The existence of L^2–normalized solutions in the L^2-critical setting

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Abstract

The note surveys the result and idea of proof in [CiGaIkTa-1]. Moreover, the existence of multiple L^2–normalized solutions is also given, which is not contained in [CiGaIkTa-1] and this result is motivated by [CiGaIkTa-2]. A proof of this multiplicity result is based on the uniqueness and nondegeneracy of positive radial solutions to $-\Delta u + u = |u|^{p-1}u$ in $\mathbf{R}^N$.

1 Introduction

The L^2–normalized problem is to find a pair $(\mu, u) \in \mathbf{R} \times H^1(\mathbf{R}^N)$ satisfying

$$-\Delta u + \mu u = g(u) \quad \text{in } \mathbf{R}^N, \quad \frac{1}{2}\int_{\mathbf{R}^N} u^2 \, \mathrm{d}x = m. \tag{1.1}$$

Here $N \geq 2$, and $g \in C(\mathbf{R})$ and $m \in (0,\infty)$ are a given nonlinearity and a constant. The study of the existence of L^2-normalized solutions and their properties are related to the stability of standing wave solutions of

$$i\partial_t \psi + \Delta_x \psi + f(|\psi|)\psi = 0. \tag{1.2}$$

Here the standing wave solutions of (1.2) are solutions of the form $\psi(t,x) = e^{i\mu t}u(x)$. For the details, we refer to Cazenave [Ca03].

Pioneer works for (1.1) are [St80, St82, CaLi82] and recently the L^2-normalized problem is actively studied. For references, we refer to [CiGaIkTa-1]. The aim of this note is to provide the result and idea of the proof in [CiGaIkTa-1] as well as to give another multiplicity result which is not given in [CiGaIkTa-1]. This multiplicity result is motivated by the function given in [CiGaIkTa-2]. To state the result in [CiGaIkTa-1], set

$$p := 1 + \frac{4}{N}.$$

This exponent plays an important role in the study of the L^2-normalized problem. In what follows, we always assume the following condition:

(g1) Set $h(s) := g(s) - |s|^{p-1}s$. Then h satisfies

$$\lim_{s\to 0^+} \frac{h(s)}{|s|^{p-1}s} = 0, \quad \lim_{s\to\infty} \frac{h(s)}{s} = 0.$$

Notice that if g satisfies (g1), then this case is included in the L^2-*critical* case. The L^2-critical case is not well studied and references for this case are limited. Here we mention the works Schino [Sc22] (the existence of the minimizer) and Jeanjean, Zhang and Zhong [JeZhZh24] (the existence of positive solutions based on the fixed point index and continuation arguments).

The existence of positive solutions to (1.1) is delicate in the L^2-critical case. In fact, it is known (cf. Kwong [Kw89]) that the equation

$$-\Delta u + u = |u|^{p-1}u \quad \text{in } \mathbf{R}^N, \quad u \in H^1(\mathbf{R}^N) \tag{1.3}$$

has a unique positive radial solution and we denote it by ω_1. For any $\mu > 0$, the equation

$$-\Delta u + \mu u = u^p \quad \text{in } \mathbf{R}^N, \quad u \in H^1(\mathbf{R}^N)$$

admits a unique positive radial solution given by $\omega_\mu(x) := \mu^{1/(p-1)}\omega_1(\mu^{1/2}x) = \mu^{N/4}\omega_1(\mu^{1/2}x)$. Notice that

$$m_1 := \frac{1}{2}\|\omega_1\|^2_{L^2(\mathbf{R}^N)} = \frac{1}{2}\|\omega_\mu\|^2_{L^2(\mathbf{R}^N)} \quad \text{for every } \mu > 0.$$

On the other hand, if $(\mu, u) \in \mathbf{R} \times H^1(\mathbf{R}^N)$ is a solution of (1.1) with $g(s) = |s|^{p-1}s$, then u satisfies the Pohozaev identity (see Berestycki and Lions [BeLi83, Proposition 1]):

$$0 = \frac{N-2}{2}\|\nabla u\|^2_{L^2(\mathbf{R}^N)} + N\left(\frac{\mu}{2}\|u\|^2_{L^2(\mathbf{R}^N)} - \frac{1}{p+1}\|u\|^{p+1}_{L^{p+1}(\mathbf{R}^N)}\right).$$

Since $\|\nabla u\|^2_{L^2(\mathbf{R}^N)} + \mu\|u\|^2_{L^2(\mathbf{R}^N)} = \|u\|^{p+1}_{L^{p+1}(\mathbf{R}^N)}$, it follows that

$$\mu\|u\|^2_{L^2(\mathbf{R}^N)} = \left(\frac{N}{p+1} - \frac{N-2}{2}\right)\|u\|^{p+1}_{L^{p+1}(\mathbf{R}^N)} > 0,$$

which yields $\mu > 0$. Thus, (1.1) with $g(s) = |s|^{p-1}s$ admits a positive radial solution if and only if $m = m_1$.

By the above consideration, in [CiGaIkTa-1], the existence of positive solutions to (1.1) with $m = m_1$ is discussed and the following result is obtained:

Theorem 1.1 ([CiGaIkTa-1])**.** *Suppose* (g1) *and the following condition:*

(g2) *There is no positive radial solution to $-\Delta u = g(u)$ in $\mathbf{R}^N$ with $\nabla u \in L^2(\mathbf{R}^N)$ and $u \in L^{p+1}(\mathbf{R}^N)$.*

Then (1.1) *with $m = m_1$ admits a solution $(\mu, u) \in (0, \infty) \times H^1_{\mathrm{rad}}(\mathbf{R}^N)$ such that $u > 0$ in $\mathbf{R}^N$.*

Remark 1.2. (i) According to (g1) and the result by Alarcón, García-Melián and Quaas [AlGaQu16], when $2 \le N \le 4$ and $g(s) > 0$ for all $s > 0$, the equation

$$-\Delta u = g(u) \quad \text{in } \mathbf{R}^N$$

has no positive solution. Thus, in this case, (g2) is not necessary.

(ii) A similar condition to (g2) is used in [JeZhZh24].

(iii) One simple condition to verify (g2) is

$$0 \le \frac{N-2}{2}g(s)s - NG(s) \quad \text{in } [0, \infty).$$

For the details, see [CiGaIkTa-1].

1.1 Idea of proof of Theorem 1.1

To prove Theorem 1.1, without loss of generality, we may assume that g is odd. Indeed, since we are interested in positive solutions, we modify the values $g(s)$ for $s \leq 0$ to obtain the odd extension $\widetilde{g}$ of g and use $\widetilde{g}$ instead of g. If the existence of positive solutions to (1.1) is shown with $\widetilde{g}$, then these are also positive solutions of (1.1) with g. Therefore, from now on, we assume that g is odd in addition to (g1) and (g2).

