Weak dissipativity in derivative nonlinear Schrödinger equations

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This short article is an announcement of the forthcoming paper [20] with Chunhua Li, Yoshinori Nishii and Yuji Sagawa, which concerns L^2-decay properties of small solutions to a class of cubic derivative nonlinear Schrödinger equations in one space dimension. Throughout this article, we denotes by $\mathcal{L}$ the standard free Schrödinger operator $i\partial_t + \frac{1}{2}\partial_x^2$ for $(t,x) \in \mathbb{R} \times \mathbb{R}$ with $i = \sqrt{-1}$. The function space H^k stands for the L^2-based Sobolev space of order k equipped with the norm $\|\phi\|_{H^k} = \sum_{j \le k} \|\partial_x^j \phi\|_{L^2}$, and the weighted Sobolev space $H^{k,m}$ is defined by $\{\phi \in L^2 \mid \langle \cdot \rangle^m \phi \in H^k\}$ with $\langle x \rangle = \sqrt{1+x^2}$.

1 Backgrounds

First of all, let us recall some of well-known results on large-time behavior of small data solutions to the cubic power-type nonlinear Schrödinger equation in the form

$$\mathcal{L}u = \lambda |u|^2 u, \qquad t > 0,\ x \in \mathbb{R}, \tag{1.1}$$

where λ is a constant. What is interesting in (1.1) is that the large-time behavior of the solution is actually affected by the coefficient λ even if the initial data is sufficiently small, smooth and decaying fast as $|x| \to \infty$. If $\lambda \in \mathbb{R}$, it is shown in [5] that the solution to (1.1) with small initial data behaves like

$$u(t,x) = \frac{1}{\sqrt{it}} \alpha(x/t) e^{i\{\frac{x^2}{2t} - \lambda |\alpha(x/t)|^2 \log t\}} + o(t^{-1/2}) \quad \text{as} \quad t \to \infty$$

with a suitable $\mathbb{C}$-valued function $\alpha(y)$. An important consequence of this asymptotic expression is that the solution decays like $O(t^{-1/2})$ in $L^\infty(\mathbb{R}_x)$, while it does not behave like the free solution unless $\lambda = 0$. In other words, the additional logarithmic factor in the phase reflects the long-range character of the cubic nonlinear Schrödinger equations in one space dimension. If $\lambda \in \mathbb{C}$ in (1.1), another kind of long-range effect can be observed. For instance, according to [26] (see also [16], [9], [3], etc.), the small data solution $u(t,x)$ to (1.1) decays like $O(t^{-1/2}(\log t)^{-1/2})$ in $L^\infty(\mathbb{R}_x)$ as $t \to +\infty$ if $\operatorname{Im} \lambda < 0$. This gain of additional logarithmic time decay should be interpreted as another kind of long-range effect (see also [1], [2], [3], [4], [6], [7], [8], [9], [10], [11], [13], [14], [16], [17], [18], [21], [22], [24], [25], and so on). Time decay in L^2-norm is also investigated by several authors. Among others, it is pointed out by Kita-Sato [15] that the optimal L^2-decay rate is $O((\log t)^{-1/2})$ in the case of (1.1) with $\operatorname{Im} \lambda < 0$. We intend to extend this kind of L^2-decay results to the case where the nonlinear term depends also on $\partial_x u$.

2 Derivative nonlinear Schrödinger equations

2.1 Weak dissipativity

From now on, we turn our attention to the initial value problem in the form

$$\mathcal{L}u = N(u, \partial_x u), \qquad t > 0,\ x \in \mathbb{R} \tag{2.1}$$

with

$$u(0,x) = \varphi(x), \qquad x \in \mathbb{R}, \tag{2.2}$$

where φ is a prescribed $\mathbb{C}$-valued function on $\mathbb{R}$. The nonlinear term $N(u, \partial_x u)$ is a cubic homogeneous polynomial in $(u, \overline{u}, \partial_x u, \overline{\partial_x u})$ with complex coefficients. If φ is $O(\varepsilon)$ in $H^3 \cap H^{2,1}$ with $0 < \varepsilon \ll 1$, what we can expect for general cubic nonlinear Schrödinger equations in $\mathbb{R}$ is the lower estimate for the lifespan T_ε in the form $T_\varepsilon \geq \exp(c/\varepsilon^2)$ with some $c > 0$ not depending on ε, and this is best possible in general (see [12] for an example of small data blow-up). More precise information on the lifespan is available under the restriction

$$N(e^{i\theta}, 0) = e^{i\theta} N(1, 0), \qquad \theta \in \mathbb{R} \tag{2.3}$$

and the initial condition

$$u(0,x) = \varepsilon\psi(x), \qquad x \in \mathbb{R}, \tag{2.4}$$

instead of (2.2), where $\psi \in H^3 \cap H^{2,1}$ is independent of ε. In fact we have the following.

