Global well-posedness of compressible Navier-Stokes equations with BV data

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1 Introduction

1.1 Problems and background

This article is to summarize our recent work [12] joint with Shih-Hsien Yu and Xiongtao Zhang on the well-posedness theory of 1-D compressible Navier-Stokes equations for initial data having small total variations. It is an extension of [8] from isentropic gas to the full system including the energy conservation law. Consider 1-D compressible Navier-Stokes equations in Lagrangian coordinates,

$$\begin{cases} v_t - u_x = 0, \\ u_t + p_x = \Big(\dfrac{\mu u_x}{v}\Big)_x, \\ (e + \dfrac{1}{2}u^2)_t + (pu)_x = \Big(\dfrac{\kappa}{v}\theta_x + \dfrac{\mu}{v}uu_x\Big)_x. \end{cases} \tag{1}$$

Here t denotes time, x denotes the Lagrangian coordinate which labels the fluid element in terms of mass, v is the specific volume, u is the velocity, p is the pressure, e is the specific internal energy, θ is the temperature. μ and κ are the viscosity and heat conductivity coefficients, respectively, and are assumed to be positive constants. For the derivation of this system from the usual Eulerian form, one is referred to [11]. We consider the ideal gases

$$p(v,\theta) = \frac{K\theta}{v}, \qquad e = c_v\theta, \tag{2}$$

where K, and heat capacity c_v are both positive constants. System (1) can also be written in terms of (v,u,θ):

$$\begin{cases} v_t - u_x = 0, \\ u_t + p_x = \Big(\dfrac{\mu u_x}{v}\Big)_x, \\ \theta_t + \dfrac{p}{c_v}u_x - \dfrac{\mu}{c_v v}(u_x)^2 = \Big(\dfrac{\kappa}{c_v v}\theta_x\Big)_x. \end{cases} \tag{3}$$

When the motion is adiabatic, one has the isentropic gas, and the pressure p is a function of v only, $p = v^{-\gamma}$, where γ is the adiabatic exponent and $\gamma \geq 1$ for usual medium. In this case, the first two equations in (1) form a closed system, called p-system,

$$\begin{cases} v_t - u_x = 0, \\ u_t + p_x = \left(\dfrac{\mu u_x}{v}\right)_x. \end{cases} \tag{4}$$

The main purpose of this article is to study the well-posedness and time asymptotic behavior of system (3) with initial data being a rough perturbation around a constant state. Without loss of generality, the constant state is assumed to be $(v, u, \theta) = (1, 0, 1)$. The initial data $(v, u, \theta)(x, 0) = (v_0, u_0, \theta_0)(x)$ is given to satisfy

$$\|v_0 - 1\|_{L^1_x} + \|v_0\|_{BV} + \|u_0\|_{L^1_x} + \|u_0\|_{BV} + \|\theta_0 - 1\|_{L^1_x} + \|\theta_0\|_{BV} < \delta \ll 1, \tag{5}$$

where L^1_x denotes the L^1 norm in the space variable x, and $\|\cdot\|_{BV}$ denotes the total variation,

$$\|f\|_{BV} \equiv \sup_{\mathscr{P}\text{: partition of } \mathbb{R}} \sum_{x_i \in \mathscr{P}} |f(x_i) - f(x_{i-1})|. \tag{6}$$

The well-posedness theories for compressible Navier-Stokes equations are established by Nash [10] and Itaya [4] for Hölder continuous initial data. The equations are rewritten into a nonlinear parabolic system for velocity and temperature, where the fundamental solution for variable coefficient parabolic system played a key role. For Hölder data, the classical frozen coefficient and parametrix method are sufficient for the construction of fundamental solution. Based on the local existence, Kanel [5] and Kazhikhov-Shelukin [6] derived a priori energy-type estimate and thus obtained the global solution for 1-D system. The energy method is then generalized to compressible Navier-Stokes equations in 3-D to obtain the global existence by Matsumura and Nishida [9] for initial data in high order Sobolev's space, where a local existence theory is also provided based on estimates of constant coefficient linear parabolic system.