In [CiGaIkTa-1], the Lagrangian function approach in Hirata and Tanaka [HiTa19] is utilized and critical points of the following functional are found:

$$I(\lambda, u) := \int_{\mathbf{R}^N} \frac{1}{2}|\nabla u|^2 - G(u)\,\mathrm{d}x + e^\lambda\left(\frac{1}{2}\int_{\mathbf{R}^N} u^2\,\mathrm{d}x - m_1\right) : \mathbf{R}\times H^1_{\mathrm{rad}}(\mathbf{R}^N) \to \mathbf{R}.$$

It is easily seen that any critical point (λ, u) of I is a solution of (1.1) with $\mu = e^\lambda$ and $m = m_1$. Inspired by works [BaLi90, BaLio97, Ta00], two minimax values $\underline{b}$ and $\overline{b}$ are introduced to find critical points of I. To define these values, by Gagliardo–Nirenberg's inequality and (g1), we shall prove that there exists some $A > 0$ such that

$$I(\lambda, u) \geq -2Am_1 \quad \text{for each } (\lambda, u) \in \mathbf{R}\times H^1_{\mathrm{rad}}(\mathbf{R}^N) \text{ with } \frac{1}{2}\int_{\mathbf{R}^N} u^2\,\mathrm{d}x = m_1.$$

Since $I(\lambda, 0) \to -\infty$ as $\lambda \to \infty$ and $I(\lambda, tu) = -\infty$ as $t \to \infty$ when $u \not\equiv 0$, the set $\mathbf{R}\times \{\, u \in H^1_{\mathrm{rad}}(\mathbf{R}^N) \mid \frac{1}{2}\int_{\mathbf{R}^N} u^2\,\mathrm{d}x = m_1 \,\}$ separates

$$\left\{\, (\lambda, u) \in \mathbf{R}\times H^1_{\mathrm{rad}}(\mathbf{R}^N) \;\middle|\; I(\lambda, u) < -2Am_1 \,\right\}$$

into at least two parts. We next find $\zeta_0 \in C(\mathbf{R}, H^1_{\mathrm{rad}}(\mathbf{R}^N))$ which enjoys the following properties:

(i) $I(\lambda, \zeta_0(\lambda)) < -2Am_1 - 1 - e^\lambda m_1$ for all $\lambda \in \mathbf{R}$;

(ii) $\displaystyle\frac{1}{2}\int_{\mathbf{R}^N} (\zeta_0(\lambda))^2\,\mathrm{d}x > m_1$ for all $\lambda \in \mathbf{R}$;

(iii) As $|\lambda| \to \infty$, $\max_{0\leq t\leq 1} I(\lambda, t\zeta_0(\lambda)) \to 0$.

Finally, we set

$$\begin{aligned}
&\gamma_0(\lambda, t) := (\lambda, t\zeta_0(\lambda)) : \mathbf{R}\times[0,1] \to \mathbf{R}\times H^1_{\mathrm{rad}}(\mathbf{R}^N),\\
&\mathcal{C}(L) := \{((-\infty, -L]\cup[L,\infty))\times[0,1]\} \cup \left\{[-L, L]\times\left([0, L^{-1}]\cup[1-L^{-1}, 1]\right)\right\}.
\end{aligned}$$

Then the values $\underline{b}$ and $\overline{b}$ are defined as follows:

$$\underline{b} := \inf_{\gamma\in\underline{\Gamma}} \max_{0\leq t\leq 1} I(\gamma(t)), \quad \overline{b} := \inf_{\gamma\in\overline{\Gamma}} \sup_{(\lambda,t)\in\mathbf{R}\times[0,1]} I(\gamma(\lambda, t)),$$

where

$$\begin{aligned}
\underline{\Gamma} &:= \left\{\, \gamma \in C([0,1], \mathbf{R}\times H^1_{\mathrm{rad}}(\mathbf{R}^N)) \;\middle|\; I(\gamma(0)) \ll 1,\ \gamma(1) = (\lambda_\gamma, \zeta_0(\lambda_\gamma)) \text{ for some } \lambda_\gamma \in \mathbf{R} \,\right\},\\
\overline{\Gamma} &:= \left\{\, \gamma \in C(\mathbf{R}\times[0,1], \mathbf{R}\times H^1_{\mathrm{rad}}(\mathbf{R}^N)) \;\middle|\; \gamma = \gamma_0 \text{ on } \mathcal{C}(L_\gamma) \text{ for some } L_\gamma > 1 \,\right\}.
\end{aligned}$$

We aim to prove that $\underline{b}$ or $\overline{b}$ is a critical value of I. To this end, we first establish

$$\underline{b} \leq b(\lambda) \leq \overline{b} \quad \text{for every } \lambda \in \mathbf{R}. \tag{1.4}$$

Here $b(\lambda)$ is the mountain pass value of the functional $H^1_{\mathrm{rad}}(\mathbf{R}^N) \ni u \mapsto I(\lambda, u)$:

$$\begin{aligned}
b(\lambda) &:= \inf_{\gamma\in\Gamma_\lambda} \max_{0\leq t\leq 1} I(\lambda, \gamma(t)),\\
\Gamma_\lambda &:= \left\{\, \gamma \in C([0,1], H^1_{\mathrm{rad}}(\mathbf{R}^N)) \;\middle|\; \gamma(0) = 0,\ I(\lambda, \gamma(1)) < -e^\lambda m_1 \,\right\}.
\end{aligned}$$

Since it can be shown that $b(\lambda) \to 0$ as $|\lambda| \to \infty$, (1.4) yields

$$\underline{b} \leq 0 \leq \overline{b}.$$

From these two inequalities, we consider the following three cases:

$$\text{(a) } \underline{b} < 0, \quad \text{(b) } 0 < \overline{b}, \quad \text{(c) } \underline{b} = 0 = \overline{b}.$$

In case (a) (resp. (b)), the value $\underline{b}$ (resp. $\overline{b}$) becomes a critical value of I. In particular, if $\underline{b} < 0 < \overline{b}$ hold, then there are at least two positive solutions (λ_1, u_1) and (λ_2, u_2) of (1.1) with $m = m_1$ with $I(\lambda_1, u_1) = \underline{b} < 0 < \overline{b} = I(\lambda_2, u_2)$. On the other hand, in case (c), we may prove that for each $\lambda \in \mathbf{R}$, any positive mountain pass solution to

$$-\Delta u + e^{\lambda} u = g(u) \quad \text{in } \mathbf{R}^N, \quad u \in H^1_{\mathrm{rad}}(\mathbf{R}^N) \tag{1.5}$$

turns out to be a positive solution of (1.1) with $m = m_1$. More precisely, let $\lambda \in \mathbf{R}$ and $u \in H^1_{\mathrm{rad}}(\mathbf{R}^N)$ be a solution of (1.5) corresponding to $b(\lambda)$. Notice that u can be chosen as a positive function. Then $\int_{\mathbf{R}^N} u^2 \, \mathrm{d}x = 2m_1$, and hence $(\lambda, u) \in \mathbf{R} \times H^1_{\mathrm{rad}}(\mathbf{R}^N)$ is a solution of (1.1). Thus, in case (c), there are infinitely many positive solutions of (1.1) with $m = m_1$. Though we may prove that case (c) occurs when $g(s) = |s|^{p-1}s$, it is not known that there is a nontrivial g in which case (c) holds.

To implement the above argument, in [CiGaIkTa-1], *Palais–Smale–Pohozaev–Cerami sequences* ((PSPC) sequences in short) and the *Palais–Smale–Pohozaev–Cerami condition* ((PSPC) condition in short) are introduced. Here $((\lambda_j, u_j))_{j=1}^{\infty} \subset \mathbf{R} \times H^1_{\mathrm{rad}}(\mathbf{R}^N)$ is called a (PSPC) sequence at level $c \in \mathbf{R}$ ($(\mathrm{PSPC})_c$ sequence in short) provided

$$\begin{aligned} &I(\lambda_j, u_j) \to c, \quad \left(1 + \|u_j\|_{H^1(\mathbf{R}^N)}\right) \|\partial_u I(\lambda_j, u_j)\|_{(H^1_{\mathrm{rad}}(\mathbf{R}^N))^*} \to 0, \\ &|\partial_\lambda I(\lambda_j, u_j)| \to 0, \quad P(\lambda_j, u_j) \to 0, \end{aligned} \tag{1.6}$$

where P is a functional corresponding to the Pohozaev identity defined by

$$P(\lambda, u) := \frac{N-2}{2} \int_{\mathbf{R}^N} |\nabla u|^2 \, \mathrm{d}x + N \int_{\mathbf{R}^N} \frac{e^{\lambda}}{2} u^2 - G(u) \, \mathrm{d}x .$$

