Existence of the 2D stationary Navier–Stokes flow on the whole plane around a radial flow

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1 Introduction

This proceeding is based on a joint work [10] with Maekawa Yasunori (Kyoto University).

We consider the two-dimensional stationary Navier-Stokes equations

$$\begin{cases} -\Delta u + u \cdot \nabla u + \nabla p = F, & x \in \mathbb{R}^2, \\ \nabla \cdot u = 0 & x \in \mathbb{R}^2. \end{cases} \tag{1}$$

Here $u = (u_1(x), u_2(x))$ and $p = p(x)$ denote the unknown velocity vector and the unknown pressure of the fluid at the point $x = (x_1, x_2) \in \mathbb{R}^2$, respectively, while $F = (F_1(x), F_2(x))$ is the given external force. Here $u \cdot \nabla u := \sum_{j=1,2} u_j \partial_{x_j} u$.

For three or higher dimension case, the system (1) have been studied well. For instance, Leray [9] and Ladyzhenskaya [8] showed the existence of strong solutions to (1), and Heywood [4] constructed solutions of (1) as a limit of solutions of the non-stationary Navier-Stokes equations. Later on, various researchers have found scaling invariant spaces of F guaranteeing the existence of solutions in $\mathbb{R}^3$, such as Chen [1] in the Lebesgue space, Kozono-Yamazaki [6] in the Morrey space, and Kaneko-Kozono-Shimizu [7] in the Besov space. These theories, however, cannot be applied for the two dimension case, since it is hard to estimate the advection term $u \cdot \nabla u$ in scaling invariant function spaces. Hence, until now, the two dimension case has been considered independently of higher dimension ones.

For example, Yamazaki [12, 13] showed the existence of small solutions to (1) in the weak L^2 space when given external forces decay sufficiently and have some symmetric properties, such as $f_1(x_2, x_1) = f_2(x_1, x_2)$ for example. Galdi-Yamazaki [?] later showed the stability of the above solutions more precisely. For its uniqueness, Nakatsuka [11] constructed more general theory. Recently, Guillod [2] showed the existence of a pair (u, F) solving (1), where F is dependent on u and is constructed around an arbitrarily given small function k having zero integral and decaying faster than $|x|^{-3}$. Moreover, Guillod-Wittwer [3] found solutions to (1) which is scaling invariant with respect to a rotation conversion.

In this study, we will show the existence of solutions for external forces which are not necessarily symmetric. Let us suppose that a external force F is divergence-free and can be expressed as $F = \nabla^\perp \phi := (\partial_{x_2}\phi, -\partial_{x_1}\phi)$ with some flow ϕ. Actually, by the Helmholtz decomposition, we can write F as a sum of the rotation-free term and divergence-free one. Then the curl-free term is absorbed in the pressure term ∇p, and the divergence-free one is written as $\nabla^\perp \phi$ if F decays sufficiently. Under such a situation, we will show that for every small compact supported radial flow ϕ_* having non-zero integral and its smaller perturbation φ (which is not necessarily symmetric) decaying faster than $|x|^{-2}$, there exist solutions (u, p) of (1) for $F = \nabla^\perp(\phi_* + \varphi)$. In our result, we also impose the smoothness of ϕ_* and φ, so that we may obtain u as a classical solution in $C^2(\mathbb{R}^2; \mathbb{R})$.

The standard approach to (1) is to analyze the vorticity-streamfunction system as below, which is equivalent with (1) in a suitable functional framework.

$$\begin{cases} \Delta\psi = -\omega, & x \in \mathbb{R}^2, \\ \Delta\omega = \nabla \times (\nabla^\perp\psi \cdot \nabla(\nabla^\perp\psi)) + \Delta\phi, & x \in \mathbb{R}^2. \end{cases} \tag{2}$$

Here $\psi = \psi(x)$ is the stream function, which generates the divergence-free flow as $u = \nabla^\perp\psi$, while $\omega := \nabla \times u := \partial_{x_1}u_2 - \partial_{x_2}u_1$ is the vorticity field. In addition, we convert orthogonal coordinates (x_1, x_2) to polar ones (r, θ), and consider the Fourier series $\psi(r,\theta) = \sum_{n\in\mathbb{Z}} \psi_n(r)e^{in\theta}$ and $\omega(r,\theta) = \sum_{n\in\mathbb{Z}} \omega_n(r)e^{in\theta}$ with respect to the angular valuable θ. Then (2) is expressed as the system of ordinary differential equations of $\{\psi_n\}_{n\in\mathbb{Z}}$ and $\{\omega_n\}_{n\in\mathbb{Z}}$ with the radius valuable r. For this method, we especially refer to Hillairet-Wittwer [5], which studied the exterior problem outside of the unit disk centered on the origin.

This report is organized as follows. In the next section, we will define the Fourier modes of vector fields on polar coordinates and some important function spaces of them, and we will state our main theorem in the third section. After that, the policy of the proof and some required propositions are stated in the last section.

2 Preliminaries

We will analyze the equations in the polar coordinates. Actually, some key structures are found by decomposing the system (2) into the Fourier mode with respect to the angular valuable. For later use, let us introduce the Fourier series, as

$$f(r\cos\theta, r\sin\theta) = \sum_{n\in\mathbb{Z}} f_n(r)e^{in\theta}, \quad (r,\theta) \in [0,\infty) \times [0, 2\pi),$$

where f_n denotes the n-mode of f defined by

$$f_n(r) = \frac{1}{2\pi}\int_0^{2\pi} f(r\cos\theta, r\sin\theta)e^{-in\theta}d\theta, \quad 0 \le r < \infty.$$

