Long-time asymptotic expansion with pointwise error bounds for solutions to the 1D barotropic Navier–Stokes equations

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Abstract

This is a summary of our preprint [5]. We give a time-asymptotic expansion with pointwise error bounds for solutions to 1D barotropic Navier–Stokes equations. The leading-order term of the expansion is the well-known diffusion wave and the higher-order terms are newly introduced family of waves which we call *higher-order diffusion waves*. These waves provide accurate description of the long-time behavior at the "tail" of the diffusion wave, that is, around the origin $x = 0$. They complement the diffusion wave approximation and provide a precise description of the long-time asymptotics of the flow. The remainder estimate of the expansion is given in a pointwise manner similar to the work by Liu and Zeng [6]. In particular, the expansion is valid locally around $x = 0$ and also in the $L^p(\mathbb{R})$-norm for all $1 \leq p \leq \infty$.

1 Introduction

We shall consider the motion of a one-dimensional viscous compressible fluid. Let $v = v(x,t)$ be the specific volume (the reciprocal of the density ρ) and $u = u(x,t)$ the flow velocity. Here t is the time and x is the Lagrangian mass coordinate which is related to the Eulerian coordinate X by $x = \int_0^X \rho(X',t)\,dX'$. We consider barotropic flow, that is, the pressure $p = p(v)$ is a function only of the specific volume and does not depend on the temperature. Here we assume that $p'(v) > 0$ and $p''(v) \neq 0$ for $v > 0$. We also assume that the viscous coefficient $\nu > 0$ is a positive constant. Under these settings, the flow is described by the following 1D barotropic Navier–Stokes equations:

$$\begin{cases} v_t - u_x = 0, & x \in \mathbb{R},\, t > 0, \\ u_t + p(v)_x = \nu (u_x/v)_x, & x \in \mathbb{R},\, t > 0, \\ v(x,0) = v_0(x),\, u(x,0) = u_0(x), & x \in \mathbb{R}. \end{cases} \tag{1}$$

The first equation is the conservation of mass, the second is the balance of momentum, and the third equations are the initial conditions.

1.1 Long-time behavior of the flow and the diffusion waves

The objective of our study is to obtain a precise understanding of the long-time behavior of the solution (v,u) when the initial data (v_0,u_0) is a small perturbation of the steady state $(v_S,u_S) \equiv (1,0)$. We note that for the purpose of describing the long-time behavior, it is more convenient to consider

$$u_1 = \frac{p''(1)}{4c}[-(v-1) + u/c], \quad u_2 = \frac{p''(1)}{4c}[(v-1) + u/c] \tag{2}$$

instead of (v, u). Here $c = \sqrt{-p'(1)}$ is the speed of sound for the state (v_S, u_S). These are Riemann invariants of the inviscid and linearized counterpart of (1). A well-known fact about the long-time behavior is that u_i has the nonlinear diffusion wave θ_i as its asymptotic state. Here θ_i is the self-similar solution to the following convective viscous Burgers' equation:

$$\begin{cases} \partial_t\theta_i + \lambda_i\partial_x\theta_i + \partial_x(\theta_i^2/2) = \dfrac{\nu}{2}\partial_x^2\theta_i, & x \in \mathbb{R}, t > 0, \\ \lim\limits_{t\searrow -1} \theta_i(x,t) = \left(\displaystyle\int_{-\infty}^{\infty} u_i(x,0)\,dx\right)\delta(x), & x \in \mathbb{R}. \end{cases} \tag{3}$$

Here $\lambda_i = (-1)^{i-1}c$ and $\delta(x)$ is the Dirac delta function. An explicit formula for θ_i is obtained by the use of the Cole–Hopf transformation:

$$\theta_i(x,t) = \frac{\sqrt{\nu}}{\sqrt{2(t+1)}}\left(e^{\frac{M_i}{\nu}} - 1\right) e^{-\frac{(x-\lambda_i(t+1))^2}{2\nu(t+1)}} \left[\sqrt{\pi} + \left(e^{\frac{M_i}{\nu}} - 1\right)\int_{\frac{x-\lambda_i(t+1)}{\sqrt{2\nu(t+1)}}}^{\infty} e^{-y^2}\,dy\right]^{-1}. \tag{4}$$