Then I is said to satisfy the $(\mathrm{PSPC})_c$ condition if every $(\mathrm{PSPC})_c$ sequence is relatively compact in $\mathbf{R} \times H^1_{\mathrm{rad}}(\mathbf{R}^N)$. If we replace $(1 + \|u_j\|_{H^1(\mathbf{R}^N)})$ by 1 in (1.6), then this notion is introduced in [HiTa19]. Condition (1.6) is motivated by Cerami [Ce78] and under (g1) and (g2), I satisfies the $(\mathrm{PSPC})_c$ condition for all $c \in \mathbf{R} \setminus \{0\}$. By this compactness condition, we may show that $\underline{b}$ (resp. $\overline{b}$) is a critical value of I when $\underline{b} < 0$ (resp. $\overline{b} > 0$). On the other hand, in case (c), since $\underline{b} = 0 = \overline{b}$, this idea does not work. Instead, we use the existence of optimal path for $b(\lambda)$ due to Jeanjean and Tanaka [JeTa03].

1.2 Another multiplicity result

As pointed in Section 1.1, when $\underline{b} < 0 < \overline{b}$ and g is odd, (1.1) with $m = m_1$ has at least two positive solutions. In [CiGaIkTa-2], an example of g enjoying $\underline{b} < 0 < \overline{b}$ is also given. On the other hand, when case (c) happens, there are infinitely many positive solutions of (1.1) with $m = m_1$, however, we do not know examples of g other than $|s|^{p-1}s$ in which case (c) occurs.

In this note, we shall prove another multiplicity result motivated by [CiGaIkTa-2].

Theorem 1.3. *For any* $k \in \mathbf{N}$ *there exists* $g_k \in C(\mathbf{R})$ *verifying* (g1) *and* $g_k \not\equiv |s|^{p-1}s$ *such that* (1.1) *with* $g = g_k$ *and* $m = m_1$ *has positive solutions* $((\mu_i, u_i))_{i=1}^{k} \subset \mathbf{R} \times H^1_{\mathrm{rad}}(\mathbf{R}^N)$ *such that*

$$\begin{aligned} &0 < \mu_1 < \mu_2 < \cdots < \mu_k, \quad u_i > 0 \quad \text{in } \mathbf{R}^N \quad (1 \leq i \leq k), \\ &u_i \neq \omega_\mu \quad \text{for each } i = 1, \ldots, k \text{ and } \mu \in (0, \infty). \end{aligned}$$

Though finding g_k in Theorem 1.3 is motivated by nonlinearities treated in [CiGaIkTa-2], the proof of Theorem 1.3 is different from [CiGaIkTa-1]. Indeed, for each $k \in \mathbf{N}$, we aim to find $g_k \in C(\mathbf{R})$ such that (g1) holds and

(A) there exists $(u_\lambda)_{\lambda \in \mathbf{R}} \subset H^1_{\mathrm{rad}}(\mathbf{R}^N)$ such that $\mathbf{R} \ni \lambda \mapsto u_\lambda \in H^1_{\mathrm{rad}}(\mathbf{R}^N)$ is of class C^1 and u_λ is a positive solution of (1.5) with $I(\lambda, u_\lambda) = b(\lambda)$ for each $\lambda \in \mathbf{R}$;

(B) the function defined by $\mathbf{R} \ni \lambda \mapsto b(\lambda)$ admits critical points $-\infty < \lambda_1 < \lambda_2 < \cdots < \lambda_k < \infty$.

If (A) and (B) hold, then $((e^{\lambda_i}, u_{\lambda_i}))_{1 \le i \le k}$ are the desired solutions of (1.1) with $g = g_k$ and $m = m_1$. Indeed, since u_{λ_i} is a positive solution of (1.5), it is enough to prove $\int_{\mathbf{R}^N} |u_{\lambda_i}|^2 \, \mathrm{d}x = 2m_1$. This can be seen from

$$0 = \frac{\mathrm{d}}{\mathrm{d}\lambda} b(\lambda)|_{\lambda=\lambda_i} = \frac{\mathrm{d}}{\mathrm{d}\lambda} I(\lambda, u_\lambda)|_{\lambda=\lambda_i} = \partial_\lambda I(\lambda_i, u_{\lambda_i}) + \partial_u I(\lambda_i, u_{\lambda_i}) \frac{\mathrm{d}}{\mathrm{d}\lambda} u_\lambda|_{\lambda=\lambda_i} = \partial_\lambda I(\lambda_i, u_{\lambda_i}).$$

In the rest of this note, we shall find g_k satisfying $g_k \not\equiv |s|^{p-1}s$, (g1), (A) and (B).

2 Proof of Theorem 1.3

As pointed in the end of Section 1.2, for any given $k \in \mathbf{N}$, we shall find $g_k \in C(\mathbf{R})$ satisfying $g_k \not\equiv |s|^{p-1}s$, (g1), (A) and (B).

Notation: In the rest of this note, we shall use the following notations.

(i) For any $q \in [1, \infty]$ and domain $\Omega \subset \mathbf{R}^N$,

$$\|u\|_{q,\Omega} := \begin{cases} \displaystyle\int_\Omega |u|^{q+1} \, \mathrm{d}x & \text{when } 1 \le q < \infty, \\ \operatorname*{ess\,sup}_\Omega |u| & \text{when } q = \infty. \end{cases}$$

When $\Omega = \mathbf{R}^N$, we simply write $\|u\|_{q,\mathbf{R}^N} = \|u\|_q$ and also introduce the following notation:

$$\langle u, v \rangle_{H^1} := \int_{\mathbf{R}^N} \nabla u \cdot \nabla v + uv \, \mathrm{d}x \,, \quad \|u\|_{H^1} := \sqrt{\langle u, u \rangle_{H^1}}.$$

(ii) $H := H^1_{\mathrm{rad}}(\mathbf{R}^N)$.

(iii) For each $\lambda \in \mathbf{R}$, write $\mu = e^\lambda$. For instance, I can be written as

$$I(\lambda, u) = \frac{1}{2}\|\nabla u\|_2^2 - \int_{\mathbf{R}^N} G(u) + \mu \left(\frac{\|u\|_2^2}{2} - m_1 \right).$$

Motivated by the nonlinearities treated in [CiGaIkTa-2], we shall treat the following class of nonlinearities:

$$g_{a,\eta}(s) := (1 + \eta a(s)) s_+^p, \quad G_{a,\eta}(s) := \int_0^s g_{a,\eta}(t) \, \mathrm{d}t = \int_0^s (1 + \eta a(t)) t_+^p \, \mathrm{d}t \,. \tag{2.1}$$

Here $s_+ := \max\{0, s\}$, $\eta \in (0, 1/2]$ and a satisfies the following conditions for some $L \ge 1$:

$$\begin{aligned} & a \in C^1_c((0,\infty)), \quad -1 \le a(s) \le 1 \quad \text{for any } s \in \mathbf{R}, \\ & |a(s)| = 1 \quad \text{for all } s \in [1/L, L], \quad |sa'(s)| \le 4(e-1) \quad \text{for every } s \in \mathbf{R}. \end{aligned} \tag{2.2}$$

Denote by $\mathcal{A}_L$ the set of all a satisfying (2.2). We remark that for each $L \geq 1$, $\mathcal{A}_L \neq \emptyset$. Indeed, consider

$$a_0(s) := \begin{cases} 0 & \text{if } 0 \leq s \leq \dfrac{1}{4L}, \\ \log\left(4L(e-1)s + 2 - e\right) & \text{if } \dfrac{1}{4L} < s \leq \dfrac{1}{2L}, \\ 1 & \text{if } \dfrac{1}{2L} \leq s \leq 2L, \\ 1 - \log\left(\dfrac{e-1}{2L}s + 2 - e\right) & \text{if } 2L < s \leq 4L, \\ 0 & \text{if } 4L < s. \end{cases}$$

Since a_0 is Lipschitz continuous and $|sa_0'(s)| \leq 2(e-1)$ for any $s \in [0,\infty)\setminus\{1/4L, 1/2L, 2L, 4L\}$, using a mollifier, we may find a with $a \in \mathcal{A}_L$. Remark also that if $a \in \mathcal{A}_L$, then $-a \in \mathcal{A}_L$.

