# Solvability of the heat equation on a half-space with a dynamical boundary condition and unbounded initial data


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This is a survey article for the paper [5], which is joint work with Marek Fila (Comenius Univ.) and Kazuhiro Ishige (Univ. of Tokyo).

## 1 Introduction

Let $N \geq 2$ and $\mathbb{R}^N_+ := \mathbb{R}^{N-1} \times \mathbb{R}_+$. This paper is concerned with global solvability of the problem

$$\begin{cases} \partial_t u - \Delta u = 0, & x \in \mathbb{R}^N_+, \ \ t > 0, \\ \partial_t u + \partial_\nu u = 0, & x \in \partial\mathbb{R}^N_+, \ \ t > 0, \\ u(x,0) = \varphi(x), & x \in \mathbb{R}^N_+, \\ u(x,0) = 0, & x = (x',0) \in \partial\mathbb{R}^N_+, \quad x' := (x_1, x_2, \dots, x_{N-1}), \end{cases} \tag{1}$$

where $\partial_t := \partial/\partial t$, and $\partial_\nu := -\partial/\partial x_N$. The boundary condition from (1) describes diffusion through the boundary in processes such as thermal contact with a perfect conductor or diffusion of solute from a well-stirred fluid or vapour (see e.g. [2]). Various aspects of analysis of parabolic equations with dynamical boundary conditions have been treated by many authors (see e.g. the reference of [5]).

In this paper we focus on the simplest linear problem from a point of view which has not been considered yet (as far as we know). Namely, we are interested in an appropriate choice of the function space of initial functions $\varphi$ such that problem (1) is solvable.

Throughout this paper we often identify $\mathbb{R}^{N-1}$ with $\partial\mathbb{R}^N_+$. We introduce some notation. Let $\Gamma_D = \Gamma_D(x,y,t)$ be the Dirichlet heat kernel on $\mathbb{R}^N_+$, that is,

$$\Gamma_D(x,y,t) := (4\pi t)^{-\frac{N}{2}} \left[ \exp\left( -\frac{|x-y|^2}{4t} \right) - \exp\left( -\frac{|x-y_*|^2}{4t} \right) \right]$$

for $(x,y,t)\in\overline{\mathbb{R}^N_+}\times\mathbb{R}^N_+\times(0,\infty)$, where $y_*=(y',-y_N)$ for $y=(y',y_N)\in\mathbb{R}^N_+$. Set

$$[S_1(t)\phi](x):=\int_{\mathbb{R}^N_+}\Gamma_D(x,y,t)\phi(y)\,dy,\quad (x,t)\in\mathbb{R}^N_+\times(0,\infty), \tag{2}$$

for any measurable function $\phi$ in $\mathbb{R}^N_+$ if the right hand side of (2) is well-defined. For $x=(x',x_N)\in\overline{\mathbb{R}^N_+}$ and $t>0$, set

$$P(x',x_N,t):=C_N(x_N+t)\biggl(|x'|^2+(x_N+t)^2\biggr)^{-\frac{N}{2}},$$

where $C_N$ is the constant chosen so that

$$\int_{\mathbb{R}^{N-1}}P(x',x_N,t)dx'=1\quad\text{for all } x_N\ge 0 \text{ and } t>0.$$

Then $P=P(x',x_N,t)$ is the fundamental solution of the Laplace equation in $\mathbb{R}^N_+$ with the homogeneous dynamical boundary condition (see e.g. [1]). Set

$$[S_2(t)\psi](x):=\int_{\mathbb{R}^{N-1}}P(x'-y',x_N,t)\psi(y')\,dy',\quad (x,t)\in\mathbb{R}^N_+\times(0,\infty), \tag{3}$$

for any measurable function $\psi$ in $\mathbb{R}^{N-1}$ if the right hand side of (3) is well-defined. Then the function $\Psi(x,t):=[S_2(t)\psi](x)$ satisfies

$$\begin{cases} -\Delta\Psi=0, & x\in\mathbb{R}^N_+,\ \ t>0,\\ \partial_t\Psi+\partial_\nu\Psi=0, & x\in\partial\mathbb{R}^N_+,\ \ t>0,\\ \Psi(x,0)=\psi(x'), & x=(x',0)\in\partial\mathbb{R}^N_+. \end{cases}$$