We wrote above that the diffusion wave θ_i is the asymptotic state of the solution u_i. Let us make precise what we mean by this. By the explicit formula above and by the results in [6], we have

$$\|\theta_i(\cdot,t)\|_{L^p} \lesssim t^{-(1-1/p)/2} \quad \text{and} \quad \|(u_i - \theta_i)(\cdot,t)\|_{L^p} \lesssim t^{-(3/2-1/p)/2} \quad (1 \le p \le \infty). \tag{5}$$

These decay estimates are optimal. Therefore, θ_i describes the leading-order asymptotics of u_i in the $L^p(\mathbb{R})$-norm for $1 \le p \le \infty$. The key to prove the decay estimate above—especially for $1 \le p < 2$—is the pointwise estimates of Green's function (the fundamental solution) of the linearization of (1) around $(v_S, u_S) \equiv (1, 0)$; see [6, 7, 9]. Using the pointwise estimates for Green's function, they obtained the following pointwise estimates for the solution itself:

$$|(u_i - \theta_i)(x,t)| \lesssim [(x - \lambda_i(t+1))^2 + (t+1)]^{-3/4} + [|x + \lambda_i(t+1)|^3 + (t+1)^2]^{-1/2}. \tag{6}$$

By an integration in the spatial variable x, we get the L^p-decay estimates (5) above.

1.2 Long-time behavior at the "tail" of the diffusion waves and time-asymptotic expansion of the solution

We now turn our attention from the decay estimates in the global $L^p(\mathbb{R})$-norm to local decay estimates. Fist, we note that $\theta_i(x,t) \sim t^{-1/2}$ for $x = \lambda_i t + O(1)$ unless $M_i = 0$. Next, from the pointwise estimates (6), we see that $|(u_i - \theta_i)(x,t)| \lesssim t^{-3/4}$ for $x = \lambda_i t + O(1)$. Therefore, the diffusion wave θ_i describes the leading-order asymptotics locally around the characteristic line $x = \lambda_i t$. However, as can be seen by (4), the diffusion wave θ_i decays exponentially fast around the origin $x = 0$. And θ_i provides almost no information about the long-time behavior around $x = 0$. In this sense, the origin $x = 0$ lies at the "tail" of the diffusion wave θ_i.

We then set the goal for the current study as follows: obtain a clear picture of the long-time behavior around the origin $x = 0$. We note in passing that this problem has some implications to a fluid-structure interaction problem (see [2, 3, 4]). The difficulty lies in the fact that, as we explained above, the origin is at the tail of the diffusion wave θ_i. To resolve this issue, we need new waves describing the long-time asymptotics of the flow. A natural direction is to investigate higher-order asymptotics of the solution. In fact, van Baalen, Popović, and Wayne constructed a time-asymptotic expansion in an L^2-framework [8], and it turned out that the first higher-order term beyond the leading order term θ_i decays as $t^{-3/2}$ around the origin. This higher-order term is then expected to describe the long-time behavior of the flow around $x = 0$. Unfortunately, we cannot conclude that the solution u_i is asymptotically equivalent to this higher-order term. The reason is because the remainder estimate of the expansion is given only in the $H^1(\mathbb{R})$-norm; this implies a far from optimal decay estimate around $x = 0$. To overcome this problem, we construct a time-asymptotic expansion with pointwise error bounds for the remainder. We explain our results in the next section.

2 Time-asymptotic expansion with pointwise error bounds

2.1 Higher-order diffusion waves

We first define *higher-order diffusion waves* $\xi_{i;n}$ $(n \geq 1)$ which consist of the higher-order terms in our time-asymptotic expansion.

Let (v, u) be the solution to (1) and define u_i and θ_i by (2) and (3), respectively. We set $\xi_{i;0} = \theta_i/2$ and define $\xi_{i;n}$ inductively by the equations:

$$\begin{cases} \partial_t \xi_{i;n} + \lambda_i \partial_x \xi_{i;n} + \partial_x(\theta_i \xi_{i;n}) + \partial_x(\theta_{i'} \xi_{i';n-1}) = \dfrac{\nu}{2}\partial_x^2 \xi_{i;n}, & x \in \mathbb{R}, t > 0, \\ \xi_{i;n}(x, 0) = 0, & x \in \mathbb{R}, \end{cases} \tag{7}$$

where $i' = 3 - i$ ($1' = 2$ and $2' = 1$).