It is immediate to verify that $g_{a,\eta}$ satisfies (g1) for any $\eta \in (0,1/2]$, $L \geq 1$ and $a \in \mathcal{A}_L$. Moreover, from (2.1) and (2.2) it follows that for each $\eta > 0$ and $a \in \mathcal{A}_L$,

$$\mu^{-N/4-1} g_{a,\eta}\big(\mu^{N/4}s\big) = \big(1 + \eta a\big(\mu^{N/4}s\big)\big) s_+^p$$

and

$$\begin{aligned} \mu^{-N/2-1} G_{a,\eta}\big(\mu^{N/4}s\big) &= \mu^{-N/2-1} \int_0^{\mu^{N/4}s} (1+\eta a(\tau))\tau_+^p \, \mathrm{d}\tau \\ &= \int_0^s \big(1 + \eta a\big(\mu^{N/4}t\big)\big) t_+^p \, \mathrm{d}t = G_{a(\mu^{N/4}\cdot),\eta}(s). \end{aligned} \tag{2.3}$$

Let $a \in \mathcal{A}_L$ and set

$$I(a,\eta;\lambda,u) := \frac{1}{2}\|\nabla u\|_2^2 - \int_{\mathbf{R}^N} G_{a,\eta}(u)\,\mathrm{d}x + \mu\left(\frac{1}{2}\|u\|_2^2 - m_1\right).$$

For our aim, it is convenient to introduce a scaled functional of I. More precisely, for $u \in H$, write $u_\lambda(x) := \mu^{N/4}u(\mu^{1/2}x)$ and $a_\mu(s) := a(\mu^{N/4}s)$. Then it follows from (2.3) that

$$\begin{aligned} I(a,\eta;\lambda,u_\lambda) &= \mu\left\{\frac{1}{2}\|\nabla u\|_2^2 + \frac{1}{2}\|u\|_2^2 - \mu^{-N/2-1}\int_{\mathbf{R}^N} G_{a,\eta}\big(\mu^{N/4}u(x)\big)\,\mathrm{d}x - m_1\right\} \\ &= \mu\left\{\frac{1}{2}\|\nabla u\|_2^2 + \frac{1}{2}\|u\|_2^2 - \int_{\mathbf{R}^N} G_{a_\mu,\eta}(u)\,\mathrm{d}x - m_1\right\} \\ &=: \mu\{K(a,\eta;\lambda,u) - m_1\}. \end{aligned} \tag{2.4}$$

We shall also write $b(a,\eta;\lambda)$ for the mountain pass value of $H \ni u \mapsto K(a,\eta;\lambda,u)$:

$$\begin{aligned} b(a,\eta;\lambda) &:= \inf_{\gamma\in\Gamma(a,\eta;\lambda)} \max_{0\leq t\leq 1} K(a,\eta;\lambda,\gamma(t)), \\ \Gamma(a,\eta;\lambda) &:= \{\, \gamma \in C([0,1],H \mid \gamma(0) = 0,\ K(a,\eta;\lambda,\gamma(1)) < 0 \,\}. \end{aligned}$$

It is known that $b(a,\eta;\lambda)$ is a critical value of $K(a,\eta;\lambda,\cdot)$ for each $a \in \mathcal{A}_L$, $\eta \in [-1/2,1/2]$ and $\lambda \in \mathbf{R}$ (see [BeGaKa83, BeLi83, JeTa03]) and set

$$\mathcal{S}_{a,\eta;\lambda} := \{\, u \in H \mid \partial_u K(a,\eta;\lambda,u) = 0,\ K(a,\eta;\lambda,u) = b(a,\eta;\lambda) \,\}.$$

Since each $u \in \mathcal{S}_{a,\eta;\lambda}$ satisfies

$$0 = \partial_u K(a,\eta;\lambda,u)u^- = -\big\|u^-\big\|_{H^1}^2,$$

we have $u \geq 0$. By $K(a,\eta;\lambda,u) = b(a,\eta;\lambda) > 0$ and $u \not\equiv 0$, the strong maximum principle yields $u > 0$ in $\mathbf{R}^N$.

We next introduce

$$K_{1/2}(u) := \int_{\mathbf{R}^N} \frac{1}{2}|\nabla u|^2 + \frac{1}{2}u^2 - \frac{|u|^{p+1}}{2(p+1)} \, \mathrm{d}x\,, \quad K_{3/2}(u) := \int_{\mathbf{R}^N} \frac{1}{2}|\nabla u|^2 + \frac{1}{2}u^2 - \frac{3|u|^{p+1}}{2(p+1)} \, \mathrm{d}x$$

and write $b_{1/2}$ and $b_{3/2}$ for the mountain pass value of $K_{1/2}$ and $K_{3/2}$. Since

$$\frac{1}{2(p+1)} s_+^{p+1} \leq G_{a_\mu,\eta}(s) \leq \frac{3}{2(p+1)} s_+^{p+1} \quad \text{for all } s \in \mathbf{R},\ \eta \in \left(0, \frac{1}{2}\right],\ L \geq 1,\ a \in \mathcal{A}_L,$$

it follows that for each $(\lambda, u) \in \mathbf{R} \times H$, $\eta \in (0, 1/2]$, $L \geq 1$ and $a \in \mathcal{A}_L$,

$$K_{3/2}(u) \leq K(a,\eta;\lambda,u) \leq K_{1/2}(u),$$

which gives

$$0 < b_{3/2} \leq b(a,\eta;\lambda) \leq b_{1/2} \quad \text{for any } L \geq 1,\ a \in \mathcal{A}_L,\ \lambda \in \mathbf{R}.$$

Now we set

$$\mathcal{G}_{a,\eta;\lambda} := \left\{ u \in H \;\middle|\; K(a,\eta;\lambda,u) \in \left[\frac{b_{3/2}}{2}, 2b_{1/2}\right],\ \partial_u K(a,\eta;\lambda,u) = 0 \right\}.$$

It is easily seen that $\emptyset \neq \mathcal{S}_{a,\eta;\lambda} \subset \mathcal{G}_{a,\eta;\lambda}$.

In order to state a next result, we define Ψ_0 by

$$\Psi_0(u) := \int_{\mathbf{R}^N} \frac{1}{2}|\nabla u|^2 + \frac{1}{2}u^2 - \frac{|u|^{p+1}}{p+1} \, \mathrm{d}x \in C^2(H, \mathbf{R}).$$

Remark that Ψ_0 corresponds to (1.3) and any critical point of Ψ_0 gives a solution of (1.3). Thanks to [Kw89], Ψ_0 has only one critical point in H, which is positive in $\mathbf{R}^N$.