On the other hand, the quasi-linear and hyperbolic-parabolic nature of (1) allows the initial discontinuities in specific volume v to propagate in later time. Nash and Itaya's theory are not applicable due to the coefficients in equations of u and θ cease to be Hölder continuous. Later, the construction of weak solution are studied by Hoff [2, 3], Lions [7] and Feireisl [1], etc. The piecewise energy estimate is carried out and total variation estimate is obtained by Hoff [2]. The construction in [7] and [1] are applicable to more general data, for example, in the presence of vacuum. There is no well-posedness theory for the weak solutions obtained by these approaches.

Liu-Yu [8] initiated a new approach to establish weak solutions in a constructive way and obtained the well-posedness theory as well as properties of the solution for the isentropic Navier-Stokes equations (4). It is based on the construction of the fundamental solution to the heat equation with BV coefficient:

$$\begin{cases} (\partial_t - \partial_x \rho(x,t) \partial_x) \mathsf{H}(x, t; y; \mu) = 0, \\ \mathsf{H}(x, 0; y; \mu) = \delta(x - y). \end{cases}$$

With sharp estimates of heat kernel, the classical iteration scheme [4] originally for Hölder coefficient can be applied to BV coefficient, thus establishing the constructive proof of local well-posedness theory for isentropic Navier-Stokes (4). Precisely, the following Theorem is obatined in [8]:

Theorem 1 ([8]) *Suppose that the initial data for* (4) *satisfies*

$$\|v_0 - 1\|_{L^1} + \|v_0\|_{BV} + \|u_0\|_{L^1} + \|u_0\|_{BV} \le \delta \tag{7}$$

for $\delta \ll 1$. *Then there exist* $t_\sharp > 0$ *and* $C_\sharp > 0$ *such that the weak solution* (v,u) *for* (4) *exists for* $t < t_\sharp$ *and satisfying for* $t \in (0, t_\sharp)$,

$$\|v(\cdot,t) - 1\|_{L^1} + \|v(\cdot,t)\|_{BV} + \|\sqrt{t} u_x(\cdot,t)\|_{L^\infty} \le 2C_\sharp \delta.$$

Moreover, suppose that two initial data (v_0^a, u_0^a) *and* (v_0^b, u_0^b) *satisfy* (7). *Let* (v^a, u^a) *and* (v^b, u^b) *be two weak solutions of the isentropic Navier-Stokes equations* (4) *constructed as above, then*

$$\sup_{0<\sigma<t_\sharp} \Big(\|v^a(\cdot,\sigma) - v^b(\cdot,\sigma)\|_{L^1} + \|u^a(\cdot,\sigma) - u^b(\cdot,\sigma)\|_{L^1} \Big) < C_\sharp \Big(\|v_0^a - v_0^b\|_{L^1} + \|u_0^a - u_0^b\|_{L^1} \Big).$$

The heat kernel with BV coefficient plays a crucial role in local existence since it accurately captures the quasi-linear nature of the system (4). Concerning large time behavior, the hyperbolic-parabolic structure is important, which is, however, not reflected in heat kernel. A new ingredient, Green's function for linearized system around constant state, is introduced. An interpolation of BV heat kernel in short time and Green's function in long time, yields an "effective Green's function", which respects both quasi-linear and dissipative structure of the system. With this, one represents the weak solution of equation (4) in terms of an integral form, then a priori estimate concludes the time asymptotic behavior of the solution:

Theorem 2 ([8]) *Consider the initial value problem for Navier-Stokes equations* (4) *for the polytropic gases* $p(v) = Av^{-\gamma}$, $1 \le \gamma < e$. *Suppose that the initial data* $(v_0(x), u_0(x))$ *satisfies* (7). *Then, there is a positive constant* C, *such that for sufficiently small* δ, *the solution exists global-in-time and satisfies*

$$\begin{aligned} &\|(v-1)(\cdot,t)\|_{L^1} + \|\sqrt{t+1}(v-1)(\cdot,t)\|_{L^\infty} + \|(v-1)(\cdot,t)\|_{BV} \\ &\quad + \|u(\cdot,t)\|_{L^1} + \|\sqrt{t+1}u(\cdot,t)\|_{L^\infty} + \|\sqrt{t}u_x(\cdot,t)\|_{L^\infty} + \|u(\cdot,t)\|_{BV} < C\delta, \qquad t > 0. \end{aligned}$$

1.2 New difficulties and novelties

We now turn to the full Navier-Stokes equations (1). Compared to isentropic gas (4), there are several new difficulties for (1).

The noteworthy one is the **regularity issue**. The second equation in (4) is a diffusion equation for u. As the equation is given in a conservative form, when using Green's function as a test function, the weak formulation automatically yields an integral representation of (v,u), which are convenient for transferring the derivative between Green's function and nonlinear source term, and thus for investigating the time-asymptotic behaviors.

However, for full system (1), it is a problem to choose whether $(v, u, e + u^2/2)$ or (v, u, θ) as unknown functions. Considering the diffusion term in the third equation of (3), temperature θ would be a good candidate, while a non-conservative form is not convenient for studying time-asymptotic behaviors. If the solution is only constructed in distribution sense, one does not have equivalence between (1) and (3), and there is a gap between local theory and global existence.

The problem is resolved by a careful investigation on the regularity of the weak solution for (3). We develop some new estimates for heat kernel and show that θ is Hölder continuos in time, which helps us to prove u_t is in $L^\infty \cap L^1$. Interestingly, this in turn improves θ from Hölder continuity to differentiable in time. With this regularity, the function $(v, u, c_v\theta + u^2/2)$ is a weak solution to conservative form (1). This serves as a basis towards the global stability. It is also worth mentioning that Hölder-type estimates are crucial even in the construction of weak solution for (3) due to the pressure term $p(v, \theta)$, unlike isentropic gas, which is not needed.

Another novelty of this article is the **uniqueness** of the solution. In Theorem 1.2 of [8], the authors proved that the constructed weak solution for isentropic model depends on initial data continuously. In this article, from the regularity result, we identify the function space of the constructed weak solution to (3), and prove **stability** of the solution in this function space, which in turn yields that given any weak solution in distribution sense, it must be identical to the one we constructed as long as it belongs to the aforementioned space.

These results largely rely on various quantitative estimates of fundamental solution for heat equation with BV variable coefficient, which captures the quasi-linear structure of the equation (3), and represents the solution accurately.

The analysis for global existence is done in a similar framework as that of [8]. One follows its procedure to replace the Green's function for a linearized 2×2 system by a 3×3 system to build "an effective Green's function", derives an integral representation, and performs a priori estimates to conclude time asymptotic behavior. The regularity in time of velocity u and temperature θ also plays a role in the a priori estimate.

1.3 Main results

Our main results for full Navier-Stokes equations (3) are stated as follows:

Theorem 3 (Local existence and regularity, [12], Theorem 1.1) *Suppose that the initial data for* (3) *satisfies* (5)*. Then, there exist positive constants* $t_\sharp$ *and* $C_\sharp$ *such that* *the system* (3) *admits a weak solution* (v,u,θ) *for* $t\in(0,t_\sharp)$ *satisfying*