Consider

$$\begin{cases} \partial_t v=\Delta v-F[v],\quad \Delta w=0, & x\in\mathbb{R}^N_+,\ \ t>0,\\ v=0,\quad \partial_t w-\partial_{x_N}w=\partial_{x_N}v, & x\in\partial\mathbb{R}^N_+,\ \ t>0,\\ v(x,0)=\varphi(x), & x\in\mathbb{R}^N_+,\\ w(x,0)=0, & x=(x',0)\in\partial\mathbb{R}^N_+, \end{cases} \tag{4}$$

where

$$\begin{aligned} F[v](x,t):=&\int_{\mathbb{R}^{N-1}}P(x'-y',x_N,0)\partial_{x_N}v(y',0,t)\,dy'\\ &+\int_0^t\int_{\mathbb{R}^{N-1}}\partial_t P(x'-y',x_N,t-s)\partial_{x_N}v(y',0,s)\,dy'\,ds. \end{aligned} \tag{5}$$

Following [4], we formulate the definition of a solution of (1).

**Definition 1.1** *Let $\varphi$ be measurable function in $\mathbb{R}^N_+$. Let $0 < T \le \infty$ and*

$$v,\ \partial_{x_N} v,\ w \in C(\overline{\mathbb{R}^N_+} \times (0,T)).$$

*We call $(v,w)$ a solution of (4) in $\mathbb{R}^N_+ \times (0,T)$ if*

$$[S_1(t)\varphi](x), \quad \int_0^t [S_1(t-s)F[v](s)](x)\,ds, \quad \int_0^t [S_2(t-s)\partial_{x_N} v(\cdot,0,s)](x)\,ds$$

*are well-defined and functions $v$ and $w$ satisfy*

$$\begin{aligned} v(x,t) &= [S_1(t)\varphi](x) - \int_0^t [S_1(t-s)F[v](s)](x)\,ds, \\ w(x,t) &= \int_0^t [S_2(t-s)\partial_{x_N} v(\cdot,0,s)](x)\,ds, \end{aligned}$$

*for $x \in \overline{\mathbb{R}^N_+}$ and $t \in (0,T)$, respectively. Then we say that $u := v + w$ is a solution of (1) in $\mathbb{R}^N_+ \times (0,T)$. In the case of $T = \infty$, we call $(v,w)$ a global-in-time solution of (4) and $u$ a global-in-time solution of (1).*

For $1 \le r \le \infty$, we write $|\cdot|_{L^r} := \|\cdot\|_{L^r(\partial\mathbb{R}^N_+)}$ and $\|\cdot\|_{L^r} := \|\cdot\|_{L^r(\mathbb{R}^N_+)}$ for simplicity. Furthermore, for $1 \le r \le \infty$ and $\alpha \ge 0$, we define

$$L^r_\alpha := \{f \in L^r(\mathbb{R}^N_+) : \|f\|_{L^r_\alpha} < \infty\},$$

where

$$\|f\|_{L^r_\alpha} := \begin{cases} \left( \displaystyle\int_{\mathbb{R}^N_+} |f(x)|^r h(x_N)^{-\alpha r}\,dx \right)^{\frac{1}{r}} & \text{if} \quad 1 \le r < \infty, \\ \|f\|_{L^\infty} & \text{if} \quad r = \infty, \end{cases} \qquad \text{with } h(x_N) := \frac{x_N}{x_N + 1}.$$

Then we can easily show that $\|f\|_{L^r_\alpha} \le \|f\|_{L^r_\beta}$ for $r \in [1,\infty]$ and $0 \le \alpha \le \beta$.

Now we are ready to state the main results of this paper.

**Theorem 1.1** *Let $N \ge 2$ and $1 \le q \le \infty$. Furthermore, let*

$$p \in (Nq/(N-1), \infty] \quad \text{if} \quad q < \infty \qquad \text{and} \qquad p = \infty \quad \text{if} \quad q = \infty.$$

*For $r \in [q,\infty]$, put*

$$\alpha(r) = (N-1)\left(\frac{1}{q} - \frac{1}{r}\right) + \frac{1}{q}. \tag{6}$$

*Assume $\varphi \in L^q_{\alpha(p)}$. Then problem (4) possesses a unique global-in-time solution $(v,w)$ with the following property: For any $T > 0$ there exists $C_T > 0$ such that*

$$\begin{aligned} &\sup_{0<t<T} \left[ t^{\frac{N}{2}(\frac{1}{q}-\frac{1}{p})} \left( \|v(t)\|_{L^p} + t^{\frac{1}{2}} \|\partial_{x_N} v(t)\|_{L^p} \right) + t^{\frac{1}{2}} |\partial_{x_N} v(t)|_{L^r} \right] \le C_T \|\varphi\|_{L^q_{\alpha(p)}}, \\ &\sup_{0<t<T} \left[ \|w(t)\|_{L^p} + |w(t)|_{L^r} \right] \le C_T \|\varphi\|_{L^q_{\alpha(p)}}, \end{aligned} \tag{7}$$