Proposition 2.1. *For any $\varepsilon > 0$ there exists $\eta_\varepsilon \in (0, 1/2)$ such that*

$$\sup \left\{ |K(a,\eta;\lambda,u) - m_1| + \|u - \omega_1\|_{H^1} \;\middle|\; \begin{array}{l} \lambda \in \mathbf{R},\ \ \eta \in (0, \eta_\varepsilon],\ \ L \geq 1, \\ a \in \mathcal{A}_L,\ \ u \in \mathcal{G}_{a,\eta;\lambda} \end{array} \right\} < \varepsilon.$$

In particular, $b(a,\eta,\lambda) \to m_1 \in [b_{3/2}, b_{1/2}]$ as $\eta \to 0^+$ uniformly with respect to $L \geq 1$, $a \in \mathcal{A}_L$ and $\lambda \in \mathbf{R}$.

Proof. We argue by contradiction and suppose that there exist $\varepsilon_0 > 0$, $(\eta_n)_{n=1}^\infty$, $(\lambda_n)_{n=1}^\infty$, $(L_n)_{n=1}^\infty$, $a_n \in \mathcal{A}_{L_n}$ and $u_n \in \mathcal{G}_{a_n,\eta_n;\lambda_n}$ such that

$$\eta_n \to 0, \quad |K(a_n,\eta_n;\lambda_n,u_n) - m_1| + \|u_n - \omega_1\|_{H^1} \geq \varepsilon_0.$$

By $u_n \in \mathcal{G}_{a_n,\eta_n;\lambda_n}$, $(K(a_n,\eta_n;\lambda_n,u_n))_{n=1}^\infty$ is bounded, and hence we may assume

$$K(a_n,\eta_n;\lambda_n,u_n) \to b_\infty \in \left[\frac{b_{3/2}}{2}, 2b_{1/2}\right].$$

Furthermore, since $\partial_u K(a_n,\eta_n;\lambda_n,u_n) = 0$, the Pohozaev identity holds:

$$0 = \frac{N-2}{2}\|\nabla u_n\|_2^2 - N\int_{\mathbf{R}^N} \frac{1}{2}u_n^2 - G_{(a_n)_{\mu_n},\eta_n}(u_n) \, \mathrm{d}x = NK(a_n,\eta_n;\lambda_n,u_n) - \|\nabla u_n\|_2^2.$$

Thus, $(\nabla u_n)_n$ is bounded in $L^2(\mathbf{R}^N)$.

Next, since $\|a_n\|_{L^\infty(\mathbf{R})} = 1$, the Gagliardo–Nirenberg's inequality gives

$$\begin{aligned} b_\infty + o(1) &= K(a_n, \eta_n; \lambda_n, u_n) - \frac{\partial_u K(a_n, \eta_n; \lambda_n, u_n) u_n}{p+1} \\ &\geq \frac{p-1}{2(p+1)} \|u_n\|_{H^1}^2 - C\eta_n \|u_n\|_{p+1}^{p+1} \geq \frac{p-1}{2(p+1)} \|u_n\|_{H^1}^2 - C\eta_n \|\nabla u_n\|_2^2 \|u_n\|_2^{4/N}. \end{aligned}$$

By $\eta_n \to 0$, $N \geq 2$ and the boundedness of $(\|\nabla u_n\|_2)_n$, we see that $(u_n)_{n=1}^\infty$ is bounded in H. Taking a subsequence if necessary, we may suppose $u_n \rightharpoonup u_\infty$ weakly in H and $u_n \to u_\infty$ strongly in $L^q(\mathbf{R}^N)$ for all $q \in (2, 2^*)$. The fact $u_n > 0$ in $\mathbf{R}^N$ implies $u_\infty \geq 0$ in $\mathbf{R}^N$. Since $\eta_n (a_n)_{\mu_n} \to 0$ strongly in $L^\infty(\mathbf{R})$, it follows that

$$\int_{\mathbf{R}^N} \nabla u_\infty \cdot \nabla \varphi + u_\infty \varphi - u_\infty^p \varphi \, \mathrm{d}x = 0 \quad \text{for any } \varphi \in H,$$

that is $\Psi_0'(u_\infty) = 0$ and u_∞ is a solution of (1.3). Moreover, notice that

$$\|u_\infty\|_{H^1}^2 \leq \liminf_{n\to\infty} \|u_n\|_{H^1}^2 = \int_{\mathbf{R}^N} \left(1 + \eta_n a_n\left(\mu_n^{N/4} u_n(x)\right)\right) u_n^{p+1} \, \mathrm{d}x \to \int_{\mathbf{R}^N} u_\infty^{p+1} \, \mathrm{d}x = \|u_\infty\|_{H^1}^2,$$

which gives $u_n \to u_\infty$ strongly in H. In particular,

$$0 < \frac{b_{3/2}}{2} \leq \Psi_0(u_\infty) = \lim_{n\to\infty} K(a_n, \eta_n; \lambda_n, u_n) \leq 2b_{1/2},$$

which means that u_∞ is a radial positive solution of (1.3) and $u_\infty = \omega_1$ holds by [Kw89]. Using the Pohozaev identity

$$0 = \frac{N-2}{2} \|\nabla \omega_1\|_2^2 + N \int_{\mathbf{R}^N} \frac{\omega_1^2}{2} - \frac{\omega_1^{p+1}}{p+1} \, \mathrm{d}x \,, \quad \frac{1}{2} \|\omega_1\|_2^2 = m_1,$$

we observe that $m_1 = \Psi_0(u_\infty)$. This leads to the following contradiction:

$$0 < \varepsilon_0 \leq \lim_{n\to\infty} \left\{ |K(a_n, \mu_n; \lambda_n, u_n) - m_1| + \|u_n - \omega_1\|_{H^1} \right\} = 0.$$

Thus, Proposition 2.1 holds. ■

To proceed, we remark that ω_1 is nondegenerate thanks to [Kw89], namely,

$$\Psi_0''(\omega_1) : H \to H^* \text{ is invertible.} \tag{2.5}$$

Thus, there exists $\rho_0 > 0$ such that for $T \in \mathcal{L}(H, H^*)$,

$$\|T - \Psi_0''(\omega_1)\|_{\mathcal{L}(H,H^*)} \leq \rho_0 \quad \Rightarrow \quad T \text{ is invertible.} \tag{2.6}$$

Proposition 2.2. *There exists $\eta_0 \in (0, 1/2)$ such that for each $\eta \in (0, \eta_0]$, $L \geq 1$, $a \in \mathcal{A}_L$ and $\lambda \in \mathbf{R}$, $\mathcal{G}_{a,\eta;\lambda} = \{ u_{a,\eta;\lambda} \} = \mathcal{S}_{a,\eta;\lambda}$ and the map $\mathbf{R} \ni \lambda \mapsto u_{a,\eta;\lambda} \in H$ is of class C^1. In particular, $\mathbf{R} \ni \lambda \mapsto b(a, \eta; \lambda) \in \mathbf{R}$ is of class C^1.*

Proof. We first prove $\mathcal{G}_{a,\eta;\lambda} = \{ u_{a,\eta;\lambda} \} (= \mathcal{S}_{a,\eta;\lambda})$ by contradiction and suppose that there exist $(\eta_n)_n$, $(L_n)_n$, $a_n \in \mathcal{A}_{L_n}$, $(\lambda_n)_n$ and $u_n, v_n \in \mathcal{G}_{a_n,\eta_n;\lambda_n}$ so that

$$\eta_n \to 0, \quad u_n \neq v_n.$$

By Proposition 2.1 we know that $\|u_n - \omega_1\|_{H^1} \to 0$ and $\|v_n - \omega_1\|_{H^1} \to 0$. Set

$$w_n(x) := \frac{u_n(x) - v_n(x)}{\|u_n - v_n\|_{H^1}}.$$

Since $\partial_u K(a_n, \eta_n; \lambda_n, u_n) = 0 = \partial_u K(a_n, \eta_n; \lambda_n, v_n)$, it follows that

$$\begin{aligned} -\Delta w_n + w_n &= \frac{1}{\|u_n - v_n\|_{H^1}}\big[(u_n^p - v_n^p) + \eta_n\big(a_n\big(\mu_n^{N/4}u_n(x)\big)u_n^p - a_n\big(\mu_n^{N/4}v_n(x)\big)v_n^p(x)\big)\big] \\ &= p\int_0^1 (\theta u_n + (1-\theta)v_n)^{p-1}\,\mathrm{d}\theta\, w_n + \frac{\eta_n}{\|u_n - v_n\|_{H^1}} f_n, \end{aligned} \tag{2.7}$$

where

$$f_n(x) := a_n\big(\mu_n^{N/4}u_n(x)\big)u_n^p(x) - a_n\big(\mu_n^{N/4}v_n(x)\big)v_n^p(x).$$