In what follows, we write $\hat{f} = (f_n)_{n\in\mathbb{Z}}$ and $\tilde{f} = (f_n)_{n\in\mathbb{Z}\setminus\{0\}}$ for the Fourier mode of f. We now introduce the following function spaces for the Fourier mode. For simplicity, we write continuous and smooth function spaces as

$$C^m := C^m([0,\infty);\mathbb{C}),\ \hat{C}^m := (C^m)^{\mathbb{Z}},\ \tilde{C}^m := (C^m)^{\mathbb{Z}\setminus\{0\}}$$

for $m \in \mathbb{N}\cup\{0\}$. Let $\alpha > 0$ and $\kappa > 1$. For a function $f \in C^l$, define the weight function $\mathcal{M}^l_{n;\alpha,\kappa}[f]$ as

$$\mathcal{M}^l_{n;\alpha,\kappa}[f](r) := (1+r)^{\alpha+l}(1+|n|)^{\kappa-l}|\partial_r^l f(r)|, \quad n \in \mathbb{Z},\ l \in \mathbb{N}\cup\{0\},\ l < \kappa.$$

We note here that α counts the decay in r and κ the decay in n. In addition, the order l of derivative makes the decay in r faster and the decay in n slower. In this report, we applied this weight $\mathcal{M}^l_{n;\alpha,\kappa}$ for the Fourier mode f_n with the same $n \in \mathbb{Z}$, so that in what follows, we use the abbreviation as

$$\mathcal{M}^l_{\alpha,\kappa}[f_n](r) := \mathcal{M}^l_{n;\alpha,\kappa}[f_n](r).$$

In association with this weight function, we define some norms and spaces as follows. As for spaces of the vorticity and external force, we define the norm

$$\|\hat{f}\|_{\mathcal{U}^m_{\alpha,\kappa}} := \sum_{l=0}^m \sup_{n\in\mathbb{Z}}\sup_{r\ge 0}\mathcal{M}^l_{\alpha,\kappa}[f_n](r), \quad \hat{f} \in \hat{C}^m,\ m \in \mathbb{N}\cup\{0\},\ m < \kappa$$

and set

$$\mathcal{U}^m_{\alpha,\kappa} := \left\{\hat{f} \in \hat{C}^m;\ \|\hat{f}\|_{\mathcal{U}^m_{\alpha,\kappa}} < \infty,\ f_{-n} = \overline{f_n}\ \forall n \in \mathbb{Z},\ f_{n'}(0) = 0\ \forall n' \in \mathbb{Z}\setminus\{0\}\right\}.$$

Here $\overline{f}$ denotes the complex conjugate of f.

On the other hand, as a space of stream functions, we set

$$\mathcal{V}^m_{\alpha,\kappa} := \left\{\hat{f} = (f_0, \tilde{f}) \in \mathcal{V}^m_0 \times \tilde{\mathcal{V}}^m_{\alpha,\kappa};\ \|\hat{f}\|_{\mathcal{U}^m_{\alpha,\kappa}} := \|f_0\|_{\mathcal{V}^m_0} + \|\tilde{f}\|_{\tilde{\mathcal{V}}^m_{\alpha,\kappa}} < \infty\right\},$$

where

$$\mathcal{V}^m_0 := \left\{f_0 \in C^m;\ \|f_0\|_{\mathcal{V}^m_0} := \sum_{l=1}^m \sup_{r\ge 0}\mathcal{M}^l_{0,0}[f_0](r) < \infty,\ f_0 = \overline{f_0}\right\},$$

$$\tilde{\mathcal{V}}^m_{\alpha,\kappa} := \left\{\tilde{f} \in \tilde{C}^m;\ \|\tilde{f}\|_{\tilde{\mathcal{V}}^m_{\alpha,\kappa}} := \sum_{l=0}^m \sup_{n\in\mathbb{Z}\setminus\{0\}}\sup_{r\ge 0}\mathcal{M}^l_{\alpha,\kappa}[f_n](r) < \infty,\ f_{-n} = \overline{f_n}\ \forall n \in \mathbb{Z}\right\}.$$

We note that as for $\hat{f} \in \mathcal{V}^m_{\alpha,\kappa}$, we do not consider the zero mode f_0 itself, but consider its differential. In addition, each differential $\partial_r^l f_0$ decays as $|\partial_r^l f_0(r)| \lesssim (1+r)^{-l}$, while that of non-zero mode $\partial_r^l f_n$ decays as $|\partial_r^l f_n(r)| \lesssim (1+r)^{-\alpha-l}$.

We easily see that the above $\mathcal{U}^m_{\alpha,\kappa}$ is a Banach space with its norm. Moreover, there hold the embeddings $\mathcal{U}^m_{\alpha,\kappa} \subset \mathcal{U}^{m'}_{\alpha',\kappa'}$ and $\mathcal{V}^m_{\alpha,\kappa} \subset \mathcal{V}^{m'}_{\alpha',\kappa'}$ for every $0 < \alpha' \le \alpha$, $1 < \kappa' \le \kappa$, and $1 \le m' \le m$ with the estimates

$$\|\hat{f}\|_{\mathcal{U}^{m'}_{\alpha',\kappa'}} \le \|\hat{f}\|_{\mathcal{U}^m_{\alpha,\kappa}}, \quad \|\hat{f}\|_{\mathcal{V}^{m'}_{\alpha',\kappa'}} \le \|\hat{f}\|_{\mathcal{V}^m_{\alpha,\kappa}}.$$

3 Main results

Our main result now reads:

Theorem 3.1. *There exists a constant $\delta > 0$ such that the following statement holds.*

