Periodic L_p estimates by $\mathcal{R}$-boundedness and applications to the Navier-Stokes equations

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1 Introduction

This is a joint work with Thomas Eiter (Weierstrass inst. Applied Analysis and Stochastics, Berlin) and Mads Kyed (Hochschule Flensburg) and this manuscript was written based on Either, Kyed and Shibata [7].

Let Ω be a C^2 exterior domain in $\mathbb{R}^3$ and Γ the boundary of Ω. At begining, I consider the Navier Stokes equations

$$\partial_t \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} - \Delta\mathbf{u} + \mathfrak{p} = \mathbf{F}, \quad \operatorname{div}\mathbf{u} = 0 \quad \text{in } \Omega\times\mathbb{T}, \quad \mathbf{u}|_\Gamma = 0. \tag{1}$$

Here, $\mathbf{F}$ is a 2π periodic external force, that is $\mathbf{F}(\cdot, t+2\pi) = \mathbf{F}(\cdot, t)$ for any $t \in \mathbb{R}$ and $\mathbb{T} = \mathbb{R}\backslash 2\pi\mathbb{Z}$. There are a lot of studies of periodic solutions to the Navier-Stokes equations (1), cf. [9, 14, 21, 22]. See [7] for more reference. By far the most popular method that emerged is based on a representation formula that arises from the principle that a solution to the initial value problem tends to a periodic orbit as $t \to \infty$ regardless of the initial value. Equivalently formulated, a solution to the initial-value problem with time-periodic right-hand side tends to a periodic orbit as $t \to \infty$. Since the Stokes operator generates a C^0 analytic semigroup, which is denoted by $\{T(t)\}_{t\geq 0}$ here, the solution is represented as

$$u(t) = \int_{-\infty}^t T(t-s)F(s)\,ds - \int_{-\infty}^t T(t-s)P(\mathbf{u}\cdot\nabla\mathbf{u})(\tau)\,ds, \tag{2}$$

where P denotes the solenoidal projection defined by (11) of Sect 2, below. Then, it is easy to verify that this integral expression indeed leads to a periodic solution of the same period as F. The challenge with the method based on (2) is to construct a framework of Banach spaces such that the integral expression is well defined. Since F is time-periodic and therefore non-decaying, this clearly requires suitable decay properties of the semigroup $\{T(t)\}_{t\geq 0}$, which is basically guaranteed by L_p-L_q decay estimates due to Iwashita [13].

In this note, I would like to propose a completely different method based on the $\mathcal{R}$-solver and operator valued de Leewe theorem. In this section, I would like to explain our method created in [6, 7] in the abstract framework. Let X, Y, Z be three Banach spaces such that $X \subset Z \subset Y$ and let $A \in \mathcal{L}(X, Y)$ and $B \in \mathcal{L}(X, Z) \cap \mathcal{L}(Z, Y)$. Here, $\mathcal{L}(E, F)$ denotes the set of all bounded linear operators from E into F. Let $\mathbb{T} = \mathbb{R} \setminus 2\pi\mathbb{Z}$ be the 2π torus and we consider time periodic problem

$$\partial_t u - Au = F, \quad Bu = G \quad \text{for } t \in \mathbb{T}, \tag{3}$$

where $F(t+2\pi) = F(t)$ and $G(t+2\pi) = G(t)$ for $t \in \mathbb{R}$. Our approach is to use $\mathcal{R}$-solvers associated with the corresponding resolvent problem:

$$\lambda v - Av = f, \quad Bv = g. \tag{4}$$

Here, λ is a complex number ranging on

$$\Sigma_{\epsilon,\lambda_0} = \{\lambda \in \mathbb{C} \mid |\arg\lambda| \leq \pi - \epsilon, \quad |\lambda| \geq \lambda_0\}$$

for some $\lambda_0 > 0$ and $\epsilon \in (0, \pi/2)$. The situation here is the case where the following generalized resolvent estimate holds:

$$|\lambda|\|v\|_Y + \|v\|_X \leq C(\|f\|_Y + \|g\|_Z + \|\lambda^{1/2} g\|_Y)$$

when $Z = (Y, X)_{1/2}$. This is an abstract version of the Agranovich and M. I. Vishik type estimate for parameter elliptic problems (cf. [1], [2], [3], [4]).

To introduce the $\mathcal{R}$-solver ($\mathcal{R}$ bounded solution operator) of problem (4), we start with the definition of $\mathcal{R}$-bounded family.

Definition 1. Let E and F be two Banach spaces. We say that an operator family $\mathcal{T} \subset \mathcal{L}(E, F)$ is $\mathcal{R}$ bounded if there exist constants $C > 0$ and $q \in [1, \infty)$ such that for any integer n, $\{T_j\}_{j=1}^n \subset \mathcal{T}$ and $\{f_j\}_{j=1}^n \subset E$, the inequality:

$$\int_0^1 \|\sum_{j=1}^n r_j(u) T_j f_j\|_F^q \, du \leq C \int_0^1 \|\sum_{j=1}^n r_j(u) f_j\|_E^q \, du$$

is valid, where the Rademacher functions r_k, $k \in \mathbb{N}$, are given by $r_k : [0, 1] \to \{-1, 1\}$; $t \mapsto \mathrm{sign}(\sin 2^k \pi t)$.

The smallest such C is called $\mathcal{R}$ bound of $\mathcal{T}$ on $\mathcal{L}(X, Y)$, which is denoted by $\mathcal{R}_{\mathcal{L}(E,F)}\mathcal{T}$.