By writing

$$A_n(x,\theta) := a_n'\big(\mu_n^{N/4}[\theta u_n(x) + (1-\theta)v_n(x)]\big)\mu_n^{N/4}[\theta u_n(x) + (1-\theta)v_n(x)],$$

it is readily checked that if $u_n(x) < v_n(x)$, then

$$\begin{aligned} f_n(x) &= \big[a_n\big(\mu_n^{N/4}u_n(x)\big) - a_n\big(\mu_n^{N/4}v_n(x)\big)\big]u_n^p(x) + a_n\big(\mu_n^{N/4}v_n(x)\big)[u_n^p(x) - v_n^p(x)] \\ &= \int_0^1 a_n'\big(\mu_n^{N/4}[\theta u_n(x) + (1-\theta)v_n(x)]\big)\,\mathrm{d}\theta\,\mu_n^{N/4}(u_n(x) - v_n(x))u_n^p(x) \\ &\quad + pa_n\big(\mu_n^{N/4}v_n(x)\big)\int_0^1 [\theta u_n(x) + (1-\theta)v_n(x)]^{p-1}\,\mathrm{d}\theta\,(u_n(x) - v_n(x)) \\ &= \int_0^1 A_n(x,\theta)\frac{u_n(x)}{(1-\theta)u_n(x) + \theta v_n(x)}\,\mathrm{d}\theta\, u_n^{p-1}(x)(u_n(x) - v_n(x)) \\ &\quad + pa_n\big(\mu_n^{N/4}v_n(x)\big)\int_0^1 [\theta u_n(x) + (1-\theta)v_n(x)]^{p-1}\,\mathrm{d}\theta\,(u_n(x) - v_n(x)). \end{aligned} \tag{2.8}$$

In a similar way, when $v_n(x) < u_n(x)$, we have

$$\begin{aligned} f_n(x) &= -\big\{a_n\big(\mu_n^{N/4}v_n(x)\big)v_n^p(x) - a_n\big(\mu_n^{N/4}u_n(x)\big)u_n^p(x)\big\} \\ &= -\big[a_n\big(\mu_n^{N/4}v_n(x)\big) - a_n\big(\mu_n^{N/4}u_n(x)\big)\big]v_n^p(x) - a_n\big(\mu_n^{N/4}u_n(x)\big)[v_n^p(x) - u_n^p(x)] \\ &= -\int_0^1 A_n(x,1-\theta)\frac{v_n(x)}{\theta v_n(x) + (1-\theta)u_n(x)}\,\mathrm{d}\theta\, v_n^{p-1}(x)(v_n(x) - u_n(x)) \\ &\quad - pa_n\big(\mu_n^{N/4}u_n(x)\big)\int_0^1 [\theta v_n(x) + (1-\theta)u_n(x)]^{p-1}\,\mathrm{d}\theta\,(v_n(x) - u_n(x)). \end{aligned} \tag{2.9}$$

Notice that (2.2) yields $|A_n(x,\theta)| \le 4(e-1)$ and $|a_n(s)| \le 1$. Moreover, from (2.8), (2.9) and

$$\begin{aligned} &0 < \frac{u_n(x)}{(1-\theta)u_n(x) + \theta v_n(x)} \le 1 \quad \text{for all } \theta \in [0,1] \text{ if } u_n(x) < v_n(x), \\ &0 < \frac{v_n(x)}{(1-\theta)v_n(x) + \theta u_n(x)} \le 1 \quad \text{for all } \theta \in [0,1] \text{ if } v_n(x) < u_n(x). \end{aligned}$$

it follows that for some $C_0 > 0$, which is independent of n,

$$|f_n(x)| \le C_0\big\{u_n(x)^{p-1} + v_n(x)^{p-1}\big\}|u_n(x) - v_n(x)|.$$

Recalling $\eta_n \to 0$ and $u_n, v_n \to \omega_1$ strongly in $H^1(\mathbf{R}^N)$, we see that

$$\left\| \frac{\eta_n}{\|u_n - v_n\|_{H^1}} f_n \right\|_{H^*} \leq C_1 \eta_n \to 0.$$

Let $w_n \rightharpoonup w_\infty \in H$ weakly in $H^1(\mathbf{R}^N)$. Then, (2.7) gives

$$-\Delta w_\infty + w_\infty = p\omega_1^{p-1} w_\infty \quad \text{in } \mathbf{R}^N,$$

which can be expressed as $\Psi_0''(\omega_1) w_\infty = 0$ in H^*. Thus, (2.5) implies $w_\infty \equiv 0$. However, this yields $w_n \to 0$ strongly in $L^q(\mathbf{R}^N)$ for any $q \in (2, 2^*)$ and (2.7) leads to the following contradiction:

$$1 = \|w_n\|_{H^1}^2 \leq p \int_{\mathbf{R}^N} \int_0^1 \{(1-\theta)u_n + \theta v_n\}^{p-1} \, \mathrm{d}\theta \, w_n^2 \, \mathrm{d}x + C_1 \eta_n \|w_n\|_{H^1} \to 0 \quad \text{as } n \to \infty.$$

Hence, there exists $\eta_0 \in (0, 1/2)$ such that $\mathcal{G}_{a,\eta;\lambda} = \{ u_{a,\eta;\lambda} \}$ holds for any $\eta \in (0, \eta_0]$, $L \geq 1$, $a \in \mathcal{A}_L$ and $\lambda \in \mathbf{R}$.

For the assertion of the regularity of $\lambda \mapsto u_{a,\eta;\lambda}$, fix $\eta \in (0, \eta_0]$, $L \geq 1$ and $a \in \mathcal{A}_L$. Notice that

$$\mathbf{R} \times H \ni (\lambda, u) \mapsto \partial_u K(a, \eta; \lambda, u) \in H^*$$

is of class C^1 and

$$\left[\partial_u^2 K(a, \eta; \lambda, u) - \Psi_0''(u)\right](\varphi, \psi) = \eta \int_{\mathbf{R}^N} \left[a'\left(\mu^{N/4} u\right) \mu^{N/4} u_+ + pa(\mu^{N/4} u)\right] u_+^{p-1} \varphi \psi \, \mathrm{d}x \, .$$

