathbf{u}(0)=\mathbf{u}_0 \text{ in } \mathbb{X}.\end{aligned}\tag{4.3b}$$

Next, we state the main results of this paper.

Main Theorem 1. *Assume (A0)–(A7), and that $U_0=[\eta_0,\mathbf{u}_0]\in\mathfrak{W}$ with $\mathbf{u}_0\in\mathbb{S}^{M-1}$ in Ω. Then, the system (P) admits at least one solution $U=[\eta,\mathbf{u}]\in L^2_{\mathrm{loc}}([0,+\infty);\mathfrak{X})$, such that*

$$\mathcal{F}(U(T))+\frac{1}{2}\int_0^T\|\partial_tU(t)\|^2_{\mathfrak{X}}\,dt\leq\mathcal{F}(U_0)+\frac{1}{2}\int_0^T\|\mathfrak{f}(t)\|^2_{\mathfrak{X}}\,dt,\quad \textit{for all } T\geq 0,$$

where $\mathcal{F}$ is the energy given by (1.1). Also, concerning the function $\mathcal{B}^*\in L^\infty(Q;\mathbb{R}^{MN})$ in (4.3), it holds that

$$\begin{cases}\mathrm{div}\left(\alpha(\eta)\mathcal{B}^*+\kappa^2\nabla\mathbf{u}\right)\in L^2_{\mathrm{loc}}([0,\infty);L^1(\Omega;\mathbb{R}^M)),\\ \mathrm{div}\left(\alpha(\eta)\mathcal{B}^*+\kappa^2\nabla\mathbf{u}\right)\wedge\mathbf{u}\in L^2_{\mathrm{loc}}([0,\infty);L^2(\Omega;\Lambda_2(\mathbb{R}^M))).\end{cases}$$

Moreover, if $\mathfrak{f}=[f_0,\mathbf{f}]\in L^\infty(Q)\times[L^\infty(Q)]^M$ and there exists $\mathbf{p}_0\in\mathbb{S}^{M-1}$ such that $\mathbf{u}_0\in\overline{B_g(\mathbf{p}_0;R)}$, with $R<\frac{\pi}{2}$ and that $\frac{\mathbf{f}}{|\mathbf{f}|}\in\overline{B_g(\mathbf{p}_0;R)}$, $|\mathbf{f}|$-a.e., then

$$\mathbf{u}\in\overline{B_g(\mathbf{p}_0;R)}\,,\quad\text{a.e. in } \Omega\,,\ \text{ for all } t\in[0,+\infty).\tag{4.4}$$

Here, $B_g(\mathbf{p}_0;R)$ is the open ball on $\mathbb{S}^{M-1}$ of centre $\mathbf{p}_0$ and radius R.

To state next result, we define the *ω-limit set* $\omega(U)$ of a solution $U = [\eta, \mathbf{u}]$ given by Main Theorem 1, as follows:

$$\omega(U) := \left\{ \bar{U} = [\bar{\eta}, \bar{\mathbf{u}}] \in \mathfrak{W} \;\middle|\; \begin{array}{l} \text{there exists a sequence } \{t_n\}_{n=1}^{\infty} \subset (0, \infty), \text{ such that } t_n \uparrow \\ \infty, \text{ and } U(t_n) = [\eta(t_n), \mathbf{u}(t_n)] \to \bar{U} = [\bar{\eta}, \bar{\mathbf{u}}] \text{ in } \mathfrak{X}, \text{ as} \\ n \to \infty. \end{array} \right\}.$$

Main Theorem 2. *In addition to the assumptions in Main Theorem 1, let us assume the following condition:*

(A8) *There exists a pair of function* $\mathfrak{f}^\infty = [f_0^\infty, \mathbf{f}^\infty] \in \mathfrak{X}$ *with* $\mathbf{f}^\infty = [f_1^\infty, \dots, f_M^\infty] \in \mathbb{X}$, *such that* $\mathfrak{f} - \mathfrak{f}^\infty \in L^2(0, \infty; \mathfrak{X})$.

Then, the ω-limit set $\omega(U)$ *is nonempty and compact in* $\mathfrak{X}$, *and moreover, any* ω-limit point $U^\infty = [\eta^\infty, \mathbf{u}^\infty] \in \omega(U)$ *solves the following variational system:*

$$\big(g(\eta^\infty) + \alpha'(\eta^\infty)|\nabla \mathbf{u}^\infty|, \varphi\big)_H + \big(\nabla \eta^\infty, \nabla \varphi\big)_H = \big(f_0^\infty, \varphi\big)_H, \text{ for any } \varphi \in V;$$

$$\int_\Omega (\alpha(\eta^\infty)\mathcal{B}^\infty + \kappa^2 \nabla \mathbf{u}^\infty) : \nabla \boldsymbol{\psi}\, dx = \int_\Omega \mu^\infty \mathbf{u}^\infty \cdot \boldsymbol{\psi}\, dx + \int_\Omega \big(\mathbf{f}^\infty - (\mathbf{f}^\infty \cdot \mathbf{u}^\infty)\mathbf{u}^\infty\big) \cdot \boldsymbol{\psi}\, dx, \text{ for any } \boldsymbol{\psi} \in C^1(\overline{\Omega}; \mathbb{R}^M), \tag{4.5a}$$

with $\mathcal{B}^\infty \in L^\infty(\Omega; \mathbb{R}^{MN})$ *and* $\mu^\infty \in L^1(\Omega)$, *fulfilling*

$$\begin{cases} \mathcal{B}^\infty \in \mathrm{Sgn}^{M,N}(\nabla \mathbf{u}^\infty) \text{ in } \mathbb{R}^{MN}, \\ \mu^\infty := (\alpha(\eta)\mathcal{B}^\infty + \kappa^2 \nabla \mathbf{u}^\infty) : \nabla \mathbf{u}^\infty, \end{cases} \quad \text{a.e. in } \Omega. \tag{4.5b}$$

In the case of no forcing in the orientation variable, and under a smallness assumption on the datum $\mathbf{u}_0$, we can characterize the ω-limit set.

Main Theorem 3. ***(Large-time behavior in the homogeneous case)*** *In addition to the assumptions as in Main Theorem 1, we assume that* $\mathbf{f} \equiv 0$ *and that there exists* $\mathbf{p}_0 \in \mathbb{S}^{M-1}$ *such that* $\mathbf{u}_0 \in \overline{B_g(\mathbf{p}_0; R)}$, *with* $0 \le R < \frac{\pi}{4}$. *Then, it holds that, there exists* $T^* \in [0, +\infty)$ *such that*

$$\int_\Omega \mathrm{dist}_g(\mathbf{u}(t,x), \mathbf{p}_c(t))^2\, dx \to 0, \text{ as } t \uparrow T^*,$$

where $\mathbf{p}_c(t)$ *is the Barycenter of* $\mu := \mathbf{u}(t)\sharp\mathcal{L}^N$, *i.e. the minimizer of*

$$\mathbf{p} \in \mathbb{S}^{M-1} \mapsto \Psi_\mu(\mathbf{p}) := \frac{1}{2} \int_{\mathbb{S}^{M-1}} \mathrm{dist}_g(\cdot, \mathbf{p})^2 d\mu.$$

In [1, Theorem 2.1.] it is proved that a unique center of mass exists for any Radon measure on $\overline{B_g(\mathbf{p}_0, R)}$ for $R < \frac{\pi}{2}$ and we have

$$0 = \mathrm{d}\Psi_\mu(\mathbf{p}_c) = \int_{\overline{B_g(\mathbf{p}_0, R)}} \exp_{\mathbf{p}_c}^{-1} d\mu, \tag{4.6}$$

where $\exp_{\mathbf{p}_c}^{-1} \colon B_g(\mathbf{p}_c, \frac{\pi}{2}) \to T_{\mathbf{p_c}}\mathbb{S}^{M-1}$ denotes the logarithmic map at $\mathbf{p_c}$.

Remark 4.2. Taking into account Main Theorems 2 and 3, we can say that if $\mathbf{f} \equiv 0$, then the component $\mathbf{u}^\infty$ of any ω-limit point is constant over Ω and that $\mathbf{u}^\infty$ is reached the constant in finite time in case the range of the initial datum $\mathbf{u}_0$, possibly after a rotation, is in the open hyperoctant of $\mathbb{S}^{M-1}$.

In this paper, we show outline of the proof of Main Theorem 1 only. For the proofs of Main Theorem 2 and 3, please see Moll-Shirakawa-W. [17].

5 Approximating problem

In this section, we consider the approximating problem to our system (P). Let us assume (A0)–(A7), and fix constants $\varepsilon \geq 0$, $\delta \geq 0$, and $\nu \geq 0$. On this basis, the approximating problem consists in the following system of parabolic PDEs:

Problem $(\mathrm{P})^\kappa_{\varepsilon,\nu,\delta}$

$$\begin{cases} \partial_t \eta - \Delta \eta + g(\eta) + \alpha'(\eta)\gamma_\varepsilon(\nabla \mathbf{u}) = f_0 \text{ in } Q, \\ \nabla \eta \cdot \mathbf{n}_\Gamma = 0 \text{ on } \Sigma, \\ \eta(0,x) = \eta_0(x), \ x \in \Omega. \end{cases}$$

$$\begin{cases} \partial_t \mathbf{u} - \mathrm{div}\big(\alpha(\eta)\partial\gamma_\varepsilon(\nabla \mathbf{u}) + \kappa^2 \nabla \mathbf{u} + \nu|\nu\nabla\mathbf{u}|^{N-1}\nu\nabla\mathbf{u}\big) + \varpi_\delta(\mathbf{u}) \ni \mathbf{f} \text{ in } Q, \\ \big(\alpha(\eta)\partial\gamma_\varepsilon(\nabla \mathbf{u}) + \kappa^2 \nabla \mathbf{u} + \nu|\nu\nabla\mathbf{u}|^{N-1}\nu\nabla\mathbf{u}\big)\mathbf{n}_\Gamma \ni 0 \text{ on } \Sigma, \\ \mathbf{u}(0,x) = \mathbf{u}_0(x), \ x \in \Omega. \end{cases}$$