Theorem 2.1 ([23], [27]). *Assume that $\psi \in H^3 \cap H^{2,1}(\mathbb{R})$. Suppose that the nonlinear term N satisfies (2.3). Let T_ε be the supremum of $T > 0$ such that the initial value problem (2.1)–(2.4) admits a unique solution in $C([0,T); H^3 \cap H^{2,1}(\mathbb{R}))$. Then it holds that*

$$\liminf_{\varepsilon \to +0} \varepsilon^2 \log T_\varepsilon \geq \frac{1}{2 \sup_{\xi \in \mathbb{R}} (|\hat{\psi}(\xi)|^2 \mathrm{Im}\, \nu(\xi))} \tag{2.5}$$

with the convention $1/0 = +\infty$, where the function $\nu : \mathbb{R} \to \mathbb{C}$ is defined by

$$\nu(\xi) = \frac{1}{2\pi i} \oint_{|z|=1} N(z, i\xi z) \frac{dz}{z^2}, \quad \xi \in \mathbb{R}, \tag{2.6}$$

and $\hat{\psi}$ denotes the Fourier transform of ψ, i.e.,

$$\hat{\psi}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-iy\xi} \psi(y)\, dy, \quad \xi \in \mathbb{R}.$$

In view of the right-hand side in (2.5), it may be natural to expect that the sign of $\mathrm{Im}\, \nu(\xi)$ has something to do with global behavior of small data solutions to (2.1). In fact, it has been pointed out in [23] that typical results on small data global existence and large-time asymptotic behavior for (2.1) under (2.3) can be summarized in terms of $\mathrm{Im}\, \nu(\xi)$ as follows:

- Small data global existence holds in $C([0,\infty); H^3 \cap H^{2,1})$ under the condition

$$\operatorname{Im} \nu(\xi) \le 0, \quad \xi \in \mathbb{R}, \tag{$\mathbf{A}$}$$

- The global solution has (at most) logarithmic phase correction if

$$\operatorname{Im} \nu(\xi) = 0, \quad \xi \in \mathbb{R}. \tag{$\mathbf{A}_0$}$$

  Also it is not difficult to see that there is no L^2-decay under ($\mathbf{A}_0$) for generic initial data of small amplitude.

- L^2-decay of the global solution occurs under the condition

$$\sup_{\xi \in \mathbb{R}} \operatorname{Im} \nu(\xi) < 0. \tag{$\mathbf{A}_+$}$$

Note that $\nu(\xi) = \lambda$ if $N = \lambda |u|^2 u$. So these results cover the results in the power-type nonlinearity case mentioned in Section 1. However, it is pointed out in [19] that an interesting case is not covered by these classifications, that is the case where ($\mathbf{A}$) is safistied but ($\mathbf{A}_0$) and ($\mathbf{A}_+$) are violated (for example, if $N = -i|u_x|^2 u$, we can easily check that $\operatorname{Im} \nu(\xi) = -\xi^2 \le 0$, while the inequality is not strict because of vanishing at $\xi = 0$). This is what we are interested in.

Before going further, let us remember the fact that, if ($\mathbf{A}$) is safistied but ($\mathbf{A}_0$) and ($\mathbf{A}_+$) are violated, then there exist $c_0 > 0$ and $\xi_0 \in \mathbb{R}$ such that $\operatorname{Im} \nu(\xi) = -c_0(\xi - \xi_0)^2$. The converse is also true. This fact naturally leads us to the following definition of *the weak dissipativity.*

Definition 2.1. We say that a cubic nonlinear term N is *weakly dissipative* if the following two conditions (i) and (ii) are satisfied:

(i) $N(e^{i\theta}, 0) = e^{i\theta} N(1, 0)$ for $\theta \in \mathbb{R}$.

(ii) There exist $c_0 > 0$ and $\xi_0 \in \mathbb{R}$ such that $\operatorname{Im} \nu(\xi) = -c_0(\xi - \xi_0)^2$.

2.2 Upper and lower L^2-decay bounds in the weakly dissipative case

The following two results are due to [20], which reveal the L^2-decay property in the weakly dissipative case.

Theorem 2.2 ([20])**.** *Suppose that N is weakly dissipative and that $\varepsilon = \|\varphi\|_{H^3 \cap H^{2,1}}$ is sufficiently small. Then there exists a positive constant C, not depending on ε, such that the global solution u to* (2.1)–(2.2) *satisfies*

$$\|u(t)\|_{L^2_x} \le \frac{C\varepsilon}{(1 + \varepsilon^2 \log(t+1))^{1/4}}$$

for $t \ge 0$.