Next, we introduce an operator valued Fourier multiplier. Let $m(\xi)$ be an $L_\infty(\mathbb{R} \setminus \{0\}$ function with value in $\mathcal{L}(E, F))$. We set

$$T_m f = \mathcal{F}_{\mathbb{R}}^{-1}[m(\xi)\mathcal{F}_{\mathbb{R}}[f](\xi)] \quad f \in \mathcal{S}(\mathbb{R}, E),$$

where $\mathcal{F}_{\mathbb{R}}$ and $\mathcal{F}_{\mathbb{R}}^{-1}$ denote respective Fourier transformation and inverse Fourier transformation on $\mathbb{R}$. The following theorem concerned with the boundedness of the operator T_m was proved by Weis [20].

Theorem 2. *Let E and F be two UMD Banach spaces. Let $m(\xi) \in C^1(\mathbb{R} \setminus \{0\}, \mathcal{L}(E, F))$ and assume that*

$$\mathcal{R}_{\mathcal{L}(E,F)}(\{m(\xi) \mid \xi \in \mathbb{R} \setminus \{0\}\}) \leq r_b$$
$$\mathcal{R}_{\mathcal{L}(E,F)}(\{\xi m'(\xi) \mid \xi \in \mathbb{R} \setminus \{0\}\}) \leq r_b$$

with some constant $r_b > 0$. Then, for any $p \in (1, \infty)$, $T_m \in \mathcal{L}(L_p(\mathbb{R}, E), L_p(\mathbb{R}, F))$ and

$$\|T_m f\|_{L_p(\mathbb{R},F)} \leq C_p r_b \|f\|_{L_p(\mathbb{R},E)}$$

with some constant C_p depending solely on p.

In view of Theorem 2, we introduce the following definition.

Definition 3 ($\mathcal{R}$-solver)**.** For every $\lambda = \gamma + i\tau \in \Sigma_{\epsilon,\lambda_0}$, there exists a map

$$\mathcal{S}(\lambda) : Y \times Y \times Z \to X \quad (F_1, F_2, F_3) \mapsto \mathcal{S}(\lambda)(F_1, F_2, F_3) \in X$$

such that
(i) $v = \mathcal{S}(\lambda)(f, \lambda^{1/2}g, g)$ is a solution of problem (4),
(ii) $\mathcal{S}(\lambda)$ satisfies

$$\mathcal{R}_{\mathcal{L}(Y\times Y\times Z,X)}(\{(\tau\partial_\tau)^\ell \mathcal{S}(\lambda) \mid \lambda \in \Sigma_{\epsilon,\lambda_0}\}) \leq r_b,$$
$$\mathcal{R}_{\mathcal{L}(Y\times Y\times Z,Y)}(\{(\tau\partial_\tau)^\ell (\lambda\mathcal{S}(\lambda)) \mid \lambda \in \Sigma_{\epsilon,\lambda_0}\}) \leq r_b,$$

for $\ell = 0, 1$ with some constant r_b.
$\mathcal{S}(\lambda)$ is called an **$\mathcal{R}$-solver** or **$\mathcal{R}$-bounded solution operator** of problem (4).

Before considering periodic problem (3), we consider

$$\partial_t \tilde{u} - A\tilde{u} = \tilde{F}, \quad B\tilde{u} = \tilde{G} \quad \text{for } t \in \mathbb{R}. \tag{5}$$

Let $\varphi(t) \in C^\infty(\mathbb{R})$ which equals 1 for $|t| \geq \lambda_0 + 1$ and 0 for $|t| \leq \lambda_0 + 1/2$. From (ii) of Definition 3 $\varphi(\tau)\mathcal{S}(i\tau)$ is $\mathcal{R}$ bounded, that is

$$\mathcal{R}_{\mathcal{L}(Y\times Y\times Z,X)}(\{(\tau\partial_\tau)^\ell \varphi(\tau)\mathcal{S}(i\tau) \mid \tau \in \mathbb{R}\}) \leq \tilde{r}_b,$$
$$\mathcal{R}_{\mathcal{L}(Y\times Y\times Z,Y)}(\{(\tau\partial_\tau)^\ell (i\tau\varphi(\tau)\mathcal{S}(i\tau)) \mid \tau \in \mathbb{R}\}) \leq \tilde{r}_b,$$

for $\ell = 0, 1$ with some constant $\tilde{r}_b$. Let

$$\mathbf{S}(\tilde{F}, \tilde{G}) = \mathcal{F}_{\mathbb{R}}^{-1}[\varphi(\tau)\mathcal{S}(i\tau)\mathcal{F}_{\mathbb{R}}[(\tilde{F}, \Lambda^{1/2}\tilde{G}, \tilde{G})](\tau)]$$

where $\mathcal{F}_\mathbb{R}$ and $\mathcal{F}_\mathbb{R}^{-1}$ denote Fourier transformation and Fourier inverse transformation on $\mathbb{R}$, and, $\Lambda^{1/2}\tilde G = \mathcal{F}_\mathbb{R}^{-1}[(i\tau)^{1/2}\mathcal{F}_\mathbb{R}[\tilde G](\tau)]$. Writing $H_\varphi = \mathcal{F}_\mathbb{R}^{-1}[\varphi(\tau)\mathcal{F}_\mathbb{R}[H](\tau)]$ for $H = \tilde F$ and $\tilde G$, we see easily that $w = \mathbf{S}(\tilde F, \tilde G)$ satisfies equations:

$$\partial_t w - A\tilde w = \tilde F_\varphi, \quad Bw = \tilde G_\varphi \quad \text{for } t \in \mathbb{R}.$$

Moreover, by Theorem 2, for any $p \in (1, \infty)$ there exists a constant C_p depending on p such that

$$\|\partial_t \mathbf{S}(\tilde F, \tilde G)\|_{L_p(\mathbb{R},Y)} + \|\mathbf{S}(\tilde F, \tilde G)\|_{L_p(\mathbb{R},X)} \leq C_p \tilde r_b(\|\tilde F_\varphi\|_{L_p(\mathbb{R},Y)} + \|\Lambda^{1/2}\tilde G_\varphi\|_{L_p(\mathbb{R},Y)} + \|\tilde G_\varphi\|_{L_p(\mathbb{R},Z)}). \tag{6}$$