The approximating problem $(\mathrm{P})^\kappa_{\varepsilon,\nu,\delta}$ is derived as a gradient descent flow of a free energy, which is defined as:

$$\begin{aligned} \mathcal{F}^\kappa_{\varepsilon,\nu,\delta} : U := [\eta, \mathbf{u}] \in \mathfrak{X} \mapsto \mathcal{F}^\kappa_{\varepsilon,\nu,\delta}(U) &= \mathcal{F}^\kappa_{\varepsilon,\nu,\delta}(\eta, \mathbf{u}) \\ := \Psi_0(\eta) + \Psi^\kappa_{\varepsilon,\nu,\delta}(U) &= \Psi_0(\eta) + \Psi^\kappa_{\varepsilon,\nu,\delta}(\eta, \mathbf{u}), \end{aligned}$$

with

$$\Psi_0 : \ \eta \in D(\Psi_0) := V \subset H \mapsto \Psi_0(\eta) := \frac{1}{2}\int_\Omega |\nabla \eta|^2 \, dx + \int_\Omega G(\eta)\, dx,$$

and

$$\begin{aligned} \Psi^\kappa_{\varepsilon,\nu,\delta} : U := [\eta, \mathbf{u}] \in D(\Psi^\kappa_{\varepsilon,\nu,\delta}) := \left\{ [\tilde{\eta}, \tilde{\mathbf{u}}] \in \mathfrak{W} \ \middle| \ \begin{matrix} \tilde{\mathbf{u}} \in L^4(\Omega; \mathbb{R}^M), \text{ and} \\ \nu\tilde{\mathbf{u}} \in W^{1,N+1}(\Omega; \mathbb{R}^M) \end{matrix} \right\} \\ \mapsto \Psi^\kappa_{\varepsilon,\nu,\delta}(U) = \Psi^\kappa_{\varepsilon,\nu,\delta}(\eta, \mathbf{u}) := \int_\Omega \alpha(\eta)\gamma_\varepsilon(\nabla \mathbf{u})\, dx + \frac{\kappa^2}{2}\int_\Omega |\nabla \mathbf{u}|^2 \, dx \\ + \frac{1}{N+1}\int_\Omega |\nu\nabla\mathbf{u}|^{N+1}\, dx + \int_\Omega \Pi_\delta(\mathbf{u})\, dx \in [0, \infty). \end{aligned}$$

Definition 5.1. A pair of functions $U := [\eta, \mathbf{u}] \in L^2_{\mathrm{loc}}([0,\infty); \mathfrak{X})$ is called a solution to the system $(\mathrm{P})^\kappa_{\varepsilon,\nu,\delta}$, if and only if

$$U = [\eta, \mathbf{u}] \in W^{1,2}_{\mathrm{loc}}([0,\infty); \mathfrak{X}) \cap L^\infty_{\mathrm{loc}}(0,\infty; \mathfrak{W}); \tag{5.1}$$

$$\big(\partial_t\eta(t) + g(\eta(t)) + \alpha'(\eta(t))\gamma_\varepsilon(\nabla \mathbf{u}(t)), \varphi\big)_H + \big(\nabla\eta(t), \nabla\varphi\big)_H = (f_0(t), \varphi)_H, \tag{5.2}$$
$$\text{for any } \varphi \in V, \text{ a.e. } t > 0, \text{ subject to } \eta(0) = \eta_0 \text{ in } H;$$

and there exists $\mathcal{B}^* \in L^\infty(Q; \mathbb{R}^{MN})$, such that

$$\mathcal{B}^* \in \partial\gamma_\varepsilon(\nabla \mathbf{u}) \text{ in } \mathbb{R}^{MN}, \text{ a.e. in } Q,$$

$$\begin{aligned}
&\left(\partial_t \mathbf{u}(t) - \frac{1}{\delta}\mathbf{u}(t), \boldsymbol{\psi}\right)_{\mathbb{X}} + \int_\Omega \big(\alpha(\eta(t))\mathcal{B}^*(t) + \kappa^2 \nabla \mathbf{u}\big) : \nabla\boldsymbol{\psi}\, dx \\
&\quad + \nu \int_\Omega |\nu\nabla \mathbf{u}(t)|^{N-1}\nu\nabla\mathbf{u}(t) : \nabla\boldsymbol{\psi}\, dx + \frac{1}{\delta}\int_\Omega |\mathbf{u}(t)|^2\mathbf{u}(t)\cdot\boldsymbol{\psi}\, dx = \big(\mathbf{f}(t), \boldsymbol{\psi}\big)_{\mathbb{X}},
\end{aligned}$$
$$\text{for any } \boldsymbol{\psi} \in W^{1,N+1}(\Omega; \mathbb{R}^M), \text{ a.e. } t > 0, \text{ subject to } \mathbf{u}(0) = \mathbf{u}_0 \text{ in } \mathbb{X}.$$

Additionally, we note that our approximating system $(\mathrm{P})^\kappa_{\varepsilon,\nu,\delta}$ can be reformulated as the following Cauchy problem of evolution equations in the Hilbert space $\mathfrak{X}$:

Cauchy problem $(\mathbf{CP})^{\boldsymbol{\kappa}}_{\boldsymbol{\varepsilon,\nu,\delta}}$

$$\begin{cases} U'(t) + \partial\Phi^\kappa_{\varepsilon,\nu,\delta}(U(t)) + \mathcal{G}^\kappa_\delta(U(t)) \ni \mathfrak{f}(t) \text{ in } \mathfrak{X}, \quad t > 0, \\ U(0) = U_0 \text{ in } \mathfrak{X}. \end{cases}$$

In this context, "$\,'\,$" denotes the time-derivative "$\frac{d}{dt}$" of an $\mathfrak{X}$-valued function (in time). For every $\kappa, \varepsilon, \nu, \delta > 0$, $\Phi^\kappa_{\varepsilon,\nu,\delta} : \mathfrak{X} \longrightarrow [0, \infty]$ is a proper l.s.c. and convex function, defined as follows:

$$\begin{aligned}
\Phi^\kappa_{\varepsilon,\nu,\delta} &: U := [\eta, \mathbf{u}] \in D(\Phi^\kappa_{\varepsilon,\nu,\delta}) \subset \mathfrak{X} \\
&\mapsto \Phi^\kappa_{\varepsilon,\nu,\delta}(U) = \Phi^\kappa_{\varepsilon,\nu,\delta}(\eta, \mathbf{u}) := \frac{1}{2}\int_\Omega |\nabla\eta|^2\, dx + \frac{1}{N+1}\int_\Omega |\nu\nabla\mathbf{u}|^{N+1}\, dx \\
&\quad + \frac{1}{2}\int_\Omega \left(\kappa\gamma_\varepsilon(\nabla\mathbf{u}) + \frac{1}{\kappa}\alpha(\eta)\right)^2 dx + \frac{1}{4\delta}\int_\Omega |\mathbf{u}|^4\, dx \in [0, \infty),
\end{aligned} \tag{5.3}$$

and $\mathcal{G}^\kappa_\delta : \mathfrak{X} \longrightarrow \mathfrak{X}$ is a non-monotone perturbation, given by

$$\mathcal{G}^\kappa_\delta :\ U := [\eta, \mathbf{u}] \in \mathfrak{X} \mapsto \mathcal{G}^\kappa_\delta(U) = \mathcal{G}^\kappa_\delta(\eta, \mathbf{u}) := \left[\ g(\eta) - \frac{1}{\kappa^2}\alpha(\eta)\alpha'(\eta),\ \ -\frac{1}{\delta}\mathbf{u}\ \right] \in \mathfrak{X}.$$

The solution to $(\mathrm{CP})^\kappa_{\varepsilon,\nu,\delta}$ is defined on the basis of the general theory of nonlinear evolution equations (cf. [3, 4, 6]).

Definition 5.2. A function $U \in L^2_{\mathrm{loc}}([0, \infty); \mathfrak{X})$ is called a solution to the Cauchy problem $(\mathrm{CP})^\kappa_{\varepsilon,\nu,\delta}$, if and only if $U \in W^{1,2}_{\mathrm{loc}}([0, \infty); \mathfrak{X})$, $\Phi^\kappa_{\varepsilon,\nu,\delta}(U) \in L^\infty(0, \infty)$, and U satisfies

$$\begin{aligned}
&\big(U'(t) + \mathcal{G}^\kappa_\delta(U(t)), U(t) - W\big)_{\mathfrak{X}} + \Phi^\kappa_{\varepsilon,\nu,\delta}(U(t)) \\
&\qquad \le \Phi^\kappa_{\varepsilon,\nu,\delta}(W) + \big(\mathfrak{f}(t), U(t) - W\big)_{\mathfrak{X}}, \text{ for any } W \in D(\Phi^\kappa_{\varepsilon,\nu,\delta}),
\end{aligned}$$

with the initial condition $U(0) = U_0$ in $\mathfrak{X}$.

Remark 5.1. Assumptions (A2) and (A3) guarantee the Lipschitz continuity of the perturbation $\mathcal{G}_\delta^\kappa$. Hence, the well-posedness of the Cauchy problem $(\mathrm{CP})_{\varepsilon,\nu,\delta}^\kappa$ is immediately verified by means of the general theory of nonlinear evolution equations, such as [6, 4]. Moreover, we will observe the continuous dependence of solution with respect to $\varepsilon \geq 0$, $\delta > 0$, $\nu \geq 0$, and $\kappa > 0$, by means of the general theory of operator-convergence as in [2, 18].

Now, on account of Remark 5.1 and the previous works [16, Theorems 1 and 2], we immediately obtain the following proposition.