Let $\phi_* \in C^2_c(\mathbb{R}^2;\mathbb{R})$ be a radial function expressed as $\phi_*(r) \in C^2_c([0,\infty);\mathbb{R})$ in polar coordinates (r,θ), and let $R_* \ge 1$ be such that $\operatorname{supp}(\phi_*) \subset [0, R_*]$. *Moreover, define*

$$\mu_* := \int_0^{R_*} s\phi_*(s)ds, \quad \rho_* := \sqrt{2}\left[\left\{1 + \left(\frac{\mu_*}{2}\right)^2\right\}^{\frac{1}{2}} + 1\right]^{\frac{1}{2}} - 2,$$

$$\nu_* := \sup_{0 \le r \le R_*}\left|\int_0^r s\phi_*(s)ds\right| + \sup_{0 \le r \le R_*}(1+r)^2|\phi_*(r)| + \sup_{0 \le r \le R_*}(1+r)^2|\partial_r\phi_*(r)|,$$

and suppose that

$$\mu_* \neq 0, \quad R_*^{\rho_*}\nu_* < \delta. \tag{3}$$

Then for every $0 < \alpha < \min\{1/2, \rho_*\}$, there exists a constant $\varepsilon = \varepsilon(R_*^{\rho_*}, \alpha) > 0$ such that for every $\varphi \in C^2(\mathbb{R}^2;\mathbb{R})$ whose Fourier mode satisfies $\|\hat{\varphi}\|_{\mathcal{U}^1_{\alpha+2,\kappa+2}} < \varepsilon$ for some $\kappa > 1$, there exists a solution $u \in C^2(\mathbb{R}^2;\mathbb{R}^2)$ of (1) for an external force $F = \nabla^\perp(\phi_* + \varphi)$ having the decay property

$$\sup_{r \ge 0}\sup_{0 \le \theta < 2\pi}(1+r)^{1+\alpha}\left|u(r,\theta) - \left(\frac{1}{r}\int_0^r s(\phi_* + \varphi_0)(s)ds\right)e_\theta\right| < \infty, \tag{4}$$

where φ_0 denotes the Fourier zero-mode of φ, and $e_\theta := (-\sin\theta, \cos\theta)$ denotes the basis for the direction of increasing angle in polar coordinates.

Here we remark that μ_* is also written as $(2\pi)^{-1}\int_{\mathbb{R}^2}\phi_*(x)dx$. On the other hand, $\rho_* = \Re[(4+2i\mu_*)^{1/2}]-2$ ($\Re[z]$ denotes the real part of z) appears in the partial linearization of the system regarding to the Fourier ± 2 modes. The asymptotic estimate (4) implies that the solution u behaves like the radial and rotational flow $cx^\perp/|x|^2$ with some $c \in \mathbb{R}$ that decays in the scale-critical order $O(|x|^{-1})$.

The key observation in the proof of Theorem 3.1 is that the system (2) has two aspects; the one is related to the analysis in the Fourier ± 2 modes with respect to the angular variable θ, where one needs to use the effect of the vorticity transport by the flow $\mu_* x^\perp/|x|^2$ to avoid the appearance of the logarithmic loss from the scale-critical decay pointed out by Guillod [2]. The other is related to the Fourier ± 1 modes, where we find the key cancellation property in the nonlinear term $\nabla \times (u \cdot \nabla u)$ that seems to be avairable only by regarding the linearized term around $\mu_* x^\perp/|x|^2$ as the perturbation. Hence, we build up the iteration scheme by taking this observation into account, that is, the transport term by the flow $\mu_* x^\perp/|x|^2$ is incorporated as the principal term for the Fourier ± 2 modes, while this term is handled as the perturbation for the other Fourier modes and we use the smallness of μ_* and the cancellation property in the Fourier ± 1 modes. The smallness condition of (3) is then needed to close the linear estimate. Another advantage of our result is that there is no restriction on φ regarding to its structure such as symmetry, while the previous studies [11, 12, 13] require such structural assumptions.

4 Proof

4.1 Outline

We now fix $\phi_* \in C_c^2([0,\infty);\mathbb{R})$ satisfying $\mathrm{supp}\,(\phi_*) \subset [0, R_*]$ and $\mu_* \neq 0$. Since $\rho_* < \mu_* < \nu_*$, and since the smallness condition in (3) should be satisfied, we assume that $0 < \rho_* < 1$ in what follows.

In terms of the polar coordinates, the vorticity-streamfunction system (2) is expressed as

$$\begin{cases} \Delta_{r,\theta}\psi = -\omega, & (r,\theta) \in [0,\infty) \times [0,2\pi), \\ \Delta_{r,\theta}\omega = G + \Delta_{r,\theta}\phi, & (r,\theta) \in [0,\infty) \times [0,2\pi). \end{cases} \tag{5}$$

Here $\Delta_{r,\theta} := \partial_r^2 + (1/r)\partial_r + (1/r^2)\partial_\theta^2$ denotes the Laplacian in the polar coordinates, and

$$G := -\frac{1}{r^2}\partial_r(r\partial_r D + \partial_\theta E) + \frac{1}{r^3}(1 + \partial_\theta^2)D, \tag{6}$$

where

$$D := \partial_r\psi\partial_\theta\psi, \quad E := \frac{(\partial_\theta\psi)^2}{r} - r(\partial_r\psi)^2.$$

On the other hand, since

$$\nabla \times (\nabla^\perp\psi \cdot \nabla(\nabla^\perp\psi)) = \nabla^\perp\psi \cdot \nabla\omega$$

by $\nabla \cdot \nabla^\perp\psi = 0$, we also see that

$$G = \frac{1}{r}(\partial_\theta\psi\partial_r\omega - \partial_r\psi\partial_\theta\omega). \tag{7}$$