$$\begin{cases}\max\left\{\begin{aligned}&\|u(\cdot,t)\|_{L^1_x},\ \|u(\cdot,t)\|_{L^\infty_x},\ \|u_x(\cdot,t)\|_{L^1_x},\ \sqrt{t}\,\|u_x(\cdot,t)\|_{L^\infty_x},\\&\sqrt{t}\,\|u_t(\cdot,t)\|_{L^1_x},\ t\,\|u_t(\cdot,t)\|_{L^\infty_x}\end{aligned}\right\}\le 2C_\sharp\delta,\\ \max\left\{\begin{aligned}&\|\theta(\cdot,t)-1\|_{L^1_x},\ \|\theta(\cdot,t)-1\|_{L^\infty_x},\ \|\theta_x(\cdot,t)\|_{L^1_x},\ \sqrt{t}\,\|\theta_x(\cdot,t)\|_{L^\infty_x},\\&\sqrt{t}\,\|\theta_t(\cdot,t)\|_{L^1_x},\ t\,\|\theta_t(\cdot,t)\|_{L^\infty_x}\end{aligned}\right\}\le 2C_\sharp\delta,\\ \max\left\{\|v(\cdot,t)\|_{BV},\ \|v(\cdot,t)-1\|_{L^1_x},\ \|v(\cdot,t)-1\|_{L^\infty_x},\ \sqrt{t}\,\|v_t(\cdot,t)\|_{L^\infty_x}\right\}\le 2C_\sharp\delta,\\ v-1=v_c^*+v_d^*,\quad v_d^*(x,t)=\sum_{z<x,z\in\mathscr{D}}[v](z)h(x-z),\quad v_c^* \text{ is continuous},\\ \left|v(\cdot,t)\Big|_{x=z^-}^{x=z^+}\right|\le 2\left|v_0^*(\cdot)\Big|_{x=z^-}^{x=z^+}\right|,\quad z\in\mathscr{D},\end{cases}$$

where $h(x)$ *is the Heaviside step function,* $\mathscr{D}$ *is the discontinuity set of* v_0*; Moreover, the fluxes of* u *and* θ*, (i.e.* $\frac{\mu u_x}{v}-p$ *and* $\frac{\kappa}{c_v v}\theta_x-\int_{-\infty}^x\left(\frac{p}{c_v}u_z-\frac{\mu}{c_v v}(u_z)^2\right)dz$ *), are both globally Lipschitz continuous with respect to* x *for any* $t>0$*; and the specific volume* $v(x,t)$ *has the following Hölder continuous properties in time for* $0\le s<t$*,*

$$\begin{cases}\|v(\cdot,t)-v(\cdot,s)\|_{BV}\le 2C_\sharp\delta\dfrac{(t-s)|\log(t-s)|}{\sqrt{t}},\\ \|v(\cdot,t)-v(\cdot,s)\|_{L^\infty}\le 2C_\sharp\delta\dfrac{t-s}{\sqrt{t}},\\ \|v(\cdot,t)-v(\cdot,s)\|_{L^1}\le 2C_\sharp\delta(t-s).\end{cases}$$

Theorem 4 (Stability and uniqueness, [12], Theorem 1.2) *Suppose there are two weak solutions* (v^a,u^a,θ^a) *and* (v^b,u^b,θ^b) *to the Navier-Stokes equations* (3) *with the regularity properties stated in* (29)*, and for a small* δ_* *their initial data both satisfy*

$$\|v_0\|_{BV}+\|u_0\|_{BV}+\|\theta_0\|_{BV}+\|v_0-1\|_{L^1_x}+\|u_0\|_{L^1_x}+\|\theta_0-1\|_{L^1_x}<\delta_*.$$

Then, there exist $t_*>0$ *and* $C_\flat>0$ *such that for* $0<t<t_*$*,*

$$\begin{aligned}\|v^a-v^b\|_{L^1_x}+\|u^a-u^b\|_{L^1_x}+\|\theta^a-\theta^b\|_{L^1_x}\le C_\flat\Big(&\|\theta^a_0-\theta^b_0\|_{L^\infty_x}+\|\theta^a_0-\theta^b_0\|_{L^1_x}\\&+\|u^a_0-u^b_0\|_{L^\infty_x}+\|u^a_0-u^b_0\|_{L^1_x}+\|v^a_0-v^b_0\|_{L^1_x}+\|v^a_0-v^b_0\|_{L^\infty_x}+\|v^a_0-v^b_0\|_{BV}\Big).\end{aligned}$$