Proposition 5.1 ([17, Proposition 4.4])**.** *Let us assume (A0)–(A7), and let us fix the constants $\varepsilon \geq 0$, $\delta > 0$, $\nu \geq 0$. Additionally, we assume that $\mathbf{u}_0 \in L^4(\Omega;\mathbb{R}^M)$ and $\nu\mathbf{u}_0 \in W^{1,N+1}(\Omega;\mathbb{R}^M)$, for the component $\mathbf{u}_0$ of the initial data $U_0 = [\eta_0, \mathbf{u}_0]$. Then, the following items hold:*

(O) *The system $(P)_{\varepsilon,\nu,\delta}^\kappa$ is equivalent to the Cauchy problem $(CP)_{\varepsilon,\nu,\delta}^\kappa$.*

(I) *The system $(P)_{\varepsilon,\nu,\delta}^\kappa$ admits a unique solution $U = [\eta, \mathbf{u}] \in L^2_{\mathrm{loc}}([0,\infty);\mathfrak{X})$, such that*

$$\begin{cases} U = [\eta, \mathbf{u}] \in W^{1,2}_{\mathrm{loc}}([0,\infty);\mathfrak{X}) \cap L^\infty_{\mathrm{loc}}(0,\infty;\mathfrak{W}), \\ \nu\mathbf{u} \in L^{N+1}_{\mathrm{loc}}([0,\infty);W^{1,N+1}(\Omega;\mathbb{R}^M)). \end{cases} \tag{5.4}$$

(II) *let $\{\varepsilon_n\}_{n=1}^\infty \subset [0,\infty)$, $\{\delta_n\}_{n=1}^\infty \subset (0,\infty)$, $\{\nu_n\}_{n=1}^\infty \subset [0,\infty)$, and $\{\kappa_n\}_{n=1}^\infty \subset (0,\infty)$ be sequences of constants, such that*

$$[\kappa_n, \varepsilon_n, \nu_n, \delta_n] \to [\kappa, \varepsilon, \nu, \delta] \text{ in } \mathbb{R}^4, \text{ as } n \to \infty. \tag{5.5}$$

Let $U_0 = [\eta_0, \mathbf{u}_0] \in \mathfrak{W}$ be the initial datum as in (A7), and let $\{U_{0,n}\}_{n=1}^\infty = \{[\eta_{0,n}, \mathbf{u}_{0,n}]\}_{n=1}^\infty \subset \mathfrak{W}$ with $\mathbf{u}_{0,n} \in L^4(\Omega;\mathbb{R}^M)$ and $\{\nu_n\mathbf{u}_{0,n}\} \subset W^{1,N+1}(\Omega;\mathbb{R}^M)$ be a sequence of initial data satisfying

$$\begin{aligned} &U_{0,n} = [\eta_{0,n}, \mathbf{u}_{0,n}] \to U_0 = [\eta_0, \mathbf{u}_0] \text{ in } \mathfrak{X}, \text{ and weakly in } \mathfrak{W}, \\ &\text{and } \nu_n\mathbf{u}_{0,n} \to \nu\mathbf{u}_0 \text{ weakly in } W^{1,N+1}(\Omega;\mathbb{R}^M), \text{ as } n \to \infty. \end{aligned} \tag{5.6}$$

Let $U = [\eta, \mathbf{u}] \in L^2_{\mathrm{loc}}([0,\infty);\mathfrak{X})$ be the solution to the system $(P)_{\varepsilon,\nu,\delta}^\kappa$. Also, for any $n \in \mathbb{N}$, let $U_n = [\eta_n, \mathbf{u}_n] \in L^2_{\mathrm{loc}}([0,\infty);\mathfrak{X})$ be the solution to the system $(P)_{\varepsilon_n,\nu_n,\delta_n}^{\kappa_n}$ corresponding to the initial data $U_{0,n} = [\eta_{0,n}, \mathbf{u}_{0,n}] \in \mathfrak{W}$. Then,

$$\begin{aligned} U_n = [\eta_n, \mathbf{u}_n] \to U = [\eta, \mathbf{u}] &\text{ in } C_{\mathrm{loc}}([0,\infty);\mathfrak{X}), \text{ in } L^2_{\mathrm{loc}}([0,\infty);\mathfrak{W}), \\ &\text{weakly in } W^{1,2}_{\mathrm{loc}}([0,\infty);\mathfrak{X}), \text{ and weakly-}* \text{ in } L^\infty_{\mathrm{loc}}(0,\infty;\mathfrak{W}), \\ &\text{and } \nu_n u_n \to \nu\mathbf{u} \text{ in } L^{N+1}_{\mathrm{loc}}([0,\infty);W^{1,N+1}(\Omega;\mathbb{R}^M)), \text{ as } n \to \infty. \end{aligned} \tag{5.7}$$

Moreover, we can prove the following theorem, concerned with the L^∞-boundedness of the component η.

Theorem 1 ([17, Theorem 4.5])**.** *Under the assumptions (A0)–(A7), let us assume $\varepsilon \geq 0$, $\delta > 0$, $\nu \geq 0$, $\mathbf{u}_0 \in L^4(\Omega;\mathbb{R}^M)$ and $\nu\mathbf{u}_0 \in W^{1,N+1}(\Omega;\mathbb{R}^M)$. Let $\sigma_* > 0$ be a positive constant such that*

$$|f_0|_{L^\infty(Q)} \leq \sigma_*, \quad g(-\sigma_*) \leq -|f_0|_{L^\infty(Q)} \quad \text{and} \quad g(\sigma_*) \geq |f_0|_{L^\infty(Q)}. \tag{5.8}$$

Then, for the component η of the solution to $(P)_{\varepsilon,\nu,\delta}^\kappa$ given by Proposition 5.1, it holds that

$$|\eta| \leq \sigma_* \quad \text{a.e. in } Q.$$

6 Outline of the proof of Main Theorem 1

In this section, we show outline of the proof of Main Theorem 1. To see this, we take the limit in $(\mathrm{P})^{\kappa}_{\varepsilon,\nu,\delta}$ as δ, ν, $\varepsilon \to 0^+$, respectively.

At first, we show an energy inequality for $\mathcal{F}^{\kappa}_{\varepsilon,\nu,\delta}$ and a priori estimates for the approximating solutions.

6.1 An energy inequality and a priori estimates

Lemma 6.1 (ref. [17, Lemma 5.1], cf. [16, Lemma 5]). *Let $U_0 = [\eta_0, \mathbf{u}_0] \in \mathfrak{W}$, with $\mathbf{u}_0 \in L^4(\Omega;\mathbb{R}^M)$, $\nu\mathbf{u}_0 \in W^{1,N+1}(\Omega;\mathbb{R}^M)$ and $U_{\varepsilon,\nu,\delta} := [\eta_{\varepsilon,\nu,\delta}, \mathbf{u}_{\varepsilon,\nu,\delta}]$ be a solution to $(\mathrm{P})^{\kappa}_{\varepsilon,\nu,\delta}$.* Then, $U_{\varepsilon,\nu,\delta}$ satisfies the following energy inequality:

$$\mathcal{F}^{\kappa}_{\varepsilon,\nu,\delta}(U_{\varepsilon,\nu,\delta}(T)) + \frac{1}{2}\int_0^T \|\partial_t U_{\varepsilon,\nu,\delta}(t)\|^2_{\mathfrak{X}}\,dt \leq \mathcal{F}^{\kappa}_{\varepsilon,\nu,\delta}(U_0) + \frac{1}{2}\int_0^T ||\mathfrak{f}(t)||^2_{\mathfrak{X}}dt \tag{6.1}$$

for all $T > 0$. Moreover, it follows that

$$\begin{cases} U_{\varepsilon,\nu,\delta} \in W^{1,2}_{loc}([0,\infty);\mathfrak{X}) \cap L^{\infty}_{loc}(0,\infty;\mathfrak{W}) \cap L^{\infty}(Q) \times L^{\infty}(Q;\mathbb{R}^M), \\ \kappa^2\nabla\mathbf{u}_{\varepsilon,\nu,\delta} \in L^{\infty}_{loc}(0,\infty;L^2(\Omega;\mathbb{R}^{MN})), \quad \nu\nabla\mathbf{u}_{\varepsilon,\nu,\delta} \in L^{\infty}_{loc}(0,\infty;L^{N+1}(\Omega;\mathbb{R}^{MN})), \end{cases} \tag{6.2}$$

and

$$\delta^{-1}\big\||\mathbf{u}_{\varepsilon,\nu,\delta}|^2 - 1\big\|^2_{L^{\infty}(0,T;H)} \leq \mathcal{F}^{\kappa}_{\varepsilon,\nu,\delta}(U_0) + \frac{1}{2}\int_0^T ||\mathfrak{f}(t)||^2_{\mathfrak{X}}dt, \tag{6.3}$$

for all $T > 0$, and

$$\big\|\alpha(\eta_{\varepsilon,\nu,\delta})[\nabla\gamma_{\varepsilon}](\nabla\mathbf{u}_{\varepsilon,\nu,\delta})\big\|_{L^{\infty}(Q;\mathbb{R}^{MN})} \leq C, \tag{6.4}$$

for some $C > 0$, independent of ε, ν, δ.

Lemma 6.2 (ref. [17, Lemma 5.2], cf. [16, Lemma 6]). *If $|\mathbf{u}_0| \leq 1$ a.e. in Ω and $\mathbf{f} \in L^{\infty}(Q;\mathbb{R}^M)$, then solutions $\mathbf{u}_{\varepsilon,\nu,\delta}$ to $(P)^{\kappa}_{\varepsilon,\nu,\delta}$ satisfy $|\mathbf{u}_{\varepsilon,\nu,\delta}| \leq C_{\delta}$ a.e. in Ω, where $C_{\delta} := \max\{||\mathbf{f}||_{L^{\infty}(Q;\mathbb{R}^M)}, \sqrt{1+\delta}\}$ for any $0 < \delta < 1$.*

At the end of this subsection, we introduce a compactness result which can be proved as [7, Theorem 2.1] and [15, Lemma 9] by using the fact that the operator

$$\mathrm{div}\,\big(\alpha(\eta_{\varepsilon,\nu,\delta})[\nabla\gamma_{\varepsilon}](\nabla\cdot) + \kappa^2(\nabla\cdot) + \nu^{N+1}|\nabla\cdot|^N(\nabla\cdot)\big)$$

is uniformly elliptic. We omit the proof of the following lemma, since it is quite similar to [5, Lemma 2.2].