Since ϕ_* depends only on r, we see that

$$(\psi,\omega)=(\psi_*(r),\omega_*(r)):=\left(-\int_0^r\frac{1}{s}\int_0^s t\phi_*(t)dtds,\ \phi_*(r)\right)$$

are exact classical solutions of (5) for $\phi=\phi_*$. Therefore, for given small $\varphi\in C^2(\mathbb{R}^2;\mathbb{R})$, we aim to construct solutions of (5) for $\phi=\phi_*+\varphi$ such as

$$\begin{cases}\psi(r,\theta)=\psi_*(r)+\gamma(r,\theta)=\psi_*(r)+\displaystyle\sum_{n\in\mathbb{Z}}\gamma_n(r)e^{in\theta},\\ \omega(r,\theta)=\omega_*(r)+w(r,\theta)=\omega_*(r)+\displaystyle\sum_{n\in\mathbb{Z}}w_n(r)e^{in\theta},\end{cases}\tag{8}$$

where (γ,w) denote the perturbations and $(\hat\gamma,\hat w)=((\gamma_n)_{n\in\mathbb{Z}},(w_n)_{n\in\mathbb{Z}})$ are those Fourier modes. Now let us suppose that (ψ,ω) in (8) are smooth enough and really solutions of (5) for the moment. Since $\Delta_{r,\theta}\psi_*=-\omega_*$ and $\Delta_{r,\theta}\omega_*=\Delta_{r,\theta}\phi_*$, the perturbations $(\hat\gamma,\hat w)$ should satisfy the following system.

$$\begin{cases}\Delta_{r,n}\gamma_n=-w_n, & n\in\mathbb{Z},\ r\geq 0,\\ \Delta_{r,n}w_n=\mathcal{G}_n^*+\Delta_{r,n}\varphi_n, & n\in\mathbb{Z},\ r\geq 0.\end{cases}\tag{9}$$

Here $\hat\varphi=(\varphi_n)_{n\in\mathbb{Z}}$ denotes the Fourier mode of φ, $\Delta_{r,n}:=\partial_r^2+(1/r)\partial_r-(n^2/r^2)I$, and

$$\mathcal{G}_n^*=\mathcal{G}_n(\psi_*,\hat\gamma):=-\frac{1}{r^2}\partial_r(r\partial_r\mathcal{D}_n(\psi_*,\hat\gamma)+in\mathcal{E}_n(\psi_*,\hat\gamma))+\frac{1}{r^3}(1-n^2)\mathcal{D}_n(\psi_*,\hat\gamma)\tag{10}$$

is the Fourier mode of G associated with (6), where

$$\begin{aligned}\mathcal{D}_n(\psi_*,\hat\gamma)&:=i\sum_{k+l=n}k\gamma_k\partial_r\gamma_l+in\gamma_n\partial_r\psi_*, && n\in\mathbb{Z},\\ \mathcal{E}_n(\psi_*,\hat\gamma)&:=-\frac{1}{r}\sum_{k+l=n}kl\gamma_k\gamma_l-r\sum_{k+l=n}\partial_r\gamma_k\partial_r\gamma_l-2r\partial_r\gamma_n\partial_r\psi_*, && n\in\mathbb{Z}\backslash\{0\}.\end{aligned}$$

On the other hand, in the notation of (7), $\mathcal{G}_n^*$ is also expressed as

$$\mathcal{G}_n^*=\mathcal{H}_n(\psi_*,\omega_*,\hat\gamma,\hat w):=\frac{i}{r}\sum_{k+l=n}(k\gamma_k\partial_r w_l-lw_l\partial_r\gamma_k)+\frac{in\partial_r\omega_*}{r}\gamma_n-\frac{in\partial_r\psi_*}{r}w_n.\tag{11}$$

Since $f(r)=r^{\pm|n|}$ are fundamental solutions of the ordinary equation $\Delta_{r,n}f=0$ for each $n\in\mathbb{Z}\backslash\{0\}$, $(\hat\gamma,\hat w)$ satisfying the system (9) are expressed as

$$\gamma_n(r)=\begin{cases}I_{|n|}^{\infty}[w_n](r)+J_{|n|}^{0}[w_n](r), & r\geq 0,\ n\in\mathbb{Z}\backslash\{0\},\\ -\displaystyle\int_0^r\frac{1}{s}\int_0^s tw_0(t)dtds, & r\geq 0,\ n=0,\end{cases}\tag{12}$$

$$w_n(r)=\begin{cases}-I_{|n|}^{\infty}[\mathcal{G}_n^*](r)-J_{|n|}^{0}[\mathcal{G}_n^*](r)+\varphi_n(r), & r\geq 0,\ n\in\mathbb{Z}\backslash\{0\},\\ -\displaystyle\int_r^\infty\frac{1}{s}\int_0^s t\mathcal{G}_0^*(t)dtds+\varphi_0(r), & r\geq 0,\ n=0.\end{cases}\tag{13}$$

Here we define integrations I_z^T and J_z^t as

$$I_z^T[f](r) := \frac{r^z}{2z}\int_r^T s^{1-z} f(s)ds,\ \ z\in\mathbb{C}\backslash\{0\},\ \ 0\le r\le T\ \ (0\le r<\infty \text{ if } T=\infty),$$
$$J_z^t[f](r) := \frac{1}{2zr^z}\int_t^r s^{1+z} f(s)ds,\ \ z\in\mathbb{C}\backslash\{0\},\ \ t\le r<\infty.$$