*for $r \in [q,p]$. Furthermore, $v$ and $w$ are bounded and smooth in $\overline{\mathbb{R}^N_+} \times I$ for any bounded interval $I \subset (0,\infty)$.*

**Remark 1.1** *We explain the role of the space $L^q_{\alpha(p)}$ in our study. Let $1 \le q \le \infty$ and take arbitrary functions $\Phi \in L^q(\mathbb{R}^{N-1})$, $\vartheta \in L^q(1,\infty)$. Now set $\varphi(x) := \Phi(x')\Psi(x_N)$ for $x = (x', x_N) \in \mathbb{R}^N_+$, where*

$$\Psi(x_N) := \begin{cases} x_N^\lambda & \text{if} \quad 0 < x_N \le 1, \quad \lambda \in \mathbb{R}, \\ \vartheta(x_N) & \text{if} \quad x_N > 1. \end{cases}$$

*Choose $p$ as in Theorem 1.1. Then it is easy to check that $\varphi \in L^q_{\alpha(p)}$ if and only if*

$$\lambda > (N-1)\left(\frac{1}{q} - \frac{1}{p}\right) (> 0 \quad \text{if } q < \infty).$$

*If $\lambda > 0$, then $\lim_{x_N \to 0} \varphi(x) = 0$ which means that the condition $u(x', 0, 0) = 0$ in (1) is satisfied. This indicates that the choice of the space of initial functions is natural and also optimal in some sense since $\lambda$ can be arbitrarily close to 0 if $q$ is large enough.*

We have not observed the importance of the behavior of $\varphi$ near $\partial\mathbb{R}^N_+$ in the $L^\infty$-setting in [4]. The main novelty of this paper consists in working in an appropriate weighted $L^q$-space by which we extend a result from [4] significantly, as we explain below.

In [4] we studied the problem

$$\begin{cases} \partial_t u - \Delta u = 0, & x \in \mathbb{R}^N_+, \ t > 0, \\ \partial_t u + \partial_\nu u = 0, & x \in \partial\mathbb{R}^N_+, \ t > 0, \\ u(x,0) = \varphi(x), & x \in \mathbb{R}^N_+, \\ u(x,0) = \varphi_b(x'), & x = (x', 0) \in \partial\mathbb{R}^N_+, \end{cases} \tag{8}$$

where $\varphi$ and $\varphi_b$ are bounded functions. A part of Theorem 1.1 in [4] reads as follows:

**Theorem 1.2** *Let $N \ge 2$, $\varphi \in L^\infty(\mathbb{R}^N_+)$ and $\varphi_b \in L^\infty(\mathbb{R}^{N-1})$. Then problem (8) possesses a unique global-in-time solution $u$ which is bounded and smooth in $\overline{\mathbb{R}^N_+} \times I$ for any bounded interval $I \subset (0,\infty)$.*

Hence, if $\varphi_b \equiv 0$ then Theorem 1.2 is a very special case of Theorem 1.1. If $\varphi_b \in L^\infty(\mathbb{R}^{N-1})$ and $\varphi \in L^q_{\alpha(p)}$ with $p, q$ as in Theorem 1.1, then we can combine Theorem 1.1 with Theorem 1.2 to obtain the existence of a solution of (8) easily, since the problem is linear.

# 2 Preliminaries

In this section we prove several lemmata on $S_1(t)\phi$ and $F[v]$, and recall some properties of $S_2(t)\psi$. In what follows, by the letter $C$ we denote generic positive constants (independent of $x$ and $t$) and they may have different values also within the same line.

We first recall some properties of $S_1(t)\phi$ (see e.g., [6] and [4, Lemma 2.1]).