Lemma 6.3. *Let $\varepsilon > 0$, $\nu > 0$ and $\kappa > 0$ be fixed. Let $\{\mathbf{w}_{\varepsilon,\nu,\delta}\}_{\delta>0}$ be bounded in $W^{1,2}_{loc}(0,\infty;\mathbb{X}) \cap L^{\infty}_{loc}(0,\infty;\mathbb{W})$, $\{\eta_{\varepsilon,\nu,\delta}\}_{\delta>0}$ be bounded in $W^{1,2}_{loc}(0,\infty;H) \cap L^{\infty}_{loc}(0,\infty;V) \cap L^{\infty}(Q)$, and $\{\mathbf{h}_{\varepsilon,\nu,\delta}\}_{\delta>0}$ be bounded in $L^1_{loc}(0,\infty;L^1(\Omega;\mathbb{R}^M))$ uniformly in δ, respectively. Suppose that they satisfy the equation, for $\delta > 0$, a.e. in Q*

$$\partial_t\mathbf{w}_{\varepsilon,\nu,\delta} - \mathrm{div}\big(\alpha(\eta_{\varepsilon,\nu,\delta})[\nabla\gamma_{\varepsilon}](\nabla\,\mathbf{w}_{\varepsilon,\nu,\delta}) + \kappa^2\nabla\mathbf{w}_{\varepsilon,\nu,\delta} + \nu|\nu\nabla\mathbf{w}_{\varepsilon,\nu,\delta}|^{N-1}\nu\nabla\mathbf{w}_{\varepsilon,\nu,\delta}\big) = \mathbf{h}_{\varepsilon,\nu,\delta}\,,$$

in the sense of distributions. Then $\{\mathbf{w}_{\varepsilon,\nu,\delta}\}_{\delta>0}$ is precompact in $L^q_{loc}(0,\infty;W^{1,q}(\Omega;\mathbb{R}^M))$ for all $1 \leq q < 2$.

We use Lemma 6.3 to take the limit in $(\mathrm{P})^{\kappa}_{\varepsilon,\nu,\delta}$ as $\delta \to 0$. When we take the limit as $\nu \to 0$ and $\varepsilon \to 0$, we use similar compactness lemmas (ref. [17, Lemma 5.3, 5.4], cf. [16, Lemma 8, 9]).

6.2 The limit as $\delta \to 0$

In this subsection we study the limit problem, as $\delta \to 0$, in $(\mathrm{P})^{\kappa}_{\varepsilon,\nu,\delta}$, assuming $0 < \varepsilon < \varepsilon_0$ for some $\varepsilon_0 > 0$; i.e. we solve the following problem. Note that the dependence on κ is not anymore needed or used. Therefore, we remove it.

Problem $(\mathbf{P})_{\varepsilon,\nu}$

$$\begin{cases} \partial_t \eta_{\varepsilon,\nu} - \Delta \eta_{\varepsilon,\nu} + g(\eta_{\varepsilon,\nu}) + \alpha'(\eta_{\varepsilon,\nu})\gamma_\varepsilon(\nabla \mathbf{u}_{\varepsilon,\nu}) = f_0 \text{ in } Q, \\ \nabla \eta_{\varepsilon,\nu} \cdot \mathbf{n}_\Gamma = 0 \text{ on } \Sigma, \\ \eta_{\varepsilon,\nu}(0,x) = \eta_0(x), \ \ x \in \Omega; \end{cases}$$

$$\begin{cases} \partial_t \mathbf{u}_{\varepsilon,\nu} - \mathrm{div}\big(\alpha(\eta_{\varepsilon,\nu})\,[\nabla\gamma_\varepsilon](\nabla \mathbf{u}_{\varepsilon,\nu}) + \kappa^2 \nabla \mathbf{u}_{\varepsilon,\nu} + \nu|\nu\nabla \mathbf{u}_{\varepsilon,\nu}|^{N-1}\nu\nabla \mathbf{u}_{\varepsilon,\nu}\big) \\ \quad = \mu_{\varepsilon,\nu}\mathbf{u}_{\varepsilon,\nu} + \mathbf{f} - (\mathbf{f}\cdot \mathbf{u}_{\varepsilon,\nu})\mathbf{u}_{\varepsilon,\nu} \text{ in } Q, \\ \big(\alpha(\eta_{\varepsilon,\nu})[\nabla\gamma_\varepsilon](\nabla \mathbf{u}_{\varepsilon,\nu}) + \kappa^2\nabla \mathbf{u}_{\varepsilon,\nu} + \nu|\nu\nabla \mathbf{u}_{\varepsilon,\nu}|^{N-1}\nu\nabla\mathbf{u}_{\varepsilon,\nu}\big)\mathbf{n}_\Gamma = 0 \text{ on } \Sigma, \\ \mathbf{u}_{\varepsilon,\nu}(0,x) = \mathbf{u}_0(x), \ \ x \in \Omega; \end{cases}$$

together with

$$\mu_{\varepsilon,\nu} := (\alpha(\eta_{\varepsilon,\nu})[\nabla\gamma_\varepsilon](\nabla \mathbf{u}_{\varepsilon,\nu}) + \kappa^2\nabla \mathbf{u}_{\varepsilon,\nu} + \nu|\nu\nabla\mathbf{u}_{\varepsilon,\nu}|^{N-1}\nu\nabla\mathbf{u}_{\varepsilon,\nu}) : \nabla \mathbf{u}_{\varepsilon,\nu}, \ \text{a.e. in } Q.$$

And we use the following energy functional:

$$\mathcal{F}_{\varepsilon,\nu}(\eta,\mathbf{u}) := \begin{cases} \dfrac{1}{2}\displaystyle\int_\Omega |\nabla\eta|^2\,dx + \int_\Omega G(\eta)\,dx + \int_\Omega \alpha(\eta)\sqrt{\varepsilon^2 + |\nabla \mathbf{u}|^2}\,dx \\ \qquad + \dfrac{1}{N+1}\displaystyle\int_\Omega |\nu\nabla\mathbf{u}|^{N+1}\,dx + \dfrac{\kappa^2}{2}\int_\Omega |\nabla\mathbf{u}|^2\,dx, \\ \qquad \text{if } \ \eta \in H^1(\Omega) \text{ and } \mathbf{u} \in H^1(\Omega;\mathbb{R}^M) \text{ with } |\mathbf{u}| = 1, \\ +\infty, \quad \text{otherwise.} \end{cases}$$

Theorem 2 (ref. [17, Theorem 5.7], cf. [16, Theorem 3])**.** *Let $U_0 = [\eta_0, \mathbf{u}_0] \in \mathfrak{W}$ with $\nu\mathbf{u}_0 \in W^{1,N+1}(\Omega;\mathbb{R}^M)$, $|\mathbf{u}_0| = 1$ in Ω and $\mathbf{f} \in L^\infty(Q;\mathbb{R}^M)$. Then, there exists $U_{\varepsilon,\nu} = [\eta_{\varepsilon,\nu}, \mathbf{u}_{\varepsilon,\nu}] \in C_{loc}([0,\infty);\mathfrak{X})$ such that $U_{\varepsilon,\nu}$ satisfies $(P)_{\varepsilon,\nu}$ in the sense of distributions and*

$$\begin{cases} U_{\varepsilon,\nu} \in W^{1,2}_{loc}([0,\infty);\mathfrak{X}) \cap L^\infty_{loc}(0,\infty;\mathfrak{W}) \cap L^\infty(Q) \times L^\infty(Q;\mathbb{R}^M), \\ \kappa^2\nabla\mathbf{u}_{\varepsilon,\nu} \in L^\infty_{loc}(0,\infty;L^2(\Omega;\mathbb{R}^{MN})), \quad \nu\nabla\mathbf{u}_{\varepsilon,\nu} \in L^\infty_{loc}(0,\infty;L^{N+1}(\Omega;\mathbb{R}^{MN})), \\ |\mathbf{u}_{\varepsilon,\nu}| = 1 \quad a.e. \ in \ Q, \end{cases} \tag{6.5}$$

and

$$\mathcal{F}_{\varepsilon,\nu}(U_{\varepsilon,\nu}(T)) + \frac{1}{2}\int_0^T \|\partial_t U_{\varepsilon,\nu}(t)\|^2_{\mathfrak{X}}dt \le \mathcal{F}_{\varepsilon,\nu}(U_0) + \frac{1}{2}\int_0^T ||\mathfrak{f}(t)||^2_{\mathfrak{X}}dt \tag{6.6}$$

$$\le \mathcal{F}_{\varepsilon_0,\nu}(U_0) + \frac{1}{2}\int_0^T ||\mathfrak{f}(t)||^2_{\mathfrak{X}}dt \quad \text{for all } T > 0,$$

$$\left\|\alpha(\eta_{\varepsilon,\nu})[\nabla\gamma_\varepsilon](\nabla \mathbf{u}_{\varepsilon,\nu})\right\|_{L^\infty(Q;\mathbb{R}^{MN})} < C, \tag{6.7}$$

$$\left\|\mathrm{div}\left(\begin{array}{r} \alpha(\eta_{\varepsilon,\nu})[\nabla\gamma_\varepsilon](\nabla \mathbf{u}_{\varepsilon,\nu}) + \kappa^2\nabla\mathbf{u}_{\varepsilon,\nu} \\ +\nu|\nu\nabla\mathbf{u}_{\varepsilon,\nu}|^{N-1}\nu\nabla\mathbf{u}_{\varepsilon,\nu}\end{array}\right)\right\|_{L^2_{loc}(0,\infty;L^1(\Omega;\mathbb{R}^M))} < C, \tag{6.8}$$

$$\left\|\mathrm{div}\left(\left(\begin{array}{r} \alpha(\eta_{\varepsilon,\nu})[\nabla\gamma_\varepsilon](\nabla \mathbf{u}_{\varepsilon,\nu}) + \kappa^2\nabla\mathbf{u}_{\varepsilon,\nu} \\ +\nu|\nu\nabla\mathbf{u}_{\varepsilon,\nu}|^{N-1}\nu\nabla\mathbf{u}_{\varepsilon,\nu}\end{array}\right)\wedge \mathbf{u}_{\varepsilon,\nu}\right)\right\|_{L^2_{loc}(0,\infty;L^2(\Omega;\Lambda_2(\mathbb{R}^M)))} < C, \tag{6.9}$$

where the constant $C > 0$ is independent of ε.