Therefore, by

$$\begin{aligned} &\left\|\partial_u^2 K(a, \eta; \lambda, u_{a,\eta;\lambda}) - \Psi_0''(\omega_1)\right\|_{\mathcal{L}(H,H^*)} \\ \leq &\left\|\partial_u^2 K(a, \eta; \lambda, u_{a,\eta;\lambda}) - \Psi_0''(u_{a,\eta;\lambda})\right\|_{\mathcal{L}(H,H^*)} + \|\Psi_0''(u_{a,\eta;\lambda}) - \Psi_0''(\omega_1)\|_{\mathcal{L}(H,H^*)} \end{aligned}$$

and (2.2), for some $C_1 > 0$, we see that

$$\begin{aligned} &\sup \left\{ \left\|\partial_u^2 K(a, \eta; \lambda, u_{a,\eta;\lambda}) - \Psi_0''(\omega_1)\right\|_{\mathcal{L}(H,H^*)} \,\middle|\, \eta \in (0, \eta_0], \ L \geq 1, \ a \in \mathcal{A}_L, \ \lambda \in \mathbf{R} \right\} \\ \leq &\, C_1 \eta_0 + \sup \left\{ \|\Psi_0''(u_{a,\eta;\lambda}) - \Psi_0''(\omega_1)\|_{\mathcal{L}(H,H^*)} \,\middle|\, \eta \in (0, \eta_0], \ L \geq 1, \ a \in \mathcal{A}_L, \ \lambda \in \mathbf{R} \right\}. \end{aligned}$$

By recalling ρ_0 in (2.6) and shrinking $\eta_0 \in (0, 1/2)$ if necessary, Proposition 2.1 implies that

$$\sup \left\{ \left\|\partial_u^2 K(a, \eta; \lambda, u_{a,\eta;\lambda}) - \Psi_0''(\omega_1)\right\|_{\mathcal{L}(H,H^*)} \,\middle|\, \eta \in (0, \eta_0], \ L \geq 1, \ a \in \mathcal{A}_L, \ \lambda \in \mathbf{R} \right\} \leq \rho_0.$$

From (2.6) we conclude that $\partial_u^2 K(a, \eta; \lambda, u_{a,\eta;\lambda})$ is invertible for every $\eta \in (0, \eta_0]$, $L \geq 1$, $a \in \mathcal{A}_L$ and $\lambda \in \mathbf{R}$. Since $\partial_u K(a, \eta; \lambda, u_{a,\eta;\lambda}) = 0$, the implicit function theorem and the fact $\mathcal{S}_{a,\eta;\lambda} = \{ u_{a,\eta;\lambda} \} = \mathcal{G}_{a,\eta;\lambda}$ with $b(a, \eta, \lambda) \in [b_{3/2}, b_{1/2}]$ yield that $\mathbf{R} \ni \lambda \mapsto u_{a,\eta;\lambda}$ is of class C^1. ■

From here we fix η_0 as in Proposition 2.2. To find a distinct k critical points of $\mathbf{R} \ni \lambda \mapsto b(a, \eta_0; \lambda)$, we notice that for $\alpha \in [-\eta_0, \eta_0]$, the equation

$$-\Delta u + u = (1 + \alpha)|u|^{p-1} u \quad \text{in } \mathbf{R}^N \tag{2.10}$$

has a unique radial positive solution given by $(1 + \alpha)^{-1/(p-1)} \omega_1$ and it is the mountain pass solution. Therefore, the mountain pass value corresponding to (2.10) is $(1 + \alpha)^{-2/(p-1)} m_1$. Now we show the following result essentially obtained in [CiGaIkTa-2]:

Proposition 2.3. *As $L \to \infty$,*

$$\sup_{\substack{a\in\mathcal{A}_L \\ a=1 \text{ on } [L^{-1},L]}} \left| b(a,\eta_0;0) - (1+\eta_0)^{-2/(p-1)} m_1 \right| + \sup_{\substack{a\in\mathcal{A}_L \\ a=-1 \text{ on } [L^{-1},L]}} \left| b(a,\eta_0;0) - (1-\eta_0)^{-2/(p-1)} m_1 \right| \to 0.$$

Proof. We may prove this proposition as in [CiGaIkTa-2] and Proposition 2.1, and hence we only give a sketch of the proof. We argue indirectly and suppose that there exist $\varepsilon_0 > 0$, $(L_n)_n$ and $a_n \in \mathcal{A}_{L_n}$ such that

$$L_n \to \infty, \quad a_n \equiv 1 \text{ on } \left[L_n^{-1}, L_n\right], \quad \varepsilon_0 \le \left| b(a_n,\eta_0;0) - (1+\eta_0)^{-2/(p-1)} m_1 \right|.$$

Let $u_n \in \mathcal{S}_{a_n,\eta_0;0}$. Then $u_n > 0$ in $\mathbf{R}^N$, $(u_n)_{n=1}^\infty$ is bounded in $H^1(\mathbf{R}^N)$ through the Pohozaev identity and we may assume $u_n \rightharpoonup u_\infty$ weakly in $H^1(\mathbf{R}^N)$. Since

$$(1+\eta_0 a_n(s))s_+^p \to (1+\eta_0)s_+^p \quad \text{in } L^\infty_{\mathrm{loc}}(\mathbf{R}),$$

u_∞ satisfies

$$\int_{\mathbf{R}^N} \nabla u_\infty \cdot \nabla \varphi + u_\infty \varphi \,\mathrm{d}x = \int_{\mathbf{R}^N} (1+\eta_0) u_\infty^p \varphi \,\mathrm{d}x \quad \text{for every } \varphi \in H$$

and

$$\|u_\infty\|_{H^1}^2 = \int_{\mathbf{R}^N} (1+\eta_0) u_\infty^{p+1} \,\mathrm{d}x = \lim_{n\to\infty} \int_{\mathbf{R}^N} (1+\eta_0 a_n(u_n)) u_n^{p+1} \,\mathrm{d}x = \lim_{n\to\infty} \|u_n\|_{H^1}^2.$$

Thus, $u_n \to u_\infty$ strongly in $H^1(\mathbf{R}^N)$ and

$$0 < b_{1/2} \le \frac{1}{2}\|\nabla u_\infty\|_2^2 + \frac{1}{2}\|u_\infty\|_2^2 - \frac{1+\eta_0}{p+1}\|u_\infty\|_{p+1}^{p+1}.$$

Hence, u_∞ is a positive radial solution of $-\Delta u + u = (1+\eta_0)u^p$ in $\mathbf{R}^N$ and $u_\infty = (1+\eta_0)^{-1/(p-1)}\omega_1$, which yields

$$\begin{aligned}(1+\eta_0)^{-2/(p-1)} m_1 = \frac{1}{2}\|\nabla u_\infty\|_2^2 + \frac{1}{2}\|u_\infty\|_2^2 - \frac{1+\eta_0}{p+1}\|u_\infty\|_{p+1}^{p+1} &= \lim_{n\to\infty} K(a_n,\eta_0;0,u_n) \\ &= \lim_{n\to\infty} b(a_n,\eta_0;0).\end{aligned}$$

This is a contradiction. We can prove other assertion similarly and Proposition 2.3 holds. ■

We now prove Theorem 1.3:

Proof of Theorem 1.3. Let $\eta = \eta_0$, $L \ge 1$ and $a \in \mathcal{A}_L$. By Proposition 2.2, $\mathbf{R} \ni \lambda \mapsto u_{a,\lambda} := u_{a,\eta_0;\lambda} \in H$ is of class C^1. Write $v_{a,\lambda}(x) := \mu^{N/4} u_{a,\lambda}(\mu^{1/2}x)$. From (2.4), it follows that

$$\begin{aligned}I(\lambda, v_{a,\lambda}) &= \mu\left\{ \frac{1}{2}\|\nabla u_{a,\lambda}\|_2^2 + \frac{1}{2}\|u_{a,\lambda}\|_2^2 - \mu^{-N/2-1}\int_{\mathbf{R}^N} G_{a,\eta_0}\big(\mu^{N/4}u_{a,\lambda}(x)\big)\,\mathrm{d}x - m_1 \right\} \\ &= \mu\{K(a,\eta_0;\lambda,u_{a,\lambda}) - m_1\}.\end{aligned} \tag{2.11}$$