Theorem 5 (Global existence, [12], Theorem 1.3) *There exist* $\delta^*>0$ *and* $\mathscr{C}>0$ *so that for any initial data* (v_0,u_0,θ_0) *of* (3) *satisfying*

$$\|v_0-1\|_{L^1_x}+\|v_0\|_{BV}+\|u_0\|_{L^1_x}+\|u_0\|_{BV}+\|\theta_0-1\|_{L^1_x}+\|\theta_0\|_{BV}\le\varepsilon<\delta^*,$$

the solution constructed in Theorem 3 satisfies

$$\begin{aligned}\left\|\sqrt{t+1}(v(\cdot,t)-1)\right\|_{L^\infty_x}+\left\|\sqrt{t+1}u(\cdot,t)\right\|_{L^\infty_x}+\left\|\sqrt{t+1}(\theta(\cdot,t)-1)\right\|_{L^\infty_x}\\+\left\|\sqrt{t}u_x(\cdot,t)\right\|_{L^\infty_x}+\left\|\sqrt{t}\theta_x(\cdot,t)\right\|_{L^\infty_x}\le\mathscr{C}\varepsilon\quad\text{for } t\in(0,+\infty).\end{aligned}$$

1.4 Organization

The rest of this paper is organized as follows. In Section 2, we demonstrate the main ideas and steps for constructing BV coefficient heat kernel. In Section 3, we construct local solution using heat kernel and prove its regularity and stability. In Section 4, we give the pointwise estimate of Green's function for system (1) linearized around constant state $(1, 0, 1)$. In Section 5, we represent the solution in terms of an integral equation by using "effective Green's function", and derive a priori estimate to conclude global well-posedness. In last section, we give an outlook for the possible future development of our approach.

2 Heat kernel for BV coefficient

We will demonstrate the main steps for construction of heat kernel with BV coefficient. This section is mainly based on [8], and some estimates are from [12].

Consider the following equation for heat kernel H with the coefficient $\rho(x,t)$ being a BV function with respect to x,

$$\begin{cases} (\partial_t - \partial_x \rho(x,t)\partial_x)\, H(x,t;y,t_0;\rho) = 0, \quad t > t_0, \\ H(x,t_0;y,t_0;\rho) = \delta(x-y), \end{cases}$$

where the BV coefficient $\rho(x,t)$ satisfies the following properties,

$$\begin{cases} \|\rho(\cdot) - \bar{\rho}\|_{L^1} \le \delta_*, \quad \|\rho(\cdot,t)\|_{BV} \le \delta_*, \quad \|\rho_t(\cdot,t)\|_\infty \le \delta_* \max\left(\dfrac{1}{\sqrt{t}}, 1\right), \quad 0 < \delta_* \ll 1, \\ \mathscr{D} \equiv \{z \mid \rho(z,t) \text{ is not continuous at } z\} \text{ is invariant in } t. \end{cases} \tag{8}$$

Here $\rho(x,t)$ plays the role as $1/v(x,t)$ in the Navier-Stokes equations (3).

To construct $H(x,t;y,t_0;\rho)$, the strategy is as follows: we first treat the case that ρ is a step function in space and independent of time; then we use step function to approximate a general BV function (still time-independent); lastly, we use time-independent solution and time-frozen technique to construct the heat kernel for time-dependent BV coefficient.