We now consider time periodic equations (3). Let $\mathcal{F}_\mathbb{T}$ and $\mathcal{F}_\mathbb{T}^{-1}$ denote Fourier transformation and Fourier inverse transformation on $\mathbb{T}$, that is

$$\mathcal{F}_\mathbb{T}[f](k) = \frac{1}{2\pi}\int_0^{2\pi} e^{-ikt} f(t)\, dt, \quad \mathcal{F}_\mathbb{T}^{-1}[(a_k)_{k\in\mathbb{Z}}](t) = \sum_{k\in\mathbb{Z}} e^{ikt} a_k,$$

where $\mathbb{Z}$ is the set of all integers and $(a_k)_{k\in\mathbb{Z}}$ denotes a sequence. Let $L_p(\mathbb{T}, X)$ be the set of all $L_p(\mathbb{R})$ function f valued with X such that $f(t + 2\pi) = f(t)$ for any $t \in \mathbb{R}$ and let $W_p^1(\mathbb{T}, X) = \{f \in L_p(\mathbb{T}, X) \mid \partial_t f \in L_p(\mathbb{T}, X)\}$. Set

$$\|f\|_{L_p(\mathbb{T},X)} = \left\{\int_0^{2\pi} \|f(t)\|_X^p \, dt\right\}^{1/p}.$$

To treat time periodic problem, we use the operator valued de Leewe transference theorem stated as follows:

Theorem 4. *Let E and F be Banach spaces and let $p \in (1, \infty)$. Let $m(\xi)$ be an $L_\infty(\mathbb{R} \setminus \{0\})$ function valued in $\mathcal{L}(E, F))$ and T_m is a bounded linear operator from $L_p(\mathbb{R}, E)$ to $L_p(\mathbb{R}, F)$. Suppose that for all $f \in E$ the point $k \in \mathbb{Z}^d$ is a Lebesgure point of $\xi \mapsto m(\xi)x$, and set $m_k f = m(ik)f$. Then, $(m_k)_{k\in\mathbb{Z}^d}$ is a Fourier multiplier from $L_p(\mathbb{T}^d, E)$ to $L_p(\mathbb{T}^d, F)$, and in fact,*

$$\|\tilde T_{(m_k)_{k\in\mathbb{Z}^d}}\|_{\mathcal{L}(L_p(\mathbb{T}^d,E),L_p(\mathbb{T}^d,F))} \leq \|T_m\|_{\mathcal{L}(L_p(\mathbb{R}^d,E),L_p(\mathbb{R}^d,F))}.$$

Here,

$$\tilde T_{(m_k)_{k\in\mathbb{Z}^d}}[f] = \mathcal{F}_\mathbb{T}^{-1}[m_k \mathcal{F}_\mathbb{T}[f](k))_{k\in\mathbb{Z}}](t) = \sum_{k\in\mathbb{Z}} e^{ikt} m_k \mathcal{F}_\mathbb{T}[f](k).$$

Proof. For the proof, refer Proposition 5.7.1 in [12]. □

Now, we consider (3), and set

$$u_1 = \mathcal{F}_\mathbb{T}^{-1}[\varphi(\tau)\mathcal{S}(i\tau)\mathcal{F}_\mathbb{T}[(F, \Lambda^\alpha G, G](\tau)].$$

Then, $\mathbf{u}_1$ satisfies periodic problem:

$$\partial_t u_1 - Au_1 = F_\varphi \quad Bu_1 = G_\varphi \quad \text{for } t \in \mathbb{T}.$$

Here, $H_\varphi = \mathcal{F}_{\mathbb{T}}^{-1}[\varphi(\tau)\mathcal{F}_{\mathbb{T}}[H](i\tau)]$ for $H \in \{F, G\}$. Moreover, combining (6) and Theorem 4, we have

$$\|\partial_t u_1\|_{L_p(\mathbb{T},Y)} + \|u_1\|_{L_p(\mathbb{T},X)} \leq C(\|F_\varphi\|_{L_p(\mathbb{T},Y)} + \|\Lambda^{1/2}G_\varphi\|_{L_p(\mathbb{T},Y)} + \|G_\varphi\|_{L_p(\mathbb{T},Z)}).$$

Thus, problem (3) is reduced to show the existence of finite number of solutions v_k of equations:

$$ikv_k - Av_k = \mathcal{F}_{\mathbb{T}}[F](ik), \quad Bv_k = \mathcal{F}_{\mathbb{T}}[G](ik).$$

And then,

$$u = u_1 + \sum_{|k| \leq \lambda_0 + 1/2} v_k \tag{7}$$

is a solution of (3). This is our strategy to solve periodic evolution problem (3).

2 A time periodic problem for the Stokes equations in exterior domains

Detailed arguments below are refered to Eiter, Kyed and Shibata [7]. Let Ω be a uniform C^2 bounded domain or exterior domain in $\mathbb{R}^N$ $(N \geq 2)$ and let Γ be the boundary of Ω. Let $L_q(\Omega)$ be the set of all Lebesgue measurable functions f defined on Ω such that

$$\|f\|_{L_q(\Omega)} = \Big\{\int_\Omega |f(x)|^q \, dx\Big\}^{1/q} < \infty$$

for $1 \leq q < \infty$. And let

$$W_q^m(\Omega) = \{f \in L_q(\Omega) \mid D_x^\alpha f(x) \in L_q(\Omega) \ \ (|\alpha| \leq m)\}, \quad \|f\|_{W_q^m(\Omega)} = \sum_{|\alpha| \leq m} \|D_x^\alpha f\|_{L_q(\Omega)}.$$

We consider time periodic problem for the Stokes equations:

$$\partial_t \mathbf{u} - \Delta \mathbf{u} + \nabla \mathfrak{p} = \mathbf{f}, \quad \operatorname{div} \mathbf{u} = 0 \quad \text{in } \Omega \times \mathbb{T}, \quad \mathbf{u}|_\Gamma = 0. \tag{8}$$