Proof. For $0 < \delta < 1$, we take

$$C_\delta := \max\{||\mathbf{f}||_{L^\infty(Q;\mathbb{R}^M)}, \sqrt{1+\delta}\} \quad \text{and} \quad C_1 := \max\{||\mathbf{f}||_{L^\infty(Q;\mathbb{R}^M)}, \sqrt{2}\}.$$

Since $|\mathbf{u}_0| = 1$, we observe that $\Pi_\delta(\mathbf{u}_0) = 0$. Hence,

$$\mathcal{F}^\kappa_{\varepsilon,\nu,\delta}(U_0) = \mathcal{F}_{\varepsilon,\nu}(U_0) < \mathcal{F}_{\varepsilon_0,\nu}(U_0) =: C < +\infty.$$

Having in mind the energy inequality (6.1), the uniform estimates (6.2)–(6.3) and the maximum principle $|\mathbf{u}_{\varepsilon,\nu,\delta}| \le C_\delta < C_1$ on Q, it follows that there exist a subsequence $\{U_{\varepsilon,\nu,\delta_n}\}_n$ and a function $U_{\varepsilon,\nu} \in C_{loc}([0,\infty);\mathfrak{X}) \cap W^{1,2}_{loc}([0,\infty);\mathfrak{X}) \cap L^\infty_{loc}(0,\infty;\mathfrak{W}) \cap L^\infty(Q) \times L^\infty(Q;\mathbb{R}^M)$ such that

$$\begin{cases} U_{\varepsilon,\nu,\delta_n} \to U_{\varepsilon,\nu} & \text{in } C_{loc}([0,\infty);\mathfrak{X}),\ \text{weakly in } W^{1,2}_{loc}([0,\infty);\mathfrak{X}), \\ & \text{weakly-}*\ \text{in } L^\infty_{loc}(0,\infty;\mathfrak{W}), \\ \nu\mathbf{u}_{\varepsilon,\nu,\delta_n} \to \nu\mathbf{u}_{\varepsilon,\nu} & \text{weakly-}*\ \text{in } L^\infty_{loc}(0,\infty;W^{1,N+1}(\Omega;\mathbb{R}^M)), \\ |\mathbf{u}_{\varepsilon,\nu,\delta_n}| \to 1 & \text{strongly in } L^2_{loc}(0,\infty;H), \end{cases} \tag{6.10}$$

as $n \to \infty$ by the Aubin type compactness theorem [21, Corollary 4]. Then, we also see that

$$|\mathbf{u}_{\varepsilon,\nu}| = 1 \quad \text{a.e. in } Q. \tag{6.11}$$

By $|\mathbf{u}_{\varepsilon,\nu,\delta}| < C_1$ in Q, the Lebesgue dominated convergence theorem implies that

$$\mathbf{u}_{\varepsilon,\nu,\delta_n} \to \mathbf{u}_{\varepsilon,\nu} \quad \text{strongly in } L^r_{loc}(0,\infty;L^r(\Omega;\mathbb{R}^M)) \quad \text{as } n\to\infty, \tag{6.12}$$

for all $r \in [1,\infty)$.

Let I be any bounded open interval such that $I \subset\subset (0,\infty)$. By $|\mathbf{u}_{\varepsilon,\nu,\delta}| < C_1$ in Q again, we can show that

$$\int_I\int_\Omega |\varpi_\delta(\mathbf{u}_{\varepsilon,\nu,\delta})|dxdt \le \frac{C_1}{\delta}\int_I\int_\Omega (1-|\mathbf{u}_{\varepsilon,\nu,\delta}|^2)^2 dxdt + \frac{C_1}{\delta}\int_I\int_\Omega |1-|\mathbf{u}_{\varepsilon,\nu,\delta}|^2||\mathbf{u}_{\varepsilon,\nu,\delta}|^2 dxdt.$$

By (6.3), the first term of right-hand side is uniformly bounded with respect to δ. To estimate the second term, we take a sequence $\{\mathrm{sgn}_\sigma(r)\}_{\sigma>0} \subset C^\infty(\mathbb{R})$ satisfying

$$\mathrm{sgn}_\sigma(r) \to \mathrm{sgn}(r) \quad \text{as } \sigma \to 0, \quad \mathrm{sgn}'_\sigma(r) \ge 0,$$

for all $r>0$. Multiplying both sides of the equation of $\mathbf{u}_{\varepsilon,\nu,\delta}$ by $\mathbf{u}_{\varepsilon,\nu,\delta}$, integrating by parts and applying the Hölder inequality, and (A6), (6.1), (6.2), (6.4), we see that

$$
\begin{aligned}
&\frac{1}{\delta}\int_I\int_\Omega \mathrm{sgn}_\sigma(1-|\mathbf{u}_{\varepsilon,\nu,\delta}|^2)(1-|\mathbf{u}_{\varepsilon,\nu,\delta}|^2)|\mathbf{u}_{\varepsilon,\nu,\delta}|^2dxdt\\
\le&\|\partial_t\mathbf{u}_{\varepsilon,\nu,\delta}\|_{L^2(I;\mathbb{X})}\|\mathbf{u}_{\varepsilon,\nu,\delta}\|_{L^2(I;\mathbb{X})}+||\mathbf{f}||_{L^1(I;L^1(\Omega;\mathbb{R}^M))}||\mathbf{u}_{\varepsilon,\nu,\delta}||_{L^\infty(I;L^\infty(\Omega;\mathbb{R}^M))}\\
&+\int_I\int_\Omega(\alpha(\eta_{\varepsilon,\nu,\delta})[\nabla\gamma_\varepsilon](\nabla\mathbf{u}_{\varepsilon,\nu,\delta})+\kappa^2\nabla\mathbf{u}_{\varepsilon,\nu,\delta}+\nu|\nu\nabla\mathbf{u}_{\varepsilon,\nu,\delta}|^{N-1}\nu\nabla\mathbf{u}_{\varepsilon,\nu,\delta})\\
&\qquad\qquad:(\mathrm{sgn}_\sigma(1-|\mathbf{u}_{\varepsilon,\nu,\delta}|^2)\nabla\mathbf{u}_{\varepsilon,\nu,\delta}-2\mathrm{sgn}'_\sigma(1-|\mathbf{u}_{\varepsilon,\nu,\delta}|^2)\mathbf{u}_{\varepsilon,\nu,\delta}\otimes\mathbf{u}_{\varepsilon,\nu,\delta}\nabla\mathbf{u}_{\varepsilon,\nu,\delta})dxdt\\
\le&\|\partial_t\mathbf{u}_{\varepsilon,\nu,\delta}\|_{L^2(I;\mathbb{X})}\|\mathbf{u}_{\varepsilon,\nu,\delta}\|_{L^2(I;\mathbb{X})}+||\mathbf{f}||_{L^1(I;L^1(\Omega;\mathbb{R}^M))}||\mathbf{u}_{\varepsilon,\nu,\delta}||_{L^\infty(I;L^\infty(\Omega;\mathbb{R}^M))}\\
&+\|\alpha(\eta_{\varepsilon,\nu,\delta})[\nabla\gamma_\varepsilon](\nabla\mathbf{u}_{\varepsilon,\nu,\delta})\|_{L^\infty(I;L^\infty(\Omega;\mathbb{R}^{MN}))}\|\nabla\mathbf{u}_{\varepsilon,\nu,\delta}\|_{L^1(I;L^1(\Omega;\mathbb{R}^{MN}))}\\
&+\|\kappa\nabla\mathbf{u}_{\varepsilon,\nu,\delta}\|^2_{L^2(I;L^2(\Omega;\mathbb{R}^{MN}))}+\|\nu\nabla\mathbf{u}_{\varepsilon,\nu,\delta}\|^{N+1}_{L^{N+1}(I;L^{N+1}(\Omega;\mathbb{R}^{MN}))}<\infty.
\end{aligned}
$$

Also, the Lebesgue dominated convergence theorem implies that

$$
\int_I\int_\Omega \mathrm{sgn}_\sigma(1-|\mathbf{u}_{\varepsilon,\nu,\delta}|^2)(1-|\mathbf{u}_{\varepsilon,\nu,\delta}|^2)|\mathbf{u}_{\varepsilon,\nu,\delta}|^2dxdt\to\int_I\int_\Omega|1-|\mathbf{u}_{\varepsilon,\nu,\delta}|^2||\mathbf{u}_{\varepsilon,\nu,\delta}|^2dxdt,
$$

as $\sigma\to 0$. Hence, $\mathbf{h}_{\varepsilon,\nu,\delta}:=-\varpi_\delta(\mathbf{u}_{\varepsilon,\nu,\delta})+\mathbf{f}$ is uniformly bounded with respect to δ in $L^1_{loc}(0,\infty;L^1(\Omega;\mathbb{R}^M))$. Therefore, we can apply Lemma 6.3 to get

$$
\nabla\mathbf{u}_{\varepsilon,\nu,\delta_n}\to\nabla\mathbf{u}_{\varepsilon,\nu}\quad\text{strongly in } L^q_{loc}(0,\infty;L^q(\Omega;\mathbb{R}^{MN}))\quad\text{as } n\to\infty, \tag{6.13}
$$

for all $q\in[1,2)$. The above convergence, (6.4) and (6.10) imply that

$$
\begin{aligned}
&\alpha(\eta_{\varepsilon,\nu,\delta_n})[\nabla\gamma_\varepsilon](\nabla\mathbf{u}_{\varepsilon,\nu,\delta_n})\to\alpha(\eta_{\varepsilon,\nu})[\nabla\gamma_\varepsilon](\nabla\mathbf{u}_{\varepsilon,\nu})\\
&\text{weakly-}*\ \text{ in } L^\infty(Q;\mathbb{R}^{MN}),\ \text{ as } n\to\infty.
\end{aligned}\tag{6.14}
$$

Next, we use wedge product technique. Taking the wedge product of the equation of $\mathbf{u}_{\varepsilon,\nu,\delta}$ with $\mathbf{u}_{\varepsilon,\nu,\delta}$, it follows that

$$
\begin{aligned}
\partial_t\mathbf{u}_{\varepsilon,\nu,\delta}\wedge\mathbf{u}_{\varepsilon,\nu,\delta}-&\mathrm{div}((\alpha(\eta_{\varepsilon,\nu,\delta})[\nabla\gamma_\varepsilon](\nabla\mathbf{u}_{\varepsilon,\nu,\delta})\\
&+\kappa^2\nabla\mathbf{u}_{\varepsilon,\nu,\delta}+\nu|\nu\nabla\mathbf{u}_{\varepsilon,\nu,\delta}|^{N-1}\nu\nabla\mathbf{u}_{\varepsilon,\nu,\delta})\wedge\mathbf{u}_{\varepsilon,\nu,\delta})=\mathbf{f}\wedge\mathbf{u}_{\varepsilon,\nu,\delta}.
\end{aligned}\tag{6.15}
$$