Although we do not have a simple explicit formula for $\xi_{i;n}$, we can still understand the long-time behavior of $\xi_{i;n}$ quite well. To explain this, we set

$$\alpha_n = 2 - \frac{1}{2^{n+1}}, \quad \beta_n = \frac{3}{2} - \frac{1}{2^{n+1}} \quad (n \geq -1)$$

and

$$\begin{aligned} \psi_n(x, t; \lambda) &= [(x - \lambda(t+1))^2 + (t+1)]^{-\alpha_n/2}, \\ \tilde{\psi}_n(x, t; \lambda) &= [|x - \lambda(t+1)|^{\alpha_n} + (t+1)^{\beta_n}]^{-1}, \\ \Psi_{i;n}(x, t) &= \psi_n(x, t; \lambda_i) + \tilde{\psi}_n(x, t; \lambda_{i'}). \end{aligned}$$

Then we have the following decay estimates for $\xi_{i;n}$.

Proposition 2.1. *Let n be a natural number. If $\int_{-\infty}^{\infty} u_j(x, 0)\,dx$ is sufficiently small for both $j = 1$ and 2, we have*

$$|\partial_x^k \xi_{i;n}(x, t)| \leq C_{n,k}(t+1)^{-k/2}\delta^{n+1}\Psi_{i;n-1}(x, t) \quad (k \geq 0)$$

for some positive constant $C_{n,k}$.

We conclude in particular that

$$|\xi_{i;n}(x,t)| \lesssim t^{-(2-1/2^n)} \quad \text{for } x = O(1).$$

This decay estimate is optimal. In fact, we can also write down a simple asymptotic formula for $\xi_{i;n}$ around $x = 0$. We refer to our preprint [5] for the precise statement. Also note that by setting $n = 1$, we get $|\xi_{i;1}(x,t)| \lesssim t^{-3/2}$ for $x = O(1)$. This is the decay rate obtained by setting $x = 0$ in (6). Thus it is expected that $\xi_{i;1}$ describes the leading-order asymptotic behavior around the origin $x = O(1)$. This is rigorously justified by the theorem in the next section.

2.2 Main theorem and its consequences

We now state our main theorem. To state the assumptions of the theorem, we set $\boldsymbol{u}_0 = (v_0 - 1, u_0)$ and denote its anti-derivatives by $\boldsymbol{u}_0^{\pm}$:

$$\boldsymbol{u}_0^-(x) = \int_{-\infty}^{x} \boldsymbol{u}_0(y)\, dy, \quad \boldsymbol{u}_0^+(x) = \int_{x}^{\infty} \boldsymbol{u}_0(y)\, dy.$$

Theorem 2.1. *For $\boldsymbol{u}_0 = (v_0 - 1, u_0) \in H^6(\mathbb{R}) \times H^6(\mathbb{R})$, let (v,u) be the solution to (1) and define u_i by (2). Further, we define θ_i and $\xi_{i;n}$ by (3) and (7), respectively. Set*

$$u_{i;1} = \xi_{i;1} + \gamma_{i'}\partial_x\theta_{i'}, \quad u_{i;n} = \xi_{i;n} + \gamma_{i'}\partial_x\xi_{i';n-1} \quad (n \geq 2).$$

Here, $i' = 3 - i$ and $\gamma_i = (-1)^i\nu/(4c)$. Then for $n \geq 1$, there exist positive constants δ_n and C_n such that if

$$\begin{aligned}\delta := \|\boldsymbol{u}_0\|_6 &+ \sup_{x\in\mathbb{R}}[(|x|+1)^{\alpha_n}|\boldsymbol{u}_0(x)| + (|x|+1)^{5/4}|\boldsymbol{u}_0'(x)|] \\ &+ \sup_{x>0}[(|x|+1)^{\beta_n}(|\boldsymbol{u}_0^-(-x)| + |\boldsymbol{u}_0^+(x)|)] \leq \delta_n,\end{aligned}$$

the solution (v,u) satisfies the following pointwise estimates:

$$\left|\left(u_i - \theta_i - \sum_{k=1}^{n} u_{i;k}\right)(x,t)\right| \leq C_n\delta\Psi_{i;n}(x,t) \tag{8}$$

for all $x \in \mathbb{R}$ and $t \geq 0$.

Let us now discuss some consequences of the theorem.