In particular, $\partial_u I(\lambda, v_{a,\lambda}) = 0$ for any $L \geq 1$, $a \in \mathcal{A}_L$ and $\lambda \in \mathbf{R}$. Furthermore, by $\mu = e^\lambda$ and (2.11),

$$\begin{aligned}
&\partial_\lambda(I(\lambda, v_{a,\lambda})) \\
&= \mu\left\{ \frac{1}{2}\|\nabla u_{a,\lambda}\|_2^2 + \frac{1}{2}\|u_{a,\lambda}\|_2^2 - \mu^{-\frac{N}{2}-1} \int_{\mathbf{R}^N} G_{a,\eta_0}\left(\mu^{\frac{N}{4}} u_{a,\lambda}\right) \mathrm{d}x - m_1 \right\} \\
&\quad + \mu\left\{ \left(\frac{N}{2}+1\right)\mu^{-\frac{N}{2}-1} \int_{\mathbf{R}^N} G_{a,\eta_0}\left(\mu^{\frac{N}{4}} u_{a,\lambda}\right) \mathrm{d}x - \frac{N}{4}\mu^{-\frac{N}{2}-1} \int_{\mathbf{R}^N} g_{a,\eta_0}\left(\mu^{\frac{N}{4}} u_{a,\lambda}\right)\mu^{\frac{N}{4}} u_{a,\lambda} \right\} \mathrm{d}x\,.
\end{aligned}$$

Since $u_{a,\lambda}$ is a solution to

$$-\Delta u + u = \mu^{-\frac{N}{2}-1} g\left(\mu^{\frac{N}{4}} u\right)\mu^{\frac{N}{4}} \quad \text{in } \mathbf{R}^N,$$

we have

$$\|u_{a,\lambda}\|_{H^1}^2 = \mu^{-\frac{N}{2}-1} \int_{\mathbf{R}^N} g\left(\mu^{\frac{N}{4}} u_{a,\lambda}\right)\mu^{\frac{N}{4}} u_{a,\lambda} \,\mathrm{d}x$$

and the Pohozaev identity holds:

$$0 = \frac{N-2}{2}\|\nabla u_{a,\lambda}\|_2^2 + N\left[\frac{1}{2}\|u_{a,\lambda}\|_2^2 - \mu^{-\frac{N}{2}-1} \int_{\mathbf{R}^N} G\left(\mu^{\frac{N}{4}} u_{a,\lambda}\right) \mathrm{d}x\right].$$

Using these two equations, we obtain

$$\begin{aligned}
\partial_\lambda(I(\lambda, v_{a,\lambda})) &= \mu\left\{ \frac{2-N}{4}\|u_{a,\lambda}\|_{H^1}^2 + \frac{N}{2}\mu^{-\frac{N}{2}-1} \int_{\mathbf{R}^N} G\left(\mu^{\frac{N}{4}} u_{a,\lambda}\right) \mathrm{d}x - m_1 \right\} \\
&= \mu\left\{ \frac{1}{2}\|u_{a,\lambda}\|_2^2 - m_1 \right\}.
\end{aligned}$$

Hence, to prove Theorem 1.3, it suffices to find suitable $L \geq 1$ and $a \in \mathcal{A}_L$ so that the function $\mathbf{R} \ni \lambda \mapsto I(\lambda, v_{a,\lambda})$ admits at least k distinct critical points.

For our aim, thanks to Proposition 2.3, there exists $L_0 > 1$ such that

$$b(a, \eta_0; 0) < m_1 < b(a, -\eta_0; 0) \quad \text{for every } a \in \mathcal{A}_{L_0}. \tag{2.12}$$

We fix $a_0 \in \mathcal{A}_{L_0}$ with $a_0 \equiv 1$ on $[L_0^{-1}, L_0]$, set $\lambda_1' := 1$ and choose $\lambda_1' = 1 \ll \lambda_2' \ll \lambda_3' \ll \cdots \ll \lambda_k'$ so that

$$\operatorname{supp} a_0\left(e^{-\frac{N}{4}\lambda_i'}\cdot\right) \cap \operatorname{supp} a_0\left(e^{-\frac{N}{4}\lambda_j'}\cdot\right) = \emptyset \quad \text{for each } i, j \text{ with } i \neq j.$$

Then consider

$$a(s) := \sum_{i=0}^{k-1} (-1)^{i-1} a_0\left(e^{-\frac{N}{4}\lambda_i'} s\right).$$

It is checked that $\widetilde{a}_i(s) := a\left(e^{N\lambda_i'/4}s\right)$ satisfies $\widetilde{a}_i(s) = (-1)^{i-1}$ on $[L_0^{-1}, L_0]$ for $i = 0, \ldots, k-1$ and $\widetilde{a}_i \in \mathcal{A}_{L_0}$. Since (2.3) gives $K(a, \eta_0; \lambda_i', u) = K(\widetilde{a}_i, \eta_0; 0, u)$, it follows that

$$b(a, \eta_0; \lambda_i') = K\left(a, \eta_0; \lambda_i', u_{a,\lambda_i'}\right) = K\left(\widetilde{a}_i, \eta_0; 0, u_{a,\lambda_i'}\right) = b(\widetilde{a}_i, \eta_0; 0).$$

Thus, we infer from (2.11) and (2.12) that

$$b(a, \eta_0; \lambda_i') \begin{cases} > m_1 & \text{if } i \text{ is even,} \\ < m_1 & \text{if } i \text{ is odd,} \end{cases} \qquad I(\lambda_i', v_{a,\lambda_i'}) \begin{cases} > 0 & \text{if } i \text{ is even,} \\ < 0 & \text{if } i \text{ is odd.} \end{cases}$$

For $i = 1, \ldots, k-1$, choose $\widetilde{\lambda}_i \in (\lambda_i', \lambda_{i+1}')$ so that $I(\widetilde{\lambda}_i, v_{a,\widetilde{\lambda}_i}) = 0$. As proved in [CiGaIkTa-1], since $I(\lambda, v_{a,\lambda}) \to 0$ as $|\lambda| \to \infty$, by setting $\widetilde{\lambda}_0 := -\infty$ and $\widetilde{\lambda}_{k+1} := \infty$, the function $(\widetilde{\lambda}_i, \widetilde{\lambda}_{i+1}) \ni \lambda \mapsto I(\lambda, v_{a,\lambda})$ takes a strictly positive maximum (resp. negative minimum) in $(\widetilde{\lambda}_i, \widetilde{\lambda}_{i+1})$ when i is even (resp. odd). Thus, let $\lambda_i \in (\widetilde{\lambda}_i, \widetilde{\lambda}_{i+1})$ be a maximum point (resp. minimum point) when i is even (resp. odd). Then

$$0 = \partial_\lambda (I(\lambda, v_{a,\lambda}))\big|_{\lambda=\lambda_i} \quad \text{for each } i = 1, \ldots, k.$$

Since $(\lambda_i, v_{a,\lambda_i})$ is a solution of

$$-\Delta u + e^{\lambda_i} u = (1 + a(u)) u^p \quad \text{in } \mathbf{R}^N, \quad \frac{1}{2} \int_{\mathbf{R}^N} u^2 \, \mathrm{d}x = m_1,$$

$(\lambda_i, (v_{a,\lambda_i}))_{i=1}^k$ are k distinct solutions of (1.1) with $m = m_1$. It is also clear that $(1+a(s))s^p \neq s^p$ and $v_{a,\lambda_i} \neq \omega_\mu$ since $I(\lambda_i, v_{a,\lambda_i}) > 0$ if i is even and $I(\lambda, v_{a,\lambda_i}) < 0$ if i is odd. This completes the proof. ■

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