2.1 Step function coefficient

One considers

$$\begin{cases} (\partial_t - \partial_x \mu(x)\partial_x)\, H(x,t;y,t_0;\mu) = 0, \quad t > t_0, \\ H(x,t_0;y,t_0;\mu) = \delta(x-y), \end{cases} \tag{9}$$

where $\mu(x)$ is a step function satisfying the following properties,

$$\begin{cases} \mathscr{D} \equiv \{z |\, [\mu](z) \neq 0\} = \{x_i | i \in \mathbb{Z}\}, \quad [\mu](z) \equiv \mu(z+0) - \mu(z-0), \\ \mu(z) = \kappa_i, \quad z \in (x_i, x_{i+1}), \quad i \in \mathbb{Z}, \\ \lim\limits_{i\to\pm\infty} x_i = \pm\infty, \quad \inf\{|x_{i+1} - x_i|, i \in \mathbb{Z}\} > \delta_*, \\ \|\mu\|_{BV} = \sum\limits_{z\in\mathscr{D}} |[\mu(z)]|. \end{cases}$$

Since the equation is time homogeneous, the solution depends only on the time difference $t - t_0$, so we may write the heat kernel as $H(x, t; y; \mu)$. It is easy to check $H(x, t; y; \mu)$ is a weak solution of (9) if and only if $(\partial_t - \partial_x \mu(x) \partial_x) H(x, t; y, \mu) = 0$ at any continuity point x of $\mu(x)$, and

$$H(x, t; y, \mu), \qquad \mu(x)\partial_x H(x, t; y, \mu) \tag{10}$$

are continuous across any jump point x of $\mu(x)$.

Assume that $y \in (x_0, x_1)$ and $x \in (x_j, x_{j+1})$. Take Laplace transform of (9) with respect to time,

$$\mathcal{L}[H](x, s; y, \mu) \equiv \int_0^\infty e^{-st} H(x, t; y, \mu) dt,$$

one has

$$(s - \partial_x \mu(x) \partial_x) \mathcal{L}[H](x, s; y, \mu) = \delta(x - y).$$

In particular, for $x \in (x_j, x_{j+1})$,

$$(s - \kappa_j \partial_x^2) \mathcal{L}[H](x, s; y, \mu) = \delta_{j0} \delta(x - y),$$

where δ_{ij} is the Kronecker symbol, $\delta_{ij} = 1$ when $i = j$ and $\delta_{ij} = 0$ when $i \neq j$. Therefore, we have, for $x \in (x_j, x_{j+1})$,

$$\mathcal{L}[H](x, s; y, \mu) = e^{\sqrt{s/\kappa_j}(x - x_{j+1})} \mathsf{U}_{j+1}(y, s) + e^{-\sqrt{s/\kappa_j}(x - x_j)} \mathsf{S}_j(y, s) + \frac{\delta_{j0}}{2\sqrt{\kappa_0 s}} e^{-\sqrt{s/\kappa_0}|x - y|}, \tag{11}$$

for some functions U_{j+1} and S_j. The continuity condition (10) at $x = x_j$ implies that the coefficients U_j and S_j satisfy

$$\begin{cases} e^{-\sqrt{s}\Delta_{j+1}} \mathsf{U}_{j+1} + \mathsf{S}_j + \dfrac{\delta_{j0}}{2\sqrt{\kappa_0 s}} e^{-\sqrt{s/\kappa_0}|x_0 - y|} = \mathsf{U}_j + e^{-\sqrt{s}\Delta_j} \mathsf{S}_{j-1} + \dfrac{\delta_{j1}}{2\sqrt{\kappa_0 s}} e^{-\sqrt{s/\kappa_0}|x_1 - y|}, \\ \sqrt{\kappa_j} \left(e^{-\sqrt{s}\Delta_{j+1}} \mathsf{U}_{j+1} - \mathsf{S}_j + \dfrac{\delta_{j0}}{2\sqrt{\kappa_0 s}} e^{-\sqrt{s/\kappa_0}|x_0 - y|} \right) = \sqrt{\kappa_{j-1}} \left(\mathsf{U}_j - e^{-\sqrt{s}\Delta_j} \mathsf{S}_{j-1} - \dfrac{\delta_{j1}}{2\sqrt{\kappa_0 s}} e^{-\sqrt{s/\kappa_0}|x_1 - y|} \right), \\ \Delta_j \equiv \dfrac{x_j - x_{j-1}}{\sqrt{\kappa_{j-1}}}. \end{cases}$$