Corollary 2.1. *Under the assumptions of Theorem 2.1, we have the optimal L^p-decay estimate*

$$\left\|\left(u_i - \theta_i - \sum_{k=1}^{n} u_{i;k}\right)(\cdot,t)\right\|_{L^p} \leq C_n\delta(t+1)^{(\alpha_n - 1/p)/2} \quad (1 \leq p \leq \infty).$$

Remark 2.1. By the Corollary above and Proposition 2.1,

$$u_i \sim \theta_i + \sum_{n=1}^{\infty} u_{i;n}$$

is a time-asymptotic expansion in the $L^p(\mathbb{R})$-norm for every $1 \le p \le \infty$. Note that

$$\|u_{i;n}(\cdot,t)\|_{L^p} \lesssim t^{-(\alpha_{n-1}-1/p)/2}.$$

This is in contrast to the result in [8]: due to the lack of pointwise estimates, their asymptotic expansion is proved to be valid only in the $L^p(\mathbb{R})$-norm for $2 \le p \le \infty$.

As we stated in the introduction, our goal is to obtain a clear understand of the long-time behavior around $x = 0$. For this, we have the following answer.

Corollary 2.2. *Under the assumptions of Theorem 2.1, we have the optimal decay estimate*

$$\left| \left(u_i - \theta_i - \sum_{k=1}^{n} u_{i;k} \right)(x,t) \right| \le C_n \delta t^{-\alpha_n}$$

for $x = O(1)$.

Remark 2.2. By the Corollary above and Proposition 2.1,

$$u_i \sim \theta_i + \sum_{n=1}^{\infty} u_{i;n}$$

is also a time-asymptotic expansion locally for $x = O(1)$. Note that

$$|u_{i;n}(x,t)| \lesssim t^{-\alpha_{n-1}}.$$

In particular, we see that u_i is asymptotically equivalent to $\xi_{i;1}$ around the origin $x = 0$. As we wrote soon after Proposition 2.1, we have a simple asymptotic formula for $\xi_{i;n}$ for $x = O(1)$. Therefore, we now have a very precise understand of the long-time behavior of the flow around the origin.

We give a remark on comparison of our time-asymptotic expansion and the one in [8].

Remark 2.3. The higher-order terms in the expansion constructed in [8] and the higher-order diffusion waves $\xi_{i;n}$ are closely related although defined in seemingly different ways. One crucial difference between them is that the higher-order terms in the former are in self-similar form but ours are not. In fact, $\xi_{i;n}$ resembles $\Psi_{i;n-1}$ and it has two waves moving with speeds λ_i and $-\lambda_i$; see Proposition 2.1. Thus we cannot approximate $\xi_{i;n}$ by a single self-similar wave. The higher-order terms in [8] are essentially the higher-order diffusion waves $\xi_{i;n}$ with the "$-\lambda_i$-part" neglected. The neglected part is higher-order in the $L^p(\mathbb{R})$-norm for $1 \le p \le \infty$ but is important for the pointwise estimates.

We finally remark on the limit $n \to \infty$.

Remark 2.4. The sum $\sum_{n=1}^{\infty} u_{i;n}$ is actually convergent but the constant C_n on the right-hand side of (8) diverges as $n \to \infty$. It seems plausible that if we add a $\log(t+1)$-factor on the right side, the pointwise estimates are also valid in the limit $n \to \infty$. To go beyond $\log t/t$-order, we need to identify the logarithmic correction. This was done for scalar generalized Burgers' equation in [1] and is an open problem for our system.

2.3 Possible extensions

We discuss some possible extensions of our theorem.

First, it should be possible to extend the theorem to general quasilinear hyperbolic-parabolic systems of conservation laws considered for example in [6]. Another possible extension is the corresponding theorem for the hyperbolic balance laws considered for example in [10].

Another natural direction is its extension to initial-boundary value problems, for example, to the free-boundary problem considered in [2]. We hope to present such results in the near future.

2.4 Outline of the proof

The proof is a higher-order analogue of the Green's function method in [6]. We refer to our preprint [5] for details.

The first step of the proof is to write down an integral formulation of (1) using Green's function (the fundamental solution) of the linearized system. Then, with the ansatz that u_i satisfies the conclusion of Theorem 2.1, we estimate the convolutions in the integral equation. Roughly speaking, if we could show that all the convolutions in the integral equation also satisfy the same ansatz, we can show that the ansatz is true. The convolutions are bounded using pointwise estimates of Green's function in [6, 7].

The important part of the proof is to understand how one should define the higher-order terms in the expansion, namely, $\xi_{i;n}$. We explain this in the next section.