We will mainly analyze the expressions (12) and (13) with $\mathcal{G}_n^*$ of the expression (10), while we will apply (11) to confirm the non-singularity at $r=0$. It should be emphasized here that the expression (10) reveals the key cancellation for $|n|=1$ in achieving the desired spatial decay, which is difficult to see if one uses only (11). However, there exists a problem in analysis of (13) when $|n|=2$. Indeed, it is difficult to derive the decay property $\lim_{r\to\infty}(1+r)^2|w_{\pm 2}(r)|=0$ from the expression (13). Therefore, in order to solve such a problem for $n=\pm 2$, we utilize the effect of ϕ_* as follows. Since

$$\partial_r\omega_*(r)=\partial_r\phi_*(r)\equiv 0,\quad \partial_r\psi_*(r)=-\frac{1}{r}\int_0^r s\phi_*(s)ds=-\frac{\mu_*}{r}$$

for every $r\ge R_*$, we can rewrite the system (9) locally as

$$\begin{cases} \Delta_{r,n}\gamma_n=-w_n, & n\in\mathbb{Z},\ r>R_*,\\ \Delta_{r,\zeta_n}w_n=\mathcal{G}_n^0+\Delta_{r,n}\varphi_n, & n\in\mathbb{Z},\ r>R_*,\end{cases}$$

where $\zeta_n:=(n^2+in\mu_*)^{\frac{1}{2}}$, $\Delta_{r,\zeta_n}:=\partial_r^2+(1/r)\partial_r-(\zeta_n^2/r^2)Id$, and

$$\mathcal{G}_n^0:=\mathcal{G}_n(0,\hat{\gamma})=\mathcal{H}_n(0,0,\hat{\gamma},\hat{w}). \tag{14}$$

Then using the fundamental solutions $f(r)=r^{\pm\zeta_n}$ for $\Delta_{r,\zeta_n}f=0$, we have the other expression of w_n such that

$$w_n(r)=-I_{\zeta_n}^{\infty}[\mathcal{G}_n^0+\Delta_{r,n}\varphi_n](r)-J_{\zeta_n}^0[\mathcal{G}_n^0+\Delta_{r,n}\varphi_n](r),\ \ r>R_*,\ \ n\in\mathbb{Z}\backslash\{0\}. \tag{15}$$

Actually, we can find the better decay property for $w_{\pm 2}$ of (15) than those of (13). Hence in the case of $|n|=2$, we set w_n as

$$w_n(r)=\begin{cases} -I_{|n|}^{R_*}[\mathcal{G}_n^*](r)-J_{|n|}^0[\mathcal{G}_n^*](r)+\varphi_n(r)+c_1r^{|n|}, & 0\le r\le R_*,\\ -I_{\zeta_n}^{\infty}[\mathcal{G}_n^0+\Delta_{r,n}\varphi_n](r)-J_{\zeta_n}^{R_*}[\mathcal{G}_n^0+\Delta_{r,n}\varphi_n](r)+c_2r^{-\zeta_n}, & R_*\le r<\infty,\end{cases} \tag{16}$$

where $c_1,c_2\in\mathbb{R}$ are constants. In this case, we should fix c_1 and c_2 so that w_n defined by (16) becomes continuous and differentiable at $r=R_*$, the detail of which will be stated later.

4.2 Key estimates

First of all, let us define the linear map $\mathcal{L}$ associated with (12), which has the following property.

Proposition 4.1. *Let $0<\alpha<1/2$ and $\kappa>1$. For every $\hat{w}\in\mathcal{U}^1_{\alpha+2,\kappa+2}$, define the map $\mathcal{L}:\hat{w}\mapsto\hat{\gamma}$ as*

$$\gamma_n(r):=\begin{cases} I^{\infty}_{|n|}[w_n](r)+J^0_{|n|}[w_n](r), & n\in\mathbb{Z}\backslash\{0\},\\ -\displaystyle\int_0^r\frac{1}{s}\int_0^s tw_0(t)dtds, & n=0.\end{cases}$$

Then $\hat{\gamma}$ belongs to $\mathcal{V}^2_{\alpha,\kappa+4}$, and there hold

$$\|\gamma_0\|_{\mathcal{V}^2_0}\lesssim\frac{1}{\alpha}\|\hat{w}\|_{\mathcal{U}^0_{\alpha+2,\kappa+2}},\quad \|\tilde{\gamma}\|_{\tilde{\mathcal{V}}^2_{\alpha,\kappa+4}}\lesssim\|\hat{w}\|_{\mathcal{U}^1_{\alpha+2,\kappa+2}}.$$

Secondly, we define the map $\mathcal{S}$ associated with (13) and (16) as follows. For fixed $0<\alpha<1/2$ and $\kappa>1$, we take $(\hat{w},\hat{\sigma})\in\mathcal{U}^1_{\alpha+2,\kappa+2}\times\mathcal{U}^1_{\alpha+2,\kappa+2}$ arbitrarily, and let $\hat{\gamma}=\mathcal{L}(\hat{w})$. Then we define $\hat{y}=\mathcal{S}(\hat{w},\hat{\sigma})$ as

$$y_n(r):=\begin{cases} -I^{\infty}_{|n|}[\mathcal{G}^*_n](r)-J^0_{|n|}[\mathcal{G}^*_n](r)+\sigma_n(r), & r\geq 0,\ n\in\mathbb{Z}\backslash\{0,\pm2\},\\ y_{n,1}(r):=\mathcal{P}_{n,1}(r)+\mathcal{Q}_{n,1}[\sigma_n](r) & 0\leq r\leq R_*,\ n=\pm2,\\ y_{n,2}(r):=\mathcal{P}_{n,2}(r)+\mathcal{Q}_{n,2}[\sigma_n](r) & r>R_*,\ n=\pm2,\\ -\displaystyle\int_r^{\infty}\frac{1}{s}\int_0^s t\mathcal{G}^*_0(t)dtds+\sigma_n(r), & r\geq 0,\ n=0,\end{cases}\tag{17}$$

where each $\mathcal{G}^*_n$ is that of (10) or (11),

$$\begin{aligned}\mathcal{P}_{n,1}(r):=&-I^{R_*}_{|n|}[\mathcal{G}^*_n](r)-J^0_{|n|}[\mathcal{G}^*_n](r)\\ &+\frac{(r/R_*)^{|n|}}{|n|+\zeta_n}\left\{(\zeta_n-|n|)J^0_{|n|}[\mathcal{G}^*_n](R_*)-2\zeta_n I^{\infty}_{\zeta_n}[\mathcal{G}^0_n](R_*)\right\},\ 0\leq r\leq R_*,\\ \mathcal{P}_{n,2}(r):=&-I^{\infty}_{\zeta_n}[\mathcal{G}^0_n](r)-J^{R_*}_{\zeta_n}[\mathcal{G}^0_n](r)\\ &+\frac{(R_*/r)^{\zeta_n}}{|n|+\zeta_n}\left\{-(\zeta_n-|n|)I^{\infty}_{\zeta_n}[\mathcal{G}^0_n](R_*)-2|n|J^0_{|n|}[\mathcal{G}^*_n](R_*)\right\},\ r>R_*\end{aligned}$$