$(G_1)$ For any $1 \le q \le r \le \infty$,

$$\|S_1(t)\phi\|_{L^r} \le c_1 t^{-\frac{N}{2}(\frac{1}{q}-\frac{1}{r})}\|\phi\|_{L^q}, \qquad t>0,$$

for all $\phi \in L^q(\mathbb{R}^N_+)$, where $c_1$ is a positive constant, independent of $q$ and $r$. In particular, if $q=r$, then

$$\sup_{t>0} \|S_1(t)\phi\|_{L^r} \le \|\phi\|_{L^r}.$$

Furthermore, for any $1 \le q \le r \le \infty$,

$$\|\partial_{x_N} S_1(t)\phi\|_{L^r} \le c_2 t^{-\frac{N}{2}(\frac{1}{q}-\frac{1}{r})-\frac{1}{2}}\|\phi\|_{L^q}, \qquad t>0,$$

for all $\phi \in L^q(\mathbb{R}^N_+)$, where $c_2$ is a positive constant, independent of $q$ and $r$.

$(G_2)$ Let $\phi \in L^q(\mathbb{R}^N_+)$ with $1 \le q \le \infty$ and $T>0$. Then $S_1(t)\phi$ is bounded and smooth with respect to $x$ and $t$ in $\overline{\mathbb{R}^N_+} \times (T,\infty)$.

Applying an argument similar to the proof of Young's inequality, we have the following.

**Lemma 2.1** *Let $1 \le q \le r \le \infty$. Assume $\phi \in L^q_{\alpha(r)}$ with $\alpha(r)$ as in (6). Then there exists $c_3 = c_3(N) > 0$ such that*

$$|\partial_{x_N}[S_1(t)\phi]|_{L^r} \le c_3 t^{-\frac{1}{2}}\|\phi\|_{L^q_{\alpha(r)}}, \qquad t>0. \tag{9}$$

Next we recall some properties of $S_2(t)\psi$.

$(P_1)$ Let $\psi \in L^r(\mathbb{R}^{N-1})$ for some $r \in [1,\infty]$ and $t$, $t'>0$. Then

$$[S_2(t)\psi](x',x_N) = [S_2(t+x_N)\psi](x',0),$$
$$[S_2(t+t')\psi](x) = [S_2(t)(S_2(t')\psi)](x),$$

for $x=(x',x_N) \in \overline{\mathbb{R}^N_+}$. Furthermore,

$$\lim_{t\to 0} |S_2(t)\psi - \psi|_r = 0 \quad \text{if } 1 \le r < \infty.$$

$(P_2)$ For any $1 \le r \le q \le \infty$,

$$|S_2(t)\psi|_{L^q} \le Ct^{-(N-1)(\frac{1}{r}-\frac{1}{q})}|\psi|_{L^r}, \qquad t>0,$$

for all $\psi \in L^r(\mathbb{R}^{N-1})$. In particular, if $q=r$, then

$$\sup_{t>0} |S_2(t)\psi|_{L^q} \le |\psi|_{L^q}.$$

$(P_3)$ Let $1 \le r < \infty$ and $Nr/(N-1) < q \le \infty$. Then

$$\|S_2(t)\psi\|_{L^q} \le C t^{-(N-1)(\frac{1}{r}-\frac{1}{q})+\frac{1}{q}} |\psi|_{L^r}, \qquad t > 0,$$

for all $\psi \in L^r(\mathbb{R}^{N-1})$. Furthermore,

$$\sup_{t>0} \|S_2(t)\psi\|_{L^q} \le C(|\psi|_{L^q} + |\psi|_{L^r}) \tag{10}$$

for all $\psi \in L^q(\mathbb{R}^{N-1}) \cap L^r(\mathbb{R}^{N-1})$.

Properties $(P_1)$, $(P_2)$, and $(P_3)$ easily follow from (3) (see e.g. [3]) and imply that

$$\sup_{t>0} \|S_2(t)\psi\|_{L^\infty} \le |\psi|_{L^\infty}$$

for all $\psi \in L^\infty(\mathbb{R}^{N-1})$. Furthermore, by an argument similar to that in the proof of property $(G_2)$ we have:

$(P_4)$ Let $\psi \in L^r(\mathbb{R}^{N-1})$ with $1 \le r \le \infty$. Then, for any $T > 0$, $S_2(t)\psi$ is bounded and smooth in $\overline{\mathbb{R}^N_+} \times (T, \infty)$.

# 3 Proof of Theorem 1.1

In this section we prove Theorem 1.1. By [4, Theorem 1.1] with $\varepsilon = 1$ and $\varphi_b \equiv 0$ we have Theorem 1.1 for the case $p = q = \infty$. So we focus on the case $q < \infty$.