The corresponding resolvent equations read for the following equations:

$$\lambda \mathbf{v} - \Delta \mathbf{v} + \nabla \mathfrak{q} = \mathbf{F}, \quad \operatorname{div} \mathbf{v} = 0 \quad \text{in } \Omega, \quad \mathbf{v}|_\Gamma = 0. \tag{9}$$

First, we introduce the Helmholtz decomposition. Let

$$\hat{H}_q^1(\Omega) = \{u \in L_{q,\mathrm{loc}}(\Omega) \mid \nabla u \in L_q(\Omega)^N\}.$$

We consider the weak Neumann problem:

$$(\nabla u, \nabla\varphi) = (\mathbf{f}, \nabla\varphi) \quad \forall\, \varphi \in \hat H^1_{q'}(\Omega). \tag{10}$$

Here, $(f, g) = \int_\Omega f(x)g(x)\,dx$. From [10, 18] we know that for any $\mathbf{f} \in L_q(\Omega)^N$ problem (10) admits a unique solution $u \in \hat H^1_q(\Omega)$ satisfying the estimate: $\|\nabla u\|_{L_q(\Omega)} \leq C\|\mathbf{f}\|_{L_q(\Omega)}$.

Let

$$J_q(\Omega) = \{\mathbf{u} \in L_q(\Omega)^N \mid (\mathbf{u}, \nabla\varphi) = 0 \quad \forall\, \varphi \in \hat H^1_{q'}(\Omega)\},$$

which is called the solenoidal space. For $\mathbf{u} \in W^1_q(\Omega)$, what $\operatorname{div}\mathbf{u} = 0$ in Ω is equivalent to that $\mathbf{u} \in J_q(\Omega)$.

For any $\mathbf{f} \in L_q(\Omega)$, let $u \in \hat H^1_q(\Omega)$ be a unique solution of (10) and set $\mathbf{g} = \mathbf{f} - \nabla u$. We see that $\mathbf{g} \in J_q(\Omega)$. $\mathbf{f} = \mathbf{g} + \nabla u$ is called the Helmholtz decomposition. Let $P : L_q(\Omega) \to J_q(\Omega)$ be the solenoidal projection defined by

$$\mathbf{g} = P\mathbf{f}. \tag{11}$$

According to Shibata [17], we know that for any $\epsilon \in (0, \pi/2)$, there exists a positive constant λ_0 and $\mathcal{R}$-solvers

$$\mathcal{T}(\lambda) \in \operatorname{Hol}(\Sigma_{\epsilon,\lambda_0}, \mathcal{L}(L_q(\Omega)^N, H^2_q(\Omega)^N), \ \ \mathcal{P}(\lambda) \in \operatorname{Hol}(\Sigma_{\epsilon,\lambda_0}, \mathcal{L}(L_q(\Omega)^N, \hat H^1_q(\Omega))$$

such that for any $\mathbf{f} \in L_q(\Omega)^N$, $\mathbf{v} = \mathcal{T}(\lambda)\mathbf{f}$ and $\mathfrak{q} = \mathcal{P}(\lambda)\mathbf{f}$ are unique solutions of equations (9) and

$$\begin{aligned}
\mathcal{R}_{\mathcal{L}(L_q(\Omega)^N, H^2_q(\Omega)^N)}(\{(\tau\partial_\tau)^\ell \mathcal{T}(\lambda) \mid \lambda \in \Sigma_{\epsilon,\lambda_0}\}) &\leq r_b \quad (\ell = 0, 1),\\
\mathcal{R}_{\mathcal{L}(L_q(\Omega)^N)}(\{(\tau\partial_\tau)^\ell (\lambda\mathcal{T}(\lambda)) \mid \lambda \in \Sigma_{\epsilon,\lambda_0}\}) &\leq r_b \quad (\ell = 0, 1),\\
\mathcal{R}_{\mathcal{L}(L_q(\Omega)^N)}(\{(\tau\partial_\tau)^\ell (\nabla\mathcal{P}(\lambda)) \mid \lambda \in \Sigma_{\epsilon,\lambda_0}\}) &\leq r_b \quad (\ell = 0, 1),
\end{aligned} \tag{12}$$

Let $\varphi(\tau)$ be a $C^\infty(\mathbb{R})$ function which equals to 1 for $|\tau| \geq \lambda_0 + 1$ and 0 for $|\tau| \leq \lambda_0 + 1/2$, and set

$$\mathbf{u}_\varphi = \mathcal{F}^{-1}_{\mathbb{T}}[\mathcal{T}(i\tau)\varphi(\tau)\mathcal{F}_{\mathbb{T}}[\mathbf{f}](\tau)], \quad \mathfrak{p}_\varphi = \mathcal{F}^{-1}_{\mathbb{T}}[\mathcal{P}(i\tau)\varphi(\tau)\mathcal{F}_{\mathbb{T}}[\mathbf{f}](\tau)].$$