with $\mathcal{G}^0_n$ of (14), and

$$\begin{aligned}\mathcal{Q}_{n,1}[\sigma_n](r):=&\sigma_n(r)\\ &+\frac{(r/R_*)^{|n|}}{|n|+\zeta_n}\left\{-2\zeta_n I^{\infty}_{\zeta_n}[\Delta_{r,n}\sigma_n](R_*)-\zeta_n\sigma_n(R_*)-R_*\partial_r\sigma_n(R_*)\right\},\ 0\leq r\leq R_*,\\ \mathcal{Q}_{n,2}[\sigma_n](r):=&-I^{\infty}_{\zeta_n}[\Delta_{r,n}\sigma_n](r)-J^{R_*}_{\zeta_n}[\Delta_{r,n}\sigma_n](r)\\ &+\frac{(R_*/r)^{\zeta_n}}{|n|+\zeta_n}\left\{-(\zeta_n-|n|)I^{\infty}_{\zeta_n}[\Delta_{r,n}\sigma_n](R_*)+|n|\sigma_n(R_*)-R_*\partial_r\sigma_n(R_*)\right\},\ r>R_*.\end{aligned}$$

In the case $|n|=2$, we see from the above expressions that if $y_{n,1}$ and $y_{n,2}$ are well-defined and differentiable in each domain, there hold

$$\partial^l_r\mathcal{P}_{n,1}(R_*)=\lim_{r\to R_*+0}\partial^l_r\mathcal{P}_{n,2}(r),\quad l=0,1\tag{18}$$

and

$$\partial_r^l \mathcal{Q}_{n,1}[\sigma_n](R_*) = \lim_{r\to R_*+0} \partial_r^l \mathcal{Q}_{n,2}[\sigma_n](r), \quad l = 0, 1, \tag{19}$$

so that each of $y_{\pm 2}$ belongs to the C^1 class on $[0, \infty)$.

For this solution map $\mathcal{S}$ with respect to the vorticity equation, we can obtain the following key estimate.

Proposition 4.2. *Let* $0 < \alpha < \min\{1/2, \rho_*\}$ *and* $\kappa > 1$*. Then the map* $\mathcal{S} : (\hat{w}, \hat{\sigma}) \mapsto \hat{y}$ *defined by* (17) *is bounded from* $\mathcal{U}^1_{\alpha+2,\kappa+2} \times \mathcal{U}^1_{\alpha+2,\kappa+2}$ *to* $\mathcal{U}^1_{\alpha+2,\kappa+2}$*. Moreover, there holds*

$$\|\mathcal{S}(\hat{w}, \hat{\sigma})\|_{\mathcal{U}^1_{\alpha+2,\kappa+2}} \lesssim R_*^{\rho_*} \left(\frac{1}{\alpha(\rho_* - \alpha)} \|\hat{w}\|^2_{\mathcal{U}^1_{\alpha+2,\kappa+2}} + \nu_* \|\hat{w}\|_{\mathcal{U}^1_{\alpha+2,\kappa+2}} + \frac{1}{\rho_* - \alpha} \|\hat{\sigma}\|_{\mathcal{U}^1_{\alpha+2,\kappa+2}} \right).$$

By Proposition 4.2, there is a constant $C_0 > 0$ such that

$$\begin{aligned} &\|\mathcal{S}(\hat{w}, \hat{\sigma})\|_{\mathcal{U}^1_{\alpha+2,\kappa+2}} \\ &\le C_0 R_*^{\rho_*} \left(\frac{1}{\alpha(\rho_* - \alpha)} \|\hat{w}\|^2_{\mathcal{U}^1_{\alpha+2,\kappa+2}} + \nu_* \|\hat{w}\|_{\mathcal{U}^1_{\alpha+2,\kappa+2}} + \frac{1}{\rho_* - \alpha} \|\hat{\sigma}\|_{\mathcal{U}^1_{\alpha+2,\kappa+2}} \right) \\ &=: K_1 \|\hat{w}\|^2_{\mathcal{U}^1_{\alpha+2,\kappa+2}} + K_2 \nu_* \|\hat{w}\|_{\mathcal{U}^1_{\alpha+2,\kappa+2}} + K_3 \|\hat{\sigma}\|_{\mathcal{U}^1_{\alpha+2,\kappa+2}} \end{aligned}$$

for every $0 < \alpha < \min\{1/2, \rho_*\}$, $\hat{w} \in \mathcal{U}^0_{\alpha+2,\kappa+2}$, and $\hat{\sigma} \in \mathcal{U}^1_{\alpha+2,\kappa+2}$. We should note that K_1 and K_3 are constants dependent on R_*, ρ_*, and α, while K_2 is a constant dependent only on R_* and ρ_*. Then in advance, we set ϕ_* so that

$$R_*^{\rho_*} \nu_* < \frac{1}{C_0} =: \delta,$$

in order to see $K_2 \nu_* < 1$. Moreover, let $\varepsilon > 0$ be such as

$$\varepsilon := \frac{(1 - K_2 \nu_*)^2}{4 K_1 K_3}$$

so that

$$(1 - K_2 \nu_*)^2 - 4 K_1 K_3 \|\hat{\sigma}\|_{\mathcal{U}^1_{\alpha+2,\kappa+2}} > 0$$

for every $\hat{\sigma} \in \mathcal{U}^1_{\alpha+2,\kappa+2}$ such that $\|\hat{\sigma}\|_{\mathcal{U}^1_{\alpha+2,\kappa+2}} < \varepsilon$. We then take $\varphi \in C^2(\mathbb{R}^2; \mathbb{R})$ such that

$$\|\hat{\varphi}\|_{\mathcal{U}^1_{\alpha+2,\kappa+2}} < \varepsilon. \tag{20}$$