Here we note that

$$
\begin{aligned}
&\mathrm{div}((\alpha(\eta_{\varepsilon,\nu,\delta})[\nabla\gamma_\varepsilon](\nabla\mathbf{u}_{\varepsilon,\nu,\delta})+\kappa^2\nabla\mathbf{u}_{\varepsilon,\nu,\delta}+\nu|\nu\nabla\mathbf{u}_{\varepsilon,\nu,\delta}|^{N-1}\nu\nabla\mathbf{u}_{\varepsilon,\nu,\delta}))\wedge\mathbf{u}_{\varepsilon,\nu,\delta}\\
&=\mathrm{div}((\alpha(\eta_{\varepsilon,\nu,\delta})[\nabla\gamma_\varepsilon](\nabla\mathbf{u}_{\varepsilon,\nu,\delta})+\kappa^2\nabla\mathbf{u}_{\varepsilon,\nu,\delta}+\nu|\nu\nabla\mathbf{u}_{\varepsilon,\nu,\delta}|^{N-1}\nu\nabla\mathbf{u}_{\varepsilon,\nu,\delta})\wedge\mathbf{u}_{\varepsilon,\nu,\delta})
\end{aligned}
$$

by (2.10) and $\mathbf{v}\wedge\mathbf{v}=0$ for any $\mathbf{v}\in\mathbb{R}^M$. Integration both side of (6.15) on $I\times\Omega$, and Setting $\delta=\delta_n$ and letting $n\to\infty$, we get

$$
\begin{aligned}
&\int_I\int_\Omega\{\langle(\partial_t\mathbf{u}_{\varepsilon,\nu}\wedge\mathbf{u}_{\varepsilon,\nu}),\boldsymbol{\omega}\rangle_2+\sum_{i=1}^N\langle((\alpha(\eta_{\varepsilon,\nu})[\nabla\gamma_\varepsilon](\partial_{x_i}\mathbf{u}_{\varepsilon,\nu})\\
&+\kappa^2\partial_{x_i}\mathbf{u}_{\varepsilon,\nu}+\nu|\nu\nabla\mathbf{u}_{\varepsilon,\nu}|^{N-1}\nu\partial_{x_i}\mathbf{u}_{\varepsilon,\nu})\wedge\mathbf{u}_{\varepsilon,\nu}),\partial_{x_i}\boldsymbol{\omega}\rangle_2\}dxdt=\int_I\int_\Omega\langle\mathbf{f}\wedge\mathbf{u}_{\varepsilon,\nu},\boldsymbol{\omega}\rangle_2dxdt,
\end{aligned}\tag{6.16}
$$

for $\boldsymbol{\omega} \in L^\infty_{loc}(0,\infty; W^{1,N+1}(\Omega,\Lambda_2(\mathbb{R}^M)))$ by (6.10), (6.12) and (6.14).

Taking $\boldsymbol{\omega} = (\mathbf{u}_{\varepsilon,\nu} \wedge \boldsymbol{\psi})$ in (6.16) for $\boldsymbol{\psi} \in C^1(\overline{I \times \Omega}; \mathbb{R}^M)$ and using (2.6) and (2.10), we have

$$\begin{aligned}
&\int_I \int_\Omega \{ (\partial_t \mathbf{u}_{\varepsilon,\nu} \wedge \mathbf{u}_{\varepsilon,\nu} \wedge *(\mathbf{u}_{\varepsilon,\nu} \wedge \boldsymbol{\psi})) + \sum_{i=1}^N (\alpha(\eta_{\varepsilon,\nu})[\nabla \gamma_\varepsilon](\partial_{x_i} \mathbf{u}_{\varepsilon,\nu}) \\
&\qquad\qquad + \kappa^2 \partial_{x_i} \mathbf{u}_{\varepsilon,\nu} + \nu |\nu \nabla \mathbf{u}_{\varepsilon,\nu}|^{N-1} \nu \partial_{x_i} \mathbf{u}_{\varepsilon,\nu}) \wedge \partial_{x_i} (\mathbf{u}_{\varepsilon,\nu} \wedge *(\mathbf{u}_{\varepsilon,\nu} \wedge \boldsymbol{\psi})) \} dxdt \\
&= \int_I \int_\Omega \mathbf{f} \wedge \mathbf{u}_{\varepsilon,\nu} \wedge *(\mathbf{u}_{\varepsilon,\nu} \wedge \boldsymbol{\psi}) dxdt.
\end{aligned} \tag{6.17}$$

Here (2.6), (2.10) and $\mathbf{v} \wedge \mathbf{v} = 0$ for any $\mathbf{v} \in \mathbb{R}^M$ imply that

$$\sum_{i=1}^N \widehat{\mathcal{B}}_i \wedge \mathbf{u}_{\varepsilon,\nu} \wedge *\partial_{x_i}(\mathbf{u}_{\varepsilon,\nu} \wedge \boldsymbol{\psi}) = \sum_{i=1}^N \widehat{\mathcal{B}}_i \wedge \partial_{x_i}(\mathbf{u}_{\varepsilon,\nu} \wedge *(\mathbf{u}_{\varepsilon,\nu} \wedge \boldsymbol{\psi})),$$

where

$$\widehat{\mathcal{B}}_i := \alpha(\eta_{\varepsilon,\nu})[\nabla \gamma_\varepsilon](\partial_{x_i} \mathbf{u}_{\varepsilon,\nu}) + \kappa^2 \partial_{x_i} \mathbf{u}_{\varepsilon,\nu} + \nu |\nu \nabla \mathbf{u}_{\varepsilon,\nu}|^{N-1} \nu \partial_{x_i} \mathbf{u}_{\varepsilon,\nu}.$$

To compute each term in (6.17), we prepare small calculations. Noting (6.11), we see that

$$\partial_t \mathbf{u}_{\varepsilon,\nu} \cdot \mathbf{u}_{\varepsilon,\nu} = 0, \quad \partial_{x_i} \mathbf{u}_{\varepsilon,\nu} \cdot \mathbf{u}_{\varepsilon,\nu} = 0, \quad \text{a.e. in } Q. \tag{6.18}$$

According to (2.4), we obtain

$$\mathbf{u}_{\varepsilon,\nu} \wedge *(\mathbf{u}_{\varepsilon,\nu} \wedge \boldsymbol{\psi}) = (\mathbf{u}_{\varepsilon,\nu} \cdot \boldsymbol{\psi}) * \mathbf{u}_{\varepsilon,\nu} - (\mathbf{u}_{\varepsilon,\nu} \cdot \mathbf{u}_{\varepsilon,\nu}) * \boldsymbol{\psi} = (\mathbf{u}_{\varepsilon,\nu} \cdot \boldsymbol{\psi}) * \mathbf{u}_{\varepsilon,\nu} - *\boldsymbol{\psi}. \tag{6.19}$$

For the first term of the right-hand side of (6.17), (6.18) and (6.19) imply that

$$\partial_t \mathbf{u}_{\varepsilon,\nu} \wedge \mathbf{u}_{\varepsilon,\nu} \wedge *(\mathbf{u}_{\varepsilon,\nu} \wedge \boldsymbol{\psi}) = (\mathbf{u}_{\varepsilon,\nu} \cdot \boldsymbol{\psi}) \partial_t \mathbf{u}_{\varepsilon,\nu} \wedge *\mathbf{u}_{\varepsilon,\nu} - \partial_t \mathbf{u}_{\varepsilon,\nu} \wedge *\boldsymbol{\psi} = -\partial_t \mathbf{u}_{\varepsilon,\nu} \cdot \boldsymbol{\psi}.$$

Note that

$$\partial_t \mathbf{u}_{\varepsilon,\nu} \wedge *\boldsymbol{\psi} = \langle \partial_t \mathbf{u}_{\varepsilon,\nu}, \boldsymbol{\psi} \rangle_1 = \partial_t \mathbf{u}_{\varepsilon,\nu} \cdot \boldsymbol{\psi},$$

by $\partial_t \mathbf{u}_{\varepsilon,\nu}, \boldsymbol{\psi} \in \mathbb{R}^M = \Lambda_1(\mathbb{R}^M)$.

Similarly, we see that

$$\begin{aligned}
&\widehat{\mathcal{B}}_i \wedge \partial_{x_i}(\mathbf{u}_{\varepsilon,\nu} \wedge *(\mathbf{u}_{\varepsilon,\nu} \wedge \boldsymbol{\psi})) \\
&= \partial_{x_i}(\mathbf{u}_{\varepsilon,\nu} \cdot \boldsymbol{\psi}) \widehat{\mathcal{B}}_i \wedge *\mathbf{u}_{\varepsilon,\nu} + (\mathbf{u}_{\varepsilon,\nu} \cdot \boldsymbol{\psi}) \widehat{\mathcal{B}}_i \wedge \partial_{x_i} * \mathbf{u}_{\varepsilon,\nu} - \widehat{\mathcal{B}}_i \wedge \partial_{x_i} * \boldsymbol{\psi} \\
&= (\mathbf{u}_{\varepsilon,\nu} \cdot \boldsymbol{\psi}) \widehat{\mathcal{B}}_i \cdot \partial_{x_i} \mathbf{u}_{\varepsilon,\nu} - \widehat{\mathcal{B}}_i \cdot \partial_{x_i} \boldsymbol{\psi},
\end{aligned}$$

and

$$\mathbf{f} \wedge \mathbf{u}_{\varepsilon,\nu} \wedge *(\mathbf{u}_{\varepsilon,\nu} \wedge \boldsymbol{\psi}) = (\mathbf{u}_{\varepsilon,\nu} \cdot \boldsymbol{\psi}) \mathbf{f} \wedge *\mathbf{u}_{\varepsilon,\nu} - \mathbf{f} \wedge *\boldsymbol{\psi} = (\mathbf{u}_{\varepsilon,\nu} \cdot \boldsymbol{\psi}) \mathbf{f} \cdot \mathbf{u}_{\varepsilon,\nu} - \mathbf{f} \cdot \boldsymbol{\psi}.$$

Having the previous equality in mind, we can see that

$$\int_I\int_\Omega\{\partial_t\mathbf{u}_{\varepsilon,\nu}\cdot\boldsymbol{\psi}+(\alpha(\eta_{\varepsilon,\nu})[\nabla\gamma_\varepsilon](\nabla\mathbf{u}_{\varepsilon,\nu})+\kappa^2\nabla\mathbf{u}_{\varepsilon,\nu}+\nu|\nu\nabla\mathbf{u}_{\varepsilon,\nu}|^{N-1}\nu\nabla\mathbf{u}_{\varepsilon,\nu}):\nabla\boldsymbol{\psi}\}dxdt$$

$$=\int_I\int_\Omega(\mu_{\varepsilon,\nu}-\mathbf{f}\cdot\mathbf{u}_{\varepsilon,\nu})(\mathbf{u}_{\varepsilon,\nu}\cdot\boldsymbol{\psi})dxdt+\int_I\int_\Omega\mathbf{f}\cdot\boldsymbol{\psi}dxdt, \tag{6.20}$$

for $\boldsymbol{\psi}\in C^1(\overline{I\times\Omega};\mathbb{R}^M)$ by (2.1), (2.5), (2.6), (2.11).