Then, $\mathbf{u}_\varphi$ and $\mathfrak{p}_\varphi$ satisfy equations:

$$\partial_t\mathbf{u}_\varphi - \Delta\mathbf{u}_\varphi + \nabla\mathfrak{p}_\varphi = \mathbf{f}_\varphi, \quad \operatorname{div}\mathbf{u}_\varphi = 0 \quad \text{in } \Omega \times \mathbb{T}, \quad \mathbf{u}_\varphi|_\Gamma = 0, \tag{13}$$

where $\mathbf{f}_\varphi = \mathcal{F}^{-1}_{\mathbb{T}}[\varphi(\tau)\mathcal{F}_{\mathbb{T}}[\mathbf{f}](\tau)]$, and the estimate:

$$\|\partial_t\mathbf{u}_\varphi\|_{L_p(\mathbb{T}, L_q(\Omega))} + \|\mathbf{u}_\varphi\|_{L_p(\mathbb{T}, W^2_q(\Omega))} \leq C\|\mathbf{f}_\varphi\|_{L_p(\mathbb{T}, L_q(\Omega))} \leq C\|\mathbf{f}\|_{L_p(\mathbb{T}, L_q(\Omega))} \tag{14}$$

provided that $\mathbf{f} \in L_p(\mathbb{T}, L_q(\Omega)^N)$ and $1 < p, q < \infty$. Thus, we have to consider the low frequency part. Let k be an element of $\mathbb{Z}$ such that $|k| \leq \lambda_0 + 1/2$. Let $\mathbf{u}_k \in W^2_q(\Omega)^N$ and $\mathfrak{p}_k \in \hat H^1_q(\Omega)$ be unique solutions of the resolvent problem:

$$ik\mathbf{u}_k - \Delta\mathbf{u}_k + \nabla\mathfrak{p}_k = \mathcal{F}_{\mathbb{T}}[\mathbf{f}](k), \quad \operatorname{div}\mathbf{u}_k = 0 \quad \text{in } \Omega, \quad \mathbf{u}_k|_\Gamma = 0. \tag{15}$$

2.1 Ω is a bounded domain

We know that when Ω is bounded, $\mathbf{u}_k \in W^2_q(\Omega)$ and $\mathfrak{p}_k \in \hat{H}^1_q(\Omega)$ exist uniquely and they satisfy the estimate:

$$\|\mathbf{u}_k\|_{W^2_q(\Omega)} + \|\nabla \mathfrak{p}_k\|_{L_q(\Omega)} \leq C\|\mathcal{F}_{\mathbb{T}}[\mathbf{f}](k)\|_{W^2_q(\Omega)}. \leq C\int_0^{2\pi} \|\mathbf{f}(\cdot,t)\|_{L_q(\Omega)}\,dt. \tag{16}$$

Combining (14) and (16), we have the following theorem.

Theorem 5. *Let Ω be a bounded uniformly C^2 domain in $\mathbb{R}^N$ ($N \geq 2$). Let $1 < p, q < \infty$. Then, for any $\mathbf{f} \in L_p(\mathbb{T}, L_q(\Omega)^N)$, problem (8) admits unique solutions $\mathbf{u}$ and $\mathfrak{p}$ with*

$$\mathbf{u} \in W^1_p(\mathbb{T}, L_q(\Omega)^N) \cap L_p(\mathbb{T}, W^2_q(\Omega)^N), \quad \mathfrak{p} \in L_p(\mathbb{T}, \hat{H}^1_q(\Omega)),$$

which satisfy the estimate:

$$\|\partial_t \mathbf{u}\|_{L_p(\mathbb{T},L_q(\Omega))} + \|\mathbf{u}\|_{L_p(\mathbb{T},W^2_q(\Omega))} + \|\nabla\mathfrak{p}\|_{L_p(\mathbb{T},L_q(\Omega))} \leq C\|\mathbf{f}\|_{L_p(\mathbb{T},L_q(\Omega))}.$$

2.2 Ω is a three dimensional exterior domain

We now continue the argument in the case where Ω is an exterior domain. We assume that Ω is a three dimensional exterior domain. Let $b > 0$ be a large number such that $\Omega^c \subset B_b = \{x \in \mathbb{R}^3 \mid |x| < b\}$. We will discuss the unique existence theorem of 2π-periodic solutions of (8).

We divide periodic function $\mathbf{f}$ into two parts as $\mathbf{f} = \mathbf{f}_\perp + \mathbf{f}_S$, where we have set

$$\mathbf{f}_S(x) = \frac{1}{2\pi}\int_0^{2\pi} \mathbf{f}(x,t)\,dt, \quad \mathbf{f}_\perp(x,t) = \mathbf{f}(x,t) - \mathbf{f}_S(x).$$

Obviously, $\int_0^{2\pi} \mathbf{f}_\perp(\cdot,t)\,dt = 0$. We call that $\mathbf{f}_\perp$ is the oscillatory part of $\mathbf{f}$ and $\mathbf{f}_S$ the stationary part of $\mathbf{f}$.

We know that when $k \neq 0$ and $|k| \leq \lambda_0 + 1/2$, problem (15) admits unique solutions $\mathbf{u}_k \in W^2_q(\Omega)^N$ and $\mathfrak{p}_k \in \hat{H}^1_q(\Omega)$ which satisfy the estimate (16). Thus, setting

$$\mathbf{u}_\perp = \mathbf{u}_\varphi + \sum_{1\leq|k|\leq\lambda_0+1/2} \mathbf{u}_k, \quad \mathfrak{p}_\perp = \mathfrak{p}_\varphi + \sum_{1\leq|k|\leq\lambda_0+1/2} \mathfrak{p}_k,$$

we see that

$$\mathbf{u}_\perp \in L_p(\mathbb{T}, W^2_q(\Omega)^N) \cap W^1_p(\mathbb{T}, L_q(\Omega)^N), \quad \mathfrak{p}_\perp \in L_p(\mathbb{T}, \hat{H}^1_q(\Omega))$$

and $\mathbf{u}_\perp$ and $\mathfrak{p}_\perp$ satisfy the equations:

$$\partial_t\mathbf{u}_\perp - \Delta\mathbf{u}_\perp + \nabla\mathfrak{p}_\perp = \mathbf{f}_\perp, \quad \operatorname{div}\mathbf{u}_\perp = 0 \quad \text{in } \Omega\times\mathbb{T}, \quad \mathbf{u}_\perp|_\Gamma = 0.$$