Let $T > 0$, $M \ge 1$, $1 \le q < \infty$, and $p \in (Nq/(N-1), \infty]$. Set

$$X_{T,M} := \left\{ v \,:\, v, \partial_{x_N} v \in C(\overline{\mathbb{R}^N_+} \times (0,T)), \|v\|_{X_{T,M}} < \infty \right\}, \quad \|v\|_{X_{T,M}} := \sup_{0<t<T} e^{-Mt} E[v](t),$$

where

$$E[v](t) := t^{\frac{N}{2}\left(\frac{1}{q}-\frac{1}{p}\right)} \left[ \|v(t)\|_{L^p} + t^{\frac{1}{2}} \|\partial_{x_N} v(t)\|_{L^p} \right] + \sup_{q \le r \le p} t^{\frac{1}{2}} |\partial_{x_N} v(t)|_{L^r}.$$

Then $X_{T,M}$ is a Banach space equipped with the norm $\|\cdot\|_{X_{T,M}}$. We apply the Banach contraction mapping principle in $X_{T,M}$ to find a fixed point of the functional

$$Q[v](t) := S_1(t)\varphi - D[v](t) \tag{11}$$

on $X_{T,M}$, where $D[v]$ is the function defined by

$$D[v](t) := \int_0^t S_1(t-s) F[v](s)\, ds \tag{12}$$

and $F[v]$ is the function defined by (5). For the function $F[v]$, we have the following.

**Lemma 3.1** *Let $T > 0$, $M \geq 1$, $1 \leq q < \infty$, and $p \in (Nq/(N-1), \infty]$. Assume that $v \in X_{T,M}$. Then there exists $C > 0$, independent of $T$ and $M$, such that, for $p \in (Nq/(N-1), \infty)$, it holds that*

$$\|F[v](t)\|_{L^p} \leq C(1 + t^{\frac{1}{p}})t^{-\frac{1}{2}}e^{Mt}\|v\|_{X_{T,M}}$$

*for $0 < t < T$. Furthermore, for any $r \in [q, p]$,*

$$\|F[v](\cdot, x_N, t)\|_{L^r(\mathbb{R}^{N-1})} \leq C\left(1 + (x_N^{-1}t)^{\frac{1}{2}}\right)t^{-\frac{1}{2}}e^{Mt}\|v\|_{X_{T,M}}$$

*for $x_N \in (0, \infty)$ and $0 < t < T$.*

Applying Lemma 3.1, we obtain the following estimate for the function $D[v]$.

**Lemma 3.2** *Assume the same conditions as in Lemma 3.1. Let $D[v]$ be the function defined by (12). Then there exists $M_* \geq 1$ such that*

$$\|D[v]\|_{X_{T,M}} \leq \frac{1}{2}\|v\|_{X_{T,M}} \tag{13}$$

*for $v \in X_{T,M}$ and $M \geq M_*$. Furthermore, $D[v]$ is bounded and smooth in $\overline{\mathbb{R}^N_+} \times (\tau, T)$ for any $0 < \tau < T$.*

Now we are ready to complete the proof of Theorem 1.1.

**Proof of Theorem 1.1.** Let $T > 0$, $M \geq 1$, $1 \leq q < \infty$, and $p \in (Nq/(N-1), \infty]$. Then, since $\|\varphi\|_{L^r_\alpha} \leq \|\varphi\|_{L^r_\beta}$ for $r \in [1, \infty]$ and $0 \leq \alpha \leq \beta$, by $(G_1)$ and (9) we have

$$e^{-Mt}E[S_1(t)\varphi](t) \leq (c_1 + c_2 + c_3)\|\varphi\|_{L^q_{\alpha(p)}} \tag{14}$$

for $t > 0$, where $c_1$, $c_2$, and $c_3$ are positive constants given in $(G_1)$ and Lemma 2.1, respectively, and $\alpha(p)$ is given in (6). Furthermore, by Lemma 3.2, taking a sufficiently large $M \geq 1$ if necessary, we see that

$$\|D[v]\|_{X_{T,M}} \leq \frac{1}{2}\|v\|_{X_{T,M}}, \qquad v \in X_{T,M}, \tag{15}$$

for $0 < t < T$. Set

$$m := 2(c_1 + c_2 + c_3)\|\varphi\|_{L^q_{\alpha(p)}}. \tag{16}$$

We deduce from (11), (14), (15), and (16) that

$$\begin{aligned} \|Q[v]\|_{X_{T,M}} &\leq \sup_{0<t<T} e^{-Mt}E[S_1(t)\varphi](t) + \|D[v]\|_{X_{T,M}} \\ &\leq (c_1 + c_2 + c_3)\|\varphi\|_{L^q_{\alpha(p)}} + \frac{1}{2}\|v\|_{X_{T,M}} \leq m \end{aligned} \tag{17}$$