On the other hand, (6.10) and (6.13) imply that

$$\alpha'(\eta_{\varepsilon,\nu,\delta_n})\gamma_\varepsilon(\nabla\mathbf{u}_{\varepsilon,\nu,\delta_n})\to\alpha'(\eta_{\varepsilon,\nu})\gamma_\varepsilon(\nabla\mathbf{u}_{\varepsilon,\nu})\quad\text{ in } L^1_{loc}(0,\infty;L^1(\Omega)), \tag{6.21}$$

as $n\to\infty$. By (6.10) and (6.21), we obtain

$$\int_I(\partial_t\eta_{\varepsilon,\nu}(t)+g(\eta_{\varepsilon,\nu}(t))+\alpha'(\eta_{\varepsilon,\nu}(t))\gamma_\varepsilon(\nabla\mathbf{u}_{\varepsilon,\nu}(t)),\varphi)_H dt+\int_I(\nabla\eta_{\varepsilon,\nu}(t),\nabla\varphi)_H dt$$

$$=\int_I(f_0,\varphi)_H dt,$$

for any $\varphi\in L^2(I;V)\cap L^\infty(I\times\Omega)$.

Finally, we confirm (6.6)-(6.9). By (6.1) and (6.10), the energy inequality (6.6) immediately holds. Moreover, the estimates (6.7)-(6.9) also follow by (6.6), (6.20) and (6.16), respectively. □

Remark 6.1. By Theorem 2, we see that

$$\mathbf{u}_{\varepsilon,\nu}\in W^{1,2}_{loc}([0,\infty);\mathbb{X})\cap L^\infty_{loc}(0,\infty;W^{1,N+1}(\Omega;\mathbb{R}^M))$$

if $\varepsilon,\nu>0$. Then, the Aubin type compactness theorem [21, Corollary 4] implies that

$$\mathbf{u}_{\varepsilon,\nu}\in C_{loc}([0,\infty);C(\overline{\Omega})),$$

if $\varepsilon,\nu>0$.

We also give the following invariance principle, analogous to the case where there is no forcing in the system ([16, Theorem 6]):

Theorem 3 ([17, Theorem 5.8]). *In addition to the hypothesis in Theorem 2, suppose that there is* $\mathbf{p}_0\in\mathbb{S}^{M-1}$ *such that* $\mathbf{u}_0\in\overline{B_g(\mathbf{p}_0;R)}$*, with* $R<\frac{\pi}{2}$ *and that* $\frac{\mathbf{f}}{|\mathbf{f}|}\in\overline{B_g(\mathbf{p}_0;R)}$*,* $|\mathbf{f}|$*-a.e. Then, the solution to* $(P)_{\varepsilon,\nu}$ *satisfies*

$$\mathbf{u}_{\varepsilon,\nu}\in\overline{B_g(\mathbf{p}_0;R)}\,,\quad\text{a.e. in }\Omega\,,\ \text{for all } t\in[0,+\infty).$$

Proof. First, we note that we can assume without loss of generality that $\mathbf{p}_0=(0,\dots,0,1)$. Suppose, by contradiction that there exists $T^*=\inf\{t\in[0,T[:\ \mathbf{u}_{\varepsilon,\nu}(t;\Omega)\not\subset\overline{B_g(\mathbf{p}_0;R)}\}$. Due to the continuity of $\mathbf{u}_{\varepsilon,\nu}$, there is a $\delta>0$ such that $\mathbf{u}_{\varepsilon,\nu}(t;\Omega)\subset B_g(\mathbf{p}_0;\frac{\pi}{2})$ for $t\in[0,T^*+\delta[$.

We now take the equation for $\mathbf{u}_{\varepsilon,\nu}$ in $(\mathrm{P})_{\varepsilon,\nu}$ and we take the projection $\pi_{\mathbf{u}_{\varepsilon,\nu}}$ from $\mathbb{R}^M$ to $T_{\mathbf{u}_{\varepsilon,\nu}}\mathbb{S}^{M-1}$. Noting that $\pi_{\mathbf{u}_{\varepsilon,\nu}}(\mu_{\varepsilon,\nu}\mathbf{u}_{\varepsilon,\nu})=0$, we get

$$\partial_t\mathbf{u}=\pi_{\mathbf{u}}\mathrm{div}(Z), \tag{6.22}$$

with

$$\mathbf{u} = \mathbf{u}_{\varepsilon,\nu}\,, \quad Z := \alpha(\eta_{\varepsilon,\nu})\,[\nabla\gamma_\varepsilon](\nabla\mathbf{u}_{\varepsilon,\nu}) + \kappa^2\nabla\mathbf{u}_{\varepsilon,\nu} + \nu|\nu\nabla\mathbf{u}_{\varepsilon,\nu}|^{N-1}\nu\nabla\mathbf{u}_{\varepsilon,\nu}.$$

We choose on $\overline{B_g(\mathbf{p}_0;\frac{\pi}{2})}$ a polar coordinate system $\mathbf{p} \mapsto (p^r; p^{\theta_1}, \dots, p^{\theta_{M-2}})$ centered at $\mathbf{p_0}$. The metric in the polar coordinates around the north pole is the following one:

$$g = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & \sin^2(r) & 0 & \cdots & 0 \\ 0 & 0 & \sin^2(r)\sin^2(\theta_1) & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \sin^2(r)\sin^2(\theta_1)\cdot\ldots\cdot\sin^2(\theta_{M-3}) \end{pmatrix}.$$

Therefore, the Christoffel symbols of the second kind for the variable r are

$$\Gamma^r = -\frac{\sin(2r)}{2}\begin{pmatrix} 0 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & \sin^2(\theta_1) & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & \dots & \sin^2(\theta_1)\cdot\ldots\cdot\sin^2(\theta_{M-3}) \end{pmatrix}.$$

Next, we note that (see [10])

$$\pi_{\mathbf{u}}(\mathrm{div} Z)^i = \mathrm{div}\mathbf{Z}^i + \sum_{j,k,l}\Gamma^i_{j,k}(\mathbf{u})\,\partial_{x_\ell}u^j\,\mathbf{Z}^k_\ell\,,\ \ i = r, \theta_1, \dots, \theta_{M-2}\,,$$
$$\text{for any } Z \in W^{1,1}(\Omega;\mathbb{R}^{MN}) \text{ such that } Z \in T_{\mathbf{u}}(\mathbb{S}^{M-1}).$$

Here, $j,k = r, \theta_1, \dots, \theta_{M-2}$, $\ell = 1, \dots, N$, ${}^t\mathbf{Z}^i \in \mathbb{R}^M$ is the i-th row vector of Z and $\mathbf{Z}^i_\ell \in \mathbb{R}$ is the ℓ-th component of ${}^t\mathbf{Z}^i$.

Thus, we get that the equation for the radial coordinate in (6.22) is the following one:

$$\begin{aligned} u^r_t =& \mathrm{div}\mathbf{Z}^r - \frac{\sin 2u^r}{2}\left(\frac{\alpha(\eta)}{\sqrt{\varepsilon^2 + |\nabla\mathbf{u}|^2}} + \kappa^2 + \nu^{N+1}|\nabla\mathbf{u}|^{N-1}\right)\times \\ &\times\left(|\nabla u^{\theta_1}|^2 + \sum_{i=1}^{M-3}\sin^2(u^{\theta_1})\cdot\ldots\cdot\sin^2(u^{\theta_i})|\nabla u^{\theta_{i+1}}|^2\right) + (\mathbf{f} - (\mathbf{f}\cdot\mathbf{u})\mathbf{u})^r, \end{aligned} \tag{6.23}$$

where

$$\mathbf{Z}^r := \left(\frac{\alpha(\eta)}{\sqrt{\varepsilon^2 + |\nabla\mathbf{u}|^2}} + \kappa^2 + \nu^{N+1}|\nabla\mathbf{u}|^{N-1}\right)\nabla u^r.$$

Therefore, since $u^r \in [0, \frac{\pi}{2}]$,

$$u^r_t \le \mathrm{div}\mathbf{Z}^r + (\mathbf{f} - (\mathbf{f}\cdot\mathbf{u})\mathbf{u})^r.$$

Thus,

$$\frac{1}{2}\frac{d}{dt}\int_\Omega (u^r - R)^2_+ = \int_\Omega u^r_t(u^r - R)_+ \le \int_\Omega (\mathrm{div}\mathbf{Z}^r + (\mathbf{f} - (\mathbf{f}\cdot\mathbf{u})\mathbf{u})^r)(u^r - R)_+$$

$$= -\int_{\Omega\cap\{u^r>R\}} \mathbf{Z}^r : \nabla u^r + \int_{\Omega\cap\{u^r>R\}} (\mathbf{f}-(\mathbf{f}\cdot\mathbf{u})\mathbf{u})^r (u^r - R)_+$$
$$\leq -\int_{\Omega\cap\{u^r>R\}} |\nabla u^r|^2 \left(\frac{\alpha(\eta)}{\sqrt{\varepsilon^2+|\nabla \mathbf{u}|^2}} + \kappa^2 + \nu^{N+1}|\nabla \mathbf{u}|^{N-1} \right) \leq 0,$$

where in the last but one inequality we use the fact that, if $u^r > R$ then $(\mathbf{f}-(\mathbf{f}\cdot\mathbf{u})\mathbf{u})^r \leq 0$ (remember that $\frac{\mathbf{f}}{|\mathbf{f}|} \in \overline{B_g(\mathbf{p}_0; R)}$). This finishes the proof. □

6.3 The limit as $\nu \to 0$ and $\varepsilon \to 0$

In this subsection, we first show the solvability of the limit problem in $(\mathrm{P})_{\varepsilon,\nu}$, as $\nu \to 0$ assuming $0 < \varepsilon < \varepsilon_0$ for some $\varepsilon_0 > 0$.