We note here that by continuity of φ at $r = 0$, it holds automatically that $\varphi_n(0) = 0$ for $n \in \mathbb{Z} \backslash \{0\}$. Therefore, $\tilde{\varphi}$ automatically satisfies the condition of $\mathcal{U}^1_{\alpha+2,\kappa+2}$ at $r = 0$. After that, we define the approximative sequence $(\hat{w}^{(j)})_{j\in\mathbb{N}}$ to the solution of the equation

$$\hat{w} = \mathcal{S}(\hat{w}, \hat{\varphi}) \tag{21}$$

as

$$\begin{cases} \hat{w}^{(1)} := \hat{\Phi}(\hat{\varphi}), \\ \hat{w}^{(j)} := \mathcal{S}(\hat{w}^{(j-1)}, \hat{\varphi}), \quad j \geq 2, \end{cases}$$

where $\hat{\Phi}(\hat{\varphi}) = (\Phi_n(\hat{\varphi}))_{n\in\mathbb{Z}}$ is such that

$$\Phi_n(\hat{\varphi}) := \begin{cases} \varphi_n, & n \in \mathbb{Z}\backslash\{\pm 2\}, \\ \mathcal{Q}_{n,1}[\varphi_n], & n = \pm 2, \ 0 \leq r \leq R_*, \\ \mathcal{Q}_{n,2}[\varphi_n], & n = \pm 2, \ r > R_*. \end{cases}$$

Let $M > 0$ be as

$$M := \frac{1 - K_2\nu_* - \sqrt{(1 - K_2\nu_*)^2 - 4K_1K_3\|\hat{\varphi}\|_{\mathcal{U}^1_{\alpha+2,\kappa+2}}}}{2K_1}.$$

Then we see that

$$\|\hat{w}^{(1)}\|_{\mathcal{U}^1_{\alpha+2,\kappa+2}} \leq K_3\|\hat{\varphi}\|_{\mathcal{U}^1_{\alpha+2,\kappa+2}} \leq M,$$

and if $\|\hat{w}^{(j)}\|_{\mathcal{U}^1_{\alpha+2,\kappa+2}} \leq M$ for some $j \in \mathbb{N} \cup \{0\}$, then

$$\|\hat{w}^{(j+1)}\|_{\mathcal{U}^1_{\alpha+2,\kappa+2}} \leq K_1M^2 + K_2\nu_*M + K_3\|\hat{\sigma}\|_{\mathcal{U}^1_{\alpha+2,\kappa+2}} = M.$$

Therefore, by induction, we see that the sequence $(\|\hat{w}^{(j)}\|_{\mathcal{U}^1_{\alpha+2,\kappa+2}})_{j\in\mathbb{N}}$ is uniformly bounded by M. Moreover, we have

$$\begin{aligned} \|\hat{w}^{(j+1)} - \hat{w}^{(j)}\|_{\mathcal{U}^1_{\alpha+2,\kappa+2}} &\leq \|\mathcal{S}(\hat{w}^{(j)}, \hat{\varphi}) - \mathcal{S}(\hat{w}^{(j-1)}, \hat{\varphi})\|_{\mathcal{U}^1_{\alpha+2,\kappa+2}} \\ &\leq (2K_1M + K_2\nu_*)\|\hat{w}^{(j)} - \hat{w}^{(j-1)}\|_{\mathcal{U}^1_{\alpha+2,\kappa+2}} \\ &\leq (2K_1M + K_2\nu_*)^{j-1}\|\mathcal{S}(\hat{\Phi}(\hat{\varphi}), \hat{\varphi}) - \hat{\Phi}(\hat{\varphi})\|_{\mathcal{U}^1_{\alpha+2,\kappa+2}} \\ &\leq (2K_1M + K_2\nu_*)^jM. \end{aligned}$$

Since $2K_1M + K_2\nu_* < 1$, we see

$$\sum_{j=1}^{\infty} \|\hat{w}^{(j+1)} - \hat{w}^{(j)}\|_{\mathcal{U}^1_{\alpha+2,\kappa+2}} < \infty,$$

which and the completeness of $\mathcal{U}^1_{\alpha+2,\kappa+2}$ yield that $\hat{w}^{(j)}$ converges to some $\hat{w}^\infty \in \mathcal{U}^1_{\alpha+2,\kappa+2}$ under the condition (20). Since

$$\|\mathcal{S}(\hat{w}^\infty, \hat{\varphi}) - \mathcal{S}(\hat{w}^{(j)}, \hat{\varphi})\|_{\mathcal{U}^1_{\alpha+2,\kappa+2}} \leq (2K_1M - K_2\nu_*)\|\hat{w}^\infty - \hat{w}^{(j)}\|_{\mathcal{U}^1_{\alpha+2,\kappa+2}} \to 0$$

as $j \to \infty$, we see that $\hat{w}^\infty$ is a solution of (21).