Moreover, we have

$$\|\partial_t \mathbf{u}_\perp\|_{L_p(\mathbb{T},L_q(\Omega))} + \|\mathbf{u}_\perp\|_{L_p(\mathbb{T},W^2_q(\Omega))} + \|\nabla \mathfrak{p}_\perp\|_{L_p(\mathbb{T},L_q(\Omega))} \leq C\|\mathbf{f}\|_{L_p(\mathbb{R},W^2_q(\Omega))}.$$

Let $\mathbf{u}_S$ and $\mathfrak{p}_S$ be the stationary part of $\mathbf{u}$ and $\mathfrak{p}$, and then $\mathbf{u}_S$ and $\mathfrak{p}_S$ satisy the stationary equations:

$$-\Delta \mathbf{u}_S + \nabla \mathfrak{p}_S = \mathbf{f}_S, \quad \operatorname{div} \mathbf{u}_S = 0 \quad \text{in } \Omega, \quad \mathbf{u}_S|_\Gamma = 0. \tag{17}$$

Using the fundamental solutions:

$$U_{ij}(x) = -\frac{1}{8\pi}\Big(\frac{\delta_{ij}}{|x|} + \frac{x_i x_j}{|x|^3}\Big), \quad q_j(x) = \frac{1}{3\pi}\frac{x_j}{|x|^3},$$

of the Stokes equations in $\mathbb{R}^3$ and the cut off technique, we have

Lemma 6. *Let Ω be a uniformly C^2 exterior domain in $\mathbb{R}^3$, and let $3 < q < \infty$. Let $< g >_\ell = \sup_{x\in\Omega}(1+|x|)^\ell |g(x)|$, and*

$$L_{q,3b}(\Omega) = \{g \in L_q(\Omega) \mid g(x) = 0 \ \text{for } |x| > 3b\}.$$

If $\mathbf{f}_S = \operatorname{div} \mathbf{F} + \mathbf{g}$ such that

$$< \operatorname{div} \mathbf{F} >_3 + < \mathbf{F} >_2 < \infty, \quad \mathbf{g} \in L_{q,3b}(\Omega),$$

then problem (17) admits unique solutions $\mathbf{u}_S \in W^2_q(\Omega)^3$ and $\mathfrak{p}_S \in W^1_q(\Omega)$ satisfying the estimate:

$$\begin{aligned}&\|\mathbf{u}_S\|_{W^2_q(\Omega)} + < \mathbf{u}_S >_1 + < \nabla \mathbf{u}_S >_2 + \|\mathfrak{p}_S\|_{W^1_q(\Omega)} + < \mathfrak{p}_S >_2 \\ &\quad \leq C(< \operatorname{div} \mathbf{F}_S >_3 + < \mathbf{F}_S >_2 + \|\mathbf{g}\|_{L_q(\Omega)}).\end{aligned}$$

It is important to investigate the asymptotic behaviour of oscillatory parts $\mathbf{u}_\perp$, especially to solve the Navier-Stokes equations. For this purpose we use the following lemma which shows the asymptotic behaviours of the fundamental solution $\Gamma_\perp$ of the resolvent equations:

$$ik\mathbf{v} - \Delta \mathbf{v} + \nabla \mathfrak{q} = \mathbf{f}, \quad \operatorname{div} \mathbf{v} = 0 \quad \text{in } \mathbb{R}^3.$$

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Lemma 7 (Eiter and Kyed [5])**.** *Let*

$$\Gamma_\perp = \mathcal{F}^{-1}_{\mathbb{R}^3\times\mathbb{T}}\Big[\frac{1-\delta_\mathbb{Z}}{|\xi|^2 + ik}\Big(\mathbf{I} - \frac{\xi\otimes\xi}{|\xi|^2}\Big)\Big].$$

Then, $\Gamma_\perp \in L_q(\mathbb{R}^3 \times \mathbb{T})$ for $q \in (1, 5/3)$ and $\nabla\Gamma_\perp \in L_q(\mathbb{R}^3\times\mathbb{T})^3$ for $q \in (1, 5/4)$. And for any multi-index $\alpha \in \mathbb{N}^3_0$, $\delta > 0$ and $r \in [1,\infty)$,

$$\|D^\alpha_x \Gamma_\perp(x,\cdot)\|_{L_p('\mathbb{T})} \leq \frac{C_{\alpha,\delta}}{|x|^{3+|\alpha|}} \quad (|x| > \delta).$$

We consider again equations:

$$\partial_t \mathbf{u}_\perp - \Delta \mathbf{u}_\perp + \nabla \mathfrak{p}_\perp = \mathbf{f}_\perp, \quad \operatorname{div} \mathbf{u}_\perp = 0 \quad \text{in } \Omega, \quad \mathbf{u}_\perp|_\Gamma = 0. \tag{18}$$

To state the asymptotic behaviour of $\mathbf{u}_\perp$, we introduce the norm

$$< f >_{p,\ell} = \sup_{x\in\Omega} \|f(x,\cdot)\|_{L_p(\mathbb{T})}(1+|x|)^\ell.$$

Using the fundamental solutions to the oscillatory part $\Gamma_\perp$ and the cut off technique, we have the following lemma.

Lemma 8. *Let $3 < q < \infty$ and $\ell \in (0, 3]$. Assume that $\mathbf{f}_\perp = \operatorname{div} \mathbf{F}_\perp + \mathbf{g}_\perp$ such that*

$$\int_0^{2\pi} \mathbf{F}_\perp(x,t)\,dt = 0, \quad < \mathbf{F}_\perp >_{p,\ell} + < \operatorname{div} \mathbf{F}_\perp >_{p,\ell+1} < \infty,$$
$$\int_0^{2\pi} \mathbf{g}_\perp(x,t)\,dt = 0, \quad \mathbf{g}_\perp \in L_p(\mathbb{T}, L_{q,3b}(\Omega)^3).$$

Then, $\mathbf{u}_\perp$ satisfies the estimate:

$$< \mathbf{u}_\perp >_{p,\ell} + < \nabla \mathbf{u}_\perp >_{p,\ell+1} \leq C(< \operatorname{div} \mathbf{F}_\perp >_{p,\ell+1} + < \mathbf{F}_\perp >_{p,\ell} + \|\mathbf{g}\|_{L_p(\mathbb{T}, L_q(\Omega))}).$$

Summing up, we have obtained the following theorem.