Problem (P)$_\varepsilon$

$$\begin{cases} \partial_t \eta_\varepsilon - \Delta \eta_\varepsilon + g(\eta_\varepsilon) + \alpha'(\eta_\varepsilon)\gamma_\varepsilon(\nabla \mathbf{u}_\varepsilon) = f_0 \text{ in } Q, \\ \nabla \eta_\varepsilon \cdot \mathbf{n}_\Gamma = 0 \text{ on } \Sigma, \\ \eta_\varepsilon(0,x) = \eta_0(x), \ x \in \Omega; \end{cases}$$

$$\begin{cases} \partial_t \mathbf{u}_\varepsilon - \mathrm{div}\big(\alpha(\eta_\varepsilon)\,[\nabla\gamma_\varepsilon](\nabla \mathbf{u}_\varepsilon) + \kappa^2 \nabla \mathbf{u}_\varepsilon\big) = (\mu_\varepsilon - \mathbf{f}\cdot\mathbf{u}_\varepsilon)\mathbf{u}_\varepsilon + \mathbf{f} \text{ in } Q, \\ \big(\alpha(\eta_\varepsilon)[\nabla\gamma_\varepsilon](\nabla \mathbf{u}_\varepsilon) + \kappa^2\nabla \mathbf{u}_\varepsilon\big)\mathbf{n}_\Gamma = 0 \text{ on } \Sigma, \\ \mathbf{u}_\varepsilon(0,x) = \mathbf{u}_0(x), \ x \in \Omega; \end{cases}$$

together with

$$\mu_\varepsilon := (\alpha(\eta_\varepsilon)[\nabla\gamma_\varepsilon](\nabla \mathbf{u}_\varepsilon) + \kappa^2\nabla \mathbf{u}_\varepsilon) : \nabla \mathbf{u}_\varepsilon, \text{ a.e. in } Q.$$

Theorem 4 ([17, Theorem 5.9])**.** *Let $U_0 = [\eta_0, \mathbf{u}_0] \in \mathfrak{W}$ with $\mathbf{u}_0 \in \mathbb{S}^{M-1}$ in Ω. Then, there exists $U_\varepsilon = [\eta_\varepsilon, \mathbf{u}_\varepsilon] \in C_{loc}([0,\infty); \mathfrak{X})$ such that U_ε satisfies (P)$_\varepsilon$ in the sense of distributions and*

$$\begin{cases} U_\varepsilon \in W^{1,2}_{loc}([0,\infty);\mathfrak{X}) \cap L^\infty_{loc}(0,\infty;\mathfrak{W}) \cap L^\infty(Q) \times L^\infty(Q;\mathbb{R}^M), \\ \kappa^2 \nabla \mathbf{u}_\varepsilon \in L^\infty_{loc}(0,\infty; L^2(\Omega;\mathbb{R}^{MN})), \\ \mathbf{u}_\varepsilon \in \mathbb{S}^{M-1} \quad a.e.\ in\ Q, \end{cases}$$

and

$$\mathcal{F}_\varepsilon(U_\varepsilon(T)) + \frac{1}{2}\int_0^T \|\partial_t U_\varepsilon(t)\|^2_{\mathfrak{X}} dt \leq \mathcal{F}_\varepsilon(U_0) + \frac{1}{2}\int_0^T ||\mathfrak{f}(t)||^2_{\mathfrak{X}} dt \quad for\ all\ T > 0,$$

where $\mathcal{F}_\varepsilon := \mathcal{F}_{\varepsilon,0}$, and

$$\big\|\alpha(\eta_\varepsilon)[\nabla\gamma_\varepsilon](\nabla \mathbf{u}_\varepsilon)\big\|_{L^\infty(Q;\mathbb{R}^{MN})} < C,$$

$$\big\|\mathrm{div}(\alpha(\eta_\varepsilon)[\nabla\gamma_\varepsilon](\nabla \mathbf{u}_\varepsilon) + \kappa^2\nabla \mathbf{u}_\varepsilon)\big\|_{L^2_{loc}(0,\infty;L^1(\Omega;\mathbb{R}^M))} < C,$$

$$\big\|\mathrm{div}((\alpha(\eta_\varepsilon)[\nabla\gamma_\varepsilon](\nabla \mathbf{u}_\varepsilon) + \kappa^2\nabla \mathbf{u}_\varepsilon) \wedge \mathbf{u}_\varepsilon)\big\|_{L^2_{loc}(0,\infty;L^2(\Omega;\Lambda_2(\mathbb{R}^M))} < C,$$

where C does not depend on ε.

Moreover, suppose that there is $\mathbf{p}_0 \in \mathbb{S}^{M-1}$ such that $\mathbf{u}_0 \in \overline{B_g(\mathbf{p}_0; R)}$, with $R < \frac{\pi}{2}$ and that $\frac{\mathbf{f}}{|\mathbf{f}|} \in \overline{B_g(\mathbf{p}_0; R)}$, $|\mathbf{f}|$-a.e . Then, the solution to $(P)_\varepsilon$ satisfies

$$\mathbf{u}_\varepsilon \in \overline{B_g(\mathbf{p}_0; R)}\,, \quad \text{a.e. in } \Omega\,, \text{ for all } t \in [0, +\infty).$$

Since the proof of Theorem 4 is similar to Theorem 2, we omit the proof (cf. [16, Theorem 4]).

Finally, we show the solvability of the initial system (P) by letting $\varepsilon \to 0$. As $\varepsilon \to 0$, the limit problem is formulated as follows:

Problem (P)

$$\begin{cases} \partial_t \eta - \Delta\eta + g(\eta) + \alpha'(\eta)|\nabla \mathbf{u}| = f_0 \text{ in } Q, \\ \nabla\eta \cdot \mathbf{n}_\Gamma = 0 \ \text{ on } \Sigma, \\ \eta(0, x) = \eta_0(x), \ x \in \Omega; \end{cases}$$

$$\begin{cases} \partial_t \mathbf{u} - \operatorname{div}\big(\alpha(\eta)\mathcal{B} + \kappa^2\nabla\mathbf{u}\big) = (\mu - \mathbf{f}\cdot\mathbf{u})\mathbf{u} + \mathbf{f} \text{ in } Q, \\ \big(\alpha(\eta)\mathcal{B} + \kappa^2\nabla\mathbf{u}\big)\mathbf{n}_\Gamma = 0 \text{ on } \Sigma, \\ \mathbf{u}(0, x) = \mathbf{u}_0(x), \ x \in \Omega; \end{cases}$$

together with

$$\mathcal{B} \in \mathrm{Sgn}^{M,N}(\nabla\mathbf{u}), \text{ and } \mu := (\alpha(\eta)\mathcal{B} + \kappa^2\nabla\mathbf{u}) : \nabla\mathbf{u}, \text{ a.e. in } Q.$$

Theorem 5 ([17, Theorem 5.10])**.** *Let $U_0 = [\eta_0, \mathbf{u}_0] \in \mathfrak{W}$ with $\mathbf{u}_0 \in \mathbb{S}^{M-1}$ in Ω. Then, there exist $U = [\eta, \mathbf{u}] \in L^2_{loc}(0, \infty; \mathfrak{X})$ and $\mathcal{B} \in L^\infty(Q; \mathbb{R}^{MN})$ such that*

$$\begin{cases} U \in W^{1,2}_{loc}([0, \infty); \mathfrak{X}) \cap L^\infty_{loc}(0, \infty; \mathfrak{W}) \cap L^\infty(Q) \times L^\infty(Q; \mathbb{R}^M), \\ \kappa^2\nabla\mathbf{u} \in L^\infty_{loc}(0, \infty; L^2(\Omega; \mathbb{R}^{MN})), \\ \mathbf{u} \in \mathbb{S}^{M-1} \quad a.e. \ in \ Q, \end{cases}$$

and

$$\mathcal{F}(U(T)) + \frac{1}{2}\int_0^T \|\partial_t U(t)\|^2_{\mathfrak{X}} dt \le \mathcal{F}(U_0) + \frac{1}{2}\int_0^T \|\mathfrak{f}(t)\|^2_{\mathfrak{X}} dt \quad \text{for all } T > 0.$$

Also, there exists a constant $C > 0$, independent of κ, such that:

$$\big\|\alpha(\eta)\mathcal{B}\big\|_{L^\infty(Q;\mathbb{R}^{MN})} < C,$$

$$\big\|\operatorname{div}\big(\alpha(\eta)\mathcal{B} + \kappa^2\nabla\mathbf{u}\big)\big\|_{L^2_{loc}(0,\infty;L^1(\Omega;\mathbb{R}^M))} < C,$$

$$\big\|\operatorname{div}\big(\alpha(\eta)\mathcal{B} + \kappa^2\nabla\mathbf{u}\big) \wedge \mathbf{u}\big)\big\|_{L^2_{loc}(0,\infty;L^2(\Lambda_2(\mathbb{R}^M))} < C.$$

Moreover, suppose that there is $\mathbf{p}_0 \in \mathbb{S}^{M-1}$ such that $\mathbf{u}_0 \in \overline{B_g(\mathbf{p}_0; R)}$, with $R < \frac{\pi}{2}$ and that $\frac{\mathbf{f}}{|\mathbf{f}|} \in \overline{B_g(\mathbf{p}_0; R)}$, $|\mathbf{f}|$-a.e. Then, the solution to (P) satisfies

$$\mathbf{u} \in \overline{B_g(\mathbf{p}_0; R)}\,, \quad \text{a.e. in } \Omega\,, \text{ for all } t \in [0, +\infty).$$

Since the proof of Theorem 5 is also similar to Theorem 2, we omit the proof (cf. [16, Theorem 5]).

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