Theorem 2.3 ([20]). *Suppose that N is weakly dissipative and that $\hat{\psi}$ does not vanish at the point ξ_0 coming from* (ii) *in Definition 2.1. Then we can choose $\varepsilon_0 > 0$ such that the global solution u to* (2.1)–(2.4) *satisfies*

$$\liminf_{t\to+\infty}((\log t)^{1/4}\|u(t)\|_{L^2_x}) > 0$$

for $\varepsilon \in (0, \varepsilon_0]$.

Remark 2.1. According to [15], the optimal L^2-decay rate is $O((\log t)^{-1/2})$ in the case where $N = \lambda|u|^2u$ with $\operatorname{Im}\lambda < 0$. This should be contrasted with Theorems 2.2 and 2.3, because these tell us that the optimal L^2-decay rate in the weakly dissipative case is $O((\log t)^{-1/4})$.

Now, let us explain heuristically why L^2-decay rate should be $O((\log t)^{-1/4})$ if $\hat{\psi}(\xi_0) \neq 0$. For this purpose, let us first remember the fact that the solution u^0 to the free Schrödinger equation (i.e., the case of $N = 0$) with (2.2) behaves like

$$\partial_x^k u^0(t,x) \sim \left(\frac{ix}{t}\right)^k \frac{e^{-i\pi/4}}{\sqrt{t}}\hat{\varphi}\left(\frac{x}{t}\right)e^{i\frac{x^2}{2t}} + \cdots$$

as $t \to +\infty$ for $k = 0, 1, 2, \ldots$. Viewing it as a rough approximation of the solution u for (2.1), we may expect that $\partial_x^k u(t,x)$ could be better approximated by

$$\left(\frac{ix}{t}\right)^k \frac{1}{\sqrt{t}}A\left(\log t, \frac{x}{t}\right)e^{i\frac{x^2}{2t}}$$

with a suitable function $A(\tau, \xi)$, where $\tau = \log t$, $\xi = x/t$ and $t \gg 1$. Note that

$$A(0,\xi) = e^{-i\pi/4}\,\hat{\varphi}(\xi)$$

and that the extra variable $\tau = \log t$ is responsible for possible long-range nonlinear effect. Substituting the above expression into (2.1) and keeping only the leading terms, we can see (at least formally) that $A(\tau, \xi)$ should satisfy the ordinary differential equation

$$i\partial_\tau A = \nu(\xi)|A|^2A + \cdots$$

under (2.3). If N is weakly dissipative, we see that

$$\partial_\tau|A|^2 = -2c_0(\xi - \xi_0)^2|A|^4 + \cdots .$$

Then it follows that

$$|A(\tau,\xi)|^2 = \frac{|\hat{\varphi}(\xi)|^2}{1 + 2c_0(\xi-\xi_0)^2|\hat{\varphi}(\xi)|^2\tau} + \cdots ,$$

whence

$$\|u(t)\|_{L^2_x} \sim \|A(\log t)\|_{L^2_\xi} \sim \left(\int_{\mathbb{R}} \frac{|\hat{\varphi}(\xi)|^2}{1 + 2c_0(\xi-\xi_0)^2|\hat{\varphi}(\xi)|^2\log t}\,d\xi\right)^{1/2} \quad (t \to +\infty)$$

up to harmless remainder. By considering the behavior as $t \to +\infty$ of this integral carefully, we see that L^2-decay rate in the weakly dissipative case should be just $O((\log t)^{-1/4})$. Indeed, we have the following lemma.

Lemma 2.1. *Let $\theta \in L^\infty(\mathbb{R})$, $\xi_0 \in \mathbb{R}$ and*

$$S(\tau) = \int_{\mathbb{R}} \frac{|\theta(\xi)|^2}{1 + (\xi - \xi_0)^2 |\theta(\xi)|^2 \tau} \, d\xi, \quad \tau \geq 1. \tag{2.7}$$

(1) We have

$$S(\tau) \leq 4 \|\theta\|_{L^\infty} \tau^{-1/2}, \quad \tau \geq 1.$$

(2) Assume that there exists an open interval I with $I \ni \xi_0$ such that $\inf_{\xi \in I} |\theta(\xi)| > 0$. Then we can choose a positive constant C_*, which is independent of $\tau \geq 1$ (but may depend on θ and ξ_0), such that

$$S(\tau) \geq C_* \tau^{-1/2}, \quad \tau \geq 1.$$

Our strategy of the proof of Theorems 2.2 and 2.3 is to justify the above heuristic argument. For the details, see the forthcoming paper [20].

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