Theorem 9. *Let $2 < p < \infty$ and $3 < q < \infty$, and $\ell \in (0, 3]$. For all $\mathbf{f} = \mathbf{f}_S + \mathbf{f}_\perp$ with $\mathbf{f}_S = \operatorname{Div} \mathbf{G}_S + \mathbf{g}_S$ and $\mathbf{f}_\perp = \operatorname{div} \mathbf{G}_\perp + \mathbf{g}_\perp$ such that $\mathbf{g}_S \in L_{q,3b}(\Omega)^3$, $\mathbf{g}_\perp \in L_p(\mathbb{T}, L_{q,3b}(\Omega)^3)$ and*

$$< \mathbf{G}_S >_2 + < \operatorname{div} \mathbf{G}_S >_3 + < \mathbf{G}_\perp >_{p,\ell} + < \operatorname{div} \mathbf{G}_\perp >_{p,\ell+1} < \infty$$

problem (8) admits unique solutions $\mathbf{u}$ and $\mathfrak{p}$ with

$$\mathbf{u} \in W^1_p(\mathbb{T}, L_q(\Omega)^3) \cap L_p(\mathbb{T}, W^2_q(\Omega)^3), \quad \mathfrak{p} \in L_p(\mathbb{T}, \hat{H}^1_q(\Omega))$$

satisfying the estimate:

$$\begin{aligned}
&\|\mathbf{u}_S\|_{W^2_q(\Omega)} + < \mathbf{u}_S >_1 + < \nabla \mathbf{u}_S >_2 + \|\mathfrak{p}_S\|_{W^1_q(\Omega)} + < \mathfrak{p}_S >_2 + \|\mathbf{u}_\perp\|_{L_p(\mathbb{T}, W^2_q(\Omega))} \\
&\quad + \|\partial_t \mathbf{u}_\perp\|_{L_p(\mathbb{T}, L_q(\Omega))} + < \mathbf{u}_\perp >_{p,\ell} + < \nabla \mathbf{u}_\perp >_{p,\ell+1} + \|\nabla \mathfrak{p}_\perp\|_{L_p(\mathbb{T}, L_q(\Omega))} \\
&\quad \leq C(< \operatorname{div} \mathbf{G}_S >_3 + < \mathbf{G}_S >_2 + < \operatorname{div} \mathbf{G}_\perp >_{p,\ell+1} + < \mathbf{G}_\perp >_{p,\ell} \\
&\qquad\qquad + \|\mathbf{g}_S\|_{L_q(\Omega)} + \|\mathbf{g}_\perp\|_{L_p(\mathbb{T}, L_q(\Omega))}).
\end{aligned}$$

3 Time periodic solutions to the Navier-Stokes equations

Since we already know Theorem 9, which tells us the unique existence of time periodic solutions to the Stokes equations based on the maximal regularity for the high-frequency part and the space decay properties of solutions, we can show the unique existence of strong solutions stated as follows, which is completely different approach from [9, 14, 21, 22].

Theorem 10. *Let* $2 < p < \infty$ *and* $3 < q < \infty$. *Assume that* $\mathbf{F} = \mathbf{F}_S + \mathbf{F}_\perp$ *with* $\mathbf{F}_S = \operatorname{div}\mathbf{G}_S$ *and* $\mathbf{F}_\perp = \operatorname{div}\mathbf{G}_\perp$. *Then, there exists a small constant* $\epsilon > 0$ *such that if* $\mathbf{F}$ *satisfy the smallness condition:* $< \mathbf{F}_S >_3 + < \mathbf{G}_S >_2 + < \mathbf{F}_\perp >_{p,2} + < \mathbf{G}_\perp >_{p,1} < \epsilon^2$, *then problem* (1) *admits unique solutions* $\mathbf{u} = \mathbf{u}_S + \mathbf{u}_\perp$ *and* $\mathfrak{p} = \mathfrak{p}_S + \mathfrak{p}_\perp$ *with*

$$\begin{aligned}&\mathbf{u}_S \in W^2_q(\Omega)^3, \quad \mathbf{u}_\perp \in L_p(\mathbb{T}, W^2_q(\Omega)^3) \cap W^1_p(\mathbb{T}, L_q(\Omega)^3),\\ &\mathfrak{p}_S \in W^1_q(\Omega), \quad \mathfrak{p}_\perp \in L_p(\mathbb{T}, \hat{H}^1_q(\Omega))\end{aligned}$$

satisfying the estimate:

$$\begin{aligned}&< \mathbf{u}_S >_1 + < \nabla\mathbf{u}_S >_2 + \|\mathbf{u}_S\|_{W^2_q(\Omega)} + < \mathfrak{p}_S >_2 + \|\mathfrak{p}_S\|_{W^1_q(\Omega)} + < \mathbf{u}_\perp >_{p,1}\\ &+ < \nabla\mathbf{u}_\perp >_{p,2} + \|\partial_t\mathbf{u}_\perp\|_{L_p(\mathbb{T},L_q(\Omega))} + \|\mathbf{u}_\perp\|_{L_p(\mathbb{T},W^2_q(\Omega))} + \|\nabla\mathfrak{p}_\perp\|_{L_p(\mathbb{T},L_q(\Omega))} < \epsilon.\end{aligned}$$

Proof. To move $\mathbf{u}\cdot\nabla\mathbf{u}$ to the right hand side and using the Banach fixed point theorem based on Theorem 9, we can prove Theorem 10 immediately. □