In what follows, we fix a pair $(\hat{w}, \hat{\varphi}) \in \mathcal{U}^1_{\alpha+2,\kappa+2} \times \mathcal{U}^1_{\alpha+2,\kappa+2}$ of solutions obtained as above. Then this pair satisfies

$$w_n(r) := \begin{cases} -I^{\infty}_{|n|}[\mathcal{G}^*_n](r) - J^0_{|n|}[\mathcal{G}^*_n](r) + \varphi_n(r), & r \geq 0,\ n \in \mathbb{Z}\backslash\{0, \pm 2\}, \\ w_{n,1}(r) := \mathcal{P}_{n,1}(r) + \mathcal{Q}_{n,1}[\varphi_n](r) & 0 \leq r \leq R_*,\ n = \pm 2, \\ w_{n,2}(r) := \mathcal{P}_{n,2}(r) + \mathcal{Q}_{n,2}[\varphi_n](r) & r > R_*,\ n = \pm 2, \\ -\int_r^{\infty} \frac{1}{s} \int_0^s t\mathcal{G}^*_0(t)dtds + \varphi_n(r), & r \geq 0,\ n = 0, \end{cases}$$

together with $\hat{\gamma} := \mathcal{L}(\hat{w}) \in \mathcal{V}^2_{\alpha,\kappa+4}$. Since $\hat{w} \in \mathcal{U}^1_{\alpha+2,\kappa+2}$, $\hat{\varphi} \in \hat{C}^2$, and since $\mathcal{G}^*_n = \mathcal{H}_n$ is finite at $r = 0$, we see that each of $\Delta_{r,n} w_n$ ($|n| \neq 2$), $\Delta_{r,n} w_{n,1}$ and $\Delta_{r,\zeta_n} w_{n,2}$ ($|n| = 2$) is well-defined and continuous in each domain, and is expressed as

$$\Delta_{r,n} w_n(r) = \mathcal{G}^*_n(r) + \Delta_{r,n}\varphi_n(r) \quad , \quad 0 \leq r < \infty,\ n \in \mathbb{Z}\backslash\{\pm 2\}, \tag{22}$$

$$\Delta_{r,n} w_{n,1}(r) = \mathcal{G}^*_n(r) + \Delta_{r,n}\varphi_n(r)\ , \quad 0 \leq r \leq R_*,\ |n| = 2, \tag{23}$$

$$\Delta_{r,\zeta_n} w_{n,2}(r) = \mathcal{G}^0_n(r) + \Delta_{r,n}\varphi_n(r), \quad R_* < r < \infty,\ |n| = 2. \tag{24}$$

From (23), (24), and the connecting properties (18) and (19), we see

$$\partial^l_r w_{n,1}(R_*) = \lim_{r \to R_* - 0} \partial^l_r w_{n,2}(r),\ \ l = 0, 1, 2,$$

and hence each of $\Delta_{r,n} w_{\pm 2}$ is also well-defined and continuous in $[0, \infty)$, and satisfies (22). Therefore, we see that $(\hat{\gamma}, \hat{w}) = (\mathcal{L}(\hat{w}), \hat{w})$ are strong solutions of the system (9).

Using the above solutions $(\hat{\gamma}, \hat{w})$ of (9), we set

$$\gamma(r, \theta) := \sum_{n \in \mathbb{Z}} \gamma_n(r) e^{in\theta}, \quad w(r, \theta) := \sum_{n \in \mathbb{Z}} w_n(r) e^{in\theta}$$

for every $(r, \theta) \in [0, \infty) \times [0, 2\pi)$. By the definition of $\mathcal{V}^2_{\alpha,\kappa+4}$ and $\mathcal{U}^1_{\alpha+2,\kappa+2}$, and the discussion in Step 2, we see $\gamma \in C^2(\mathbb{R}^2; \mathbb{R})$ and $w \in C^2(\mathbb{R}^2; \mathbb{R})$. Moreover, by summing up Fourier modes of the system (9), we also see that (ψ, ω) defined by

$$\psi(r, \theta) := \psi_*(r) + \gamma(r, \theta),\ \ \omega(r, \theta) := \omega_*(r) + w(r, \theta).$$

are strong solutions of the vorticity-streamfunction system (5), i.e., (2) for $\phi = \phi_* + \varphi$. We note here that by the elliptic regularity of the Poisson equation, we see that $\psi \in C^3(\mathbb{R}^2, \mathbb{R})$.

Now let $u := \nabla^{\perp}\psi$. Then we see $u \in C^2(\mathbb{R}^2, \mathbb{R})$. On the other hand, $\partial_r \gamma_0$ is actually expressed as

$$\begin{aligned} \partial_r \gamma_0(r) &= -\frac{1}{r} \int_0^r \left[-\mathcal{D}_0(s) + 2s \int_s^{\infty} \frac{\mathcal{D}_0(t)}{t^2} dt + s\varphi_0(s) \right] ds \\ &= -r \int_r^{\infty} \frac{\mathcal{D}_0(s)}{s^2} ds - \frac{1}{r} \int_0^r s\varphi_0(s) ds. \end{aligned}$$

We can easily check that the modulus of this first term is bounded by $(1+r)^{-(\alpha+1)}$. Hence together with $\tilde{\gamma} \in \tilde{V}^2_{\alpha,\kappa+4}$, we obtain the decay property (4) of u by using the formula $\nabla^\perp(\phi_* + \varphi_0) = -\partial_r(\phi_* + \varphi_0)e_\theta$. Furthermore, this u and ω satisfy

$$\begin{cases} \Delta\omega = u \cdot \nabla\omega + \Delta(\phi_* + \varphi), & x \in \mathbb{R}^2, \\ \nabla \cdot u = 0 & x \in \mathbb{R}^2, \end{cases}$$

and the decay properties

$$|u(r,\theta)| \lesssim (1+r)^{-1}, \ |\nabla u(r,\theta)| \lesssim (1+r)^{-2}, \ |\nabla\omega(r,\theta)| \lesssim (1+r)^{-3}.$$

Hence using a similar method to Hillairet-Wittwer [5, Section 3], we see that u becomes a classical solution of the Navier-Stokes system (1) for $F = \nabla^\perp(\phi_* + \varphi)$ together with some pressure p.

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