for $v \in X_{T,M}$ with $\|v\|_{X_{T,M}} \leq m$. Similarly, it follows from (15) that

$$\|Q[v_1] - Q[v_2]\|_{X_{T,M}} = \|D[v_1 - v_2]\|_{X_{T,M}} \leq \frac{1}{2}\|v_1 - v_2\|_{X_{T,M}} \tag{18}$$

for $v_i \in X_{T,M}$ $(i=1,2)$. Then, by (17) and (18), applying the contraction mapping theorem, we find a unique solution $v \in X_{T,M}$ with $\|v\|_{X_{T,M}} \le m$ such that

$$v = Q[v] = S_1(t)\varphi - D[v](t) \quad \text{in} \quad X_{T,M}.$$

In particular, we see that

$$\|v\|_{X_{T,M}} \le C\|\varphi\|_{L^q_{\alpha(p)}}.$$

Moreover, by $(G_2)$ and Lemma 3.2, we see that $v$ is bounded and smooth in $\overline{\mathbb{R}^N_+} \times (T_1, T)$ for any $0 < T_1 < T$.

Set

$$w(x,t) = \int_0^t [S_2(t-s)\partial_{x_N} v(\cdot,0,s)](x)\, ds$$

for $x \in \overline{\mathbb{R}^N_+}$ and $t \in (0,T)$. By (10) and (16) we obtain

$$\begin{aligned}
\|w(t)\|_{L^p} &\le \int_0^t \|S_2(t-s)\partial_{x_N} v(\cdot,0,s)\|_{L^p}\, ds \\
&\le C\int_0^t \Big( |\partial_{x_N} v(s)|_{L^q} + |\partial_{x_N} v(s)|_{L^p} \Big)\, ds \\
&\le C\int_0^t e^{Ms} s^{-\frac{1}{2}} \|v\|_{X_{T,M}}\, ds \le Ce^{MT}T^{\frac{1}{2}}\|\varphi\|_{L^q_{\alpha(p)}} < \infty,
\end{aligned}$$

and

$$\begin{aligned}
|w(t)|_{L^r} &\le \int_0^t |S_2(t-s)\partial_{x_N} v(\cdot,0,s)|_{L^r}\, ds \\
&\le C\int_0^t |\partial_{x_N} v(s)|_{L^r}\, ds \\
&\le C\int_0^t e^{Ms} s^{-\frac{1}{2}} \|v\|_{X_{T,M}}\, ds \le Ce^{MT}T^{\frac{1}{2}}\|\varphi\|_{L^q_{\alpha(p)}} < \infty,
\end{aligned}$$

for $0 < t < T$. Furthermore, by $(P_3)$ we apply an argument similar to that in the proof of Lemma 3.2 and see that $w$ is bounded and smooth in $\overline{\mathbb{R}^N_+} \times (T_1, T)$ for any $0 < T_1 < T$. Therefore we deduce that $(v, w)$ is a solution of (4) in $\mathbb{R}^N_+ \times (0,T)$ satisfying (7).

Let $(\tilde{v}, \tilde{w})$ be a solution of (4) in $\mathbb{R}^N_+ \times (0, T_*)$ for any $T_* > T$ and such that $\tilde{v} \in X_{T_*,M_*}$ with some $M_* > 0$. Then $\tilde{v} \in X_{T,M}$ and since

$$v - \tilde{v} = Q[v] - Q[\tilde{v}] = D[v-\tilde{v}] \quad \text{in} \quad X_{T,M},$$

by (13) we have

$$\|v-\tilde{v}\|_{X_{T,M}} \le \frac{1}{2}\|v-\tilde{v}\|_{X_{T,M}}.$$

This implies that $v = \tilde{v}$ in $X_{T,M}$. Therefore we deduce that $(v,w)$ is a unique global-in-time solution of (4) satisfying (7). Thus Theorem 1.1 holds for the case $q < \infty$. Furthermore, by [4, Theorem 1.1] with $\varepsilon = 1$ and $\varphi_b \equiv 0$ we have Theorem 1.1 for the case $p = q = \infty$, and the proof of Theorem 1.1 is complete. □

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