We now consider the Navier-Stokes equations in a periodically moving exterior domain. Let $\phi \in C^0(\mathbb{T}, C^3(\Omega)^N) \cap C^1(\mathbb{T}, C^1(\Omega)^N)$ with

$$\|\phi\|_{C^0(\mathbb{T},C^3(\Omega))} + \|\partial_t\phi\|_{C^0(\mathbb{T},C^1(\Omega))} \leq \epsilon^2 \tag{19}$$

and Ω_t and Γ_t are given by

$$\Omega_t = \{x = y + \phi(y,t) \mid y \in \Omega\}, \quad \Gamma_t = \{x = y + \phi(y,t) \mid y \in \Gamma\} \quad (t \in \mathbb{R}).$$

Consider the Navier-Stokes equations:

$$\partial_t\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} - \Delta\mathbf{u} + \mathfrak{p} = \mathbf{F}, \quad \operatorname{div}\mathbf{u} = 0 \quad \text{in } \Omega_t, \quad \mathbf{u}|_{\Gamma_t} = 0. \tag{20}$$

When ϵ is small enough and Ω is a bounded domain, Farwig, Kozono, Tsuda, and Wegmann [8, 11] proved the global well-posedness. Eiter, Kyes and Shibata [7] also proved the global well-posedness by using a perturbation method based on Theorem 5. The method in [7] is completely different from [8, 11].

Moreover, in [7], the case where Ω_t is an exterior domain of $\mathbb{R}^3$ is treated. If we assume that ϵ in (19) is small enough, we have the inverse transform: $y = x + \psi(x,t)$, and we reduce equations (20) to the following equations:

$$\partial_t\mathbf{w} - \Delta\mathbf{w} + \nabla\mathfrak{q} = \mathbf{G} + \mathcal{L}(\mathbf{w},\mathfrak{q}) + \mathcal{N}(\mathbf{w}), \quad \operatorname{div}\mathbf{w} = 0 \quad \text{in } \Omega\times\mathbb{T}, \quad \mathbf{w}|_\Gamma = 0 \tag{21}$$

with a fixed domain Ω. Here, $\mathcal{L}$ is a linear operator of the form:

$$\mathcal{L}(\mathbf{w}, \mathfrak{q}) = a(x,t)\partial_t \mathbf{w} + \sum_{|\alpha|\le 2} b_\alpha(x,t) D_x^\alpha \mathbf{w} + c(x,t)\nabla \mathfrak{q}$$

with $\|(a, b_\alpha, c)\|_{L_\infty(\Omega\times\mathbb{T})} \le C\epsilon^2$, and $\mathcal{N}(\mathbf{w})$ is a nonlinear term satisfying the estimate $|\mathcal{N}(\mathbf{w})| \le C|\mathbf{w}||\nabla \mathbf{w}|$. Using the standard iteration argument based on Theorem 9, we have the following theorem.

Theorem 11. *Let* $2 < p < \infty$ *and* $3 < q < \infty$. *Assume that* $2/p + 3/q < 2$. *Assume that* $\mathbf{G} = \mathbf{G}_S + \mathbf{G}_\perp$ *with* $\mathbf{G}_S = \operatorname{div} \mathbf{H}_S$ *and* $\mathbf{G}_\perp = \operatorname{div} \mathbf{H}_\perp$. *Then, there exists a small constant* $\epsilon > 0$ *such that if* ϕ *and* $\mathbf{G}$ *satisfy the smallness condition:*

$$\begin{aligned}&\|\phi\|_{C^0(\mathbb{T},C^3(\Omega))} + \|\partial_t\phi\|_{C^0(\mathbb{T},C^1(\Omega))} \le \epsilon^2,\\ &\|\mathbf{G}_\perp\|_{L_p(\mathbb{T},L_q(\Omega))} + < \mathbf{G}_\perp >_{p,2} + < \mathbf{H}_\perp >_{p,1} + < \mathbf{G}_S >_3 + < \mathbf{H}_S >_2 < \epsilon^2\end{aligned}$$

then problem (21) *admits unique solutions* $\mathbf{w} = \mathbf{w}_S + \mathbf{w}_\perp$ *and* $\mathfrak{q} = \mathfrak{q}_S + \mathfrak{q}_\perp$ *with*

$$\mathbf{w}_\perp \in H^1_p(\mathbb{T}, L_q(\Omega)^3) \cap L_p(\mathbb{T}, H^2_q(\Omega)^3),\ \mathbf{w}_S \in H^2_q(\Omega)^3,\ \mathfrak{q}_\perp \in L_p(\mathbb{T}, \hat H^1_q(\Omega)),\ \mathfrak{q}_S \in H^1_q(\Omega)$$

satisfying the estimate:

$$\begin{aligned}&< \mathbf{w}_\perp >_{p,1} + < \nabla\mathbf{w}_\perp >_{p,2} + \|\mathbf{w}\|_{L_p(\mathbb{T},H^2_q(\Omega))} + \|\partial_t \mathbf{w}\|_{L_p(\mathbb{T},L_q(\Omega))} + \|\nabla \mathfrak{q}_\perp\|_{L_p(\mathbb{T},L_q(\Omega))}\\ &+ < \mathbf{w}_S >_1 + < \nabla\mathbf{w}_S >_2 + \|\mathbf{w}\|_{H^2_q(\Omega)} + \|\mathfrak{q}_S\|_{H^1_q(\Omega)} \le \epsilon.\end{aligned}$$

Here,

$$< g_\perp >_{p,\ell} = \sup_{x\in\Omega} \|g_\perp(x,\cdot)\|_{L_p(\mathbb{T})}(1+|x|)^\ell, \quad < g_S >_\ell = \sup_{x\in\Omega} |g_S(x)|(1+|x|)^\ell.$$

This theorem gives us the unique existence of periodic solutions of equations (20) for small ϵ when Ω_t is an exterior domain of $\mathbb{R}^3$.

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