2.5 Heuristics behind the definition of the higher-order diffusion waves

Define Ξ_i as the infinite sum of the higher-order diffusion waves $(\xi_{i;n})_{n=1}^{\infty}$, that is,

$$\Xi_i = \sum_{n=1}^{\infty} \xi_{i;n}.$$

Then by (7), we see that

$$\begin{cases} \partial_t \Xi_i + \lambda_i \partial_x \Xi_i + \partial_x \left(\theta_{i'}^2/2 + \theta_1 \Xi_1 + \theta_2 \Xi_2 \right) = \dfrac{\nu}{2} \partial_x^2 \Xi_i, & x \in \mathbb{R}, t > 0, i \in \{1, 2\}, \\ \Xi_i(x, 0) = 0, & x \in \mathbb{R}, i \in \{1, 2\}. \end{cases} \tag{9}$$

Note that the equations for Ξ_1 and Ξ_2 are coupled. We shall see below that these equations arise naturally from (1).

We first note that (1) can be written in vector form as

$$\boldsymbol{u}_t + A\boldsymbol{u}_x = \begin{pmatrix} 0 & 0 \\ 0 & \nu \end{pmatrix} \boldsymbol{u}_{xx} + \begin{pmatrix} 0 \\ N_x \end{pmatrix}, \tag{10}$$

where

$$\boldsymbol{u} = \begin{pmatrix} v - 1 \\ u \end{pmatrix}, \quad A = \begin{pmatrix} 0 & -1 \\ -c^2 & 0 \end{pmatrix}, \quad N = -p(v) + p(1) - c^2(v - 1) - \nu \frac{v - 1}{v} u_x.$$

The matrix A has

$$l_1 = \frac{p''(1)}{4c}\begin{pmatrix}-1 & 1/c\end{pmatrix}, \quad l_2 = \frac{p''(1)}{4c}\begin{pmatrix}1 & 1/c\end{pmatrix}$$

as left eigenvectors corresponding to eigenvalues $\lambda_1 = c$ and $\lambda_2 = -c$.

Let $w_i = u_i - \theta_i$. We investigate the differential equation satisfied by w_i. First, multiply l_i to (10) from the left and apply Taylor's theorem to the nonlinear term N. Then we obtain

$$\partial_t u_i + \lambda_i \partial_x u_i + \partial_x (u_1 - u_2)^2/2 = \frac{\nu}{2}\partial_x^2 (u_1 + u_2) + \partial_x O(|\boldsymbol{u}|^3 + |\boldsymbol{u}||\partial_x \boldsymbol{u}|).$$

Next, subtract (3) from the equation above. We then observe that w_i satisfies

$$\begin{aligned} &\partial_t w_i + \lambda_i \partial_x w_i + \partial_x(\theta_{i'}^2/2) + \partial_x(\theta_1 w_1 + \theta_2 w_2) - \frac{\nu}{2}\partial_x^2 w_i \\ &= \partial_x(-w_1^2/2 - w_2^2/2 + u_1 u_2) + \frac{\nu}{2}\partial_x^2 u_{i'} + \partial_x O(|\boldsymbol{u}|^3 + |\boldsymbol{u}||\partial_x \boldsymbol{u}|). \end{aligned} \tag{11}$$

The terms w_1^2 and w_2^2 decay as $t^{-3/2}$ in the $L^\infty(\mathbb{R})$-norm by (5). So these are compatible to $O(|\boldsymbol{u}|^3 + |\boldsymbol{u}||\partial_x \boldsymbol{u}|)$ and are expected to be negligible. Next, the term $u_1 u_2$ is a product of waves of different family. Consider a similar example, namely, the product $\theta_1\theta_2$. This decays exponentially fast as can be seen from (4). By a similar reason, $u_1 u_2$ is negligible. Finally, the term $\partial_x^2 u_{i'}$ is indexed by i'; such term has lesser contribution compared to waves belonging to the index i (cf. [6, Lemma 3.4]). Dropping these terms from (11), we get the equation for Ξ_i, that is, (9).

From the consideration above, we see that w_i approximately satisfies (9). So $w_i - \Xi_i = u_i - \theta_i - \Xi_i$ should decay faster compared to $w_i = u_i - \theta_i$. This heuristically explains why $u_i - \theta_i - \sum_{n=1}^{\infty} \xi_{i;n}$ appears in Theorem 2.1. A more detailed analysis reveals that $\gamma_{i'}\partial_x(\theta_{i'} + \sum_{n=1}^{\infty} \xi_{i';n})$ is also needed. Nevertheless, the argument above at least gives a heuristic explanation of the definition of the higher-order diffusion wave $\xi_{i;n}$.

Acknowledgements

This work is supported by JSPS Grant-in-Aid for Early-Career Scientists (Grant Number 22K13938).

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