The above identities can be written as

$$\begin{pmatrix} \mathsf{U}_j \\ \mathsf{S}_j \end{pmatrix} = e^{-\sqrt{s}\Delta_j} \mathbf{R}_j \begin{pmatrix} \mathsf{U}_{j-1} \\ \mathsf{S}_{j-1} \end{pmatrix} + e^{-\sqrt{s}\Delta_{j+1}} \mathbf{L}_j \begin{pmatrix} \mathsf{U}_{j+1} \\ \mathsf{S}_{j+1} \end{pmatrix} + \begin{pmatrix} \mathsf{U}_j^0 \\ \mathsf{S}_j^0 \end{pmatrix}, \tag{12}$$

where

$$\mathbf{R}_j = \begin{pmatrix} 0 & R_{--,j} \\ 0 & T_{-+,j} \end{pmatrix}, \quad R_{--,j} \equiv \frac{\sqrt{\kappa_{j-1}} - \sqrt{\kappa_j}}{\sqrt{\kappa_{j-1}} + \sqrt{\kappa_j}}, \quad T_{-+,j} = R_{--,j} + 1,$$

$$\mathbf{L}_j = \begin{pmatrix} T_{+-,j} & 0 \\ R_{++,j} & 0 \end{pmatrix}, \quad R_{++,j} \equiv -\frac{\sqrt{\kappa_{j-1}} - \sqrt{\kappa_j}}{\sqrt{\kappa_{j-1}} + \sqrt{\kappa_j}}, \quad T_{+-,j} = R_{++,j} + 1,$$

$$(\mathsf{U}_j^0, \mathsf{S}_j^0) = \begin{cases} \frac{e^{\sqrt{s/\kappa_0}(x_0 - y)}}{2\sqrt{\kappa_0 s}} (T_{+-,0}, R_{++,0}) & \text{if } j = 0, \\ \frac{e^{-\sqrt{s/\kappa_0}(x_1 - y)}}{2\sqrt{\kappa_0 s}} (R_{--,1}, T_{-+,1}) & \text{if } j = 1, \\ (0, 0) & \text{if } j \neq 0, 1. \end{cases}$$

Here R_* and T_* can be viewed as reflection and transmission coefficients at the jumps. This yields an infinitely many algebraic equations for $(\mathsf{U}_j, \mathsf{S}_j), j \in \mathbb{Z}$, which is difficult to solve directly. We construct $(\mathsf{U}_j, \mathsf{S}_j)$ by recurrence,

$$\begin{pmatrix} \mathsf{U}_j^{n+1} \\ \mathsf{S}_j^{n+1} \end{pmatrix} = e^{-\sqrt{s}\Delta_j} \mathbf{R}_j \begin{pmatrix} \mathsf{U}_{j-1}^{n} \\ \mathsf{S}_{j-1}^{n} \end{pmatrix} + e^{-\sqrt{s}\Delta_{j+1}} \mathbf{L}_j \begin{pmatrix} \mathsf{U}_{j+1}^{n} \\ \mathsf{S}_{j+1}^{n} \end{pmatrix}.$$

Then $\mathsf{U}_j = \sum_{n=0}^{\infty} \mathsf{U}_j^n$ and $\mathsf{S}_j = \sum_{n=0}^{\infty} \mathsf{S}_j^n$ give the solution to (12) whenever they are convergent. This recurrence relation looks like a random walk starting from x_0 and x_1. Plugging them into (11), we obtain

$$\mathcal{L}[H](x, s; y, \mu) = \sum_{\gamma \in \Omega_{y \to x}} \frac{\mathbf{m}[\gamma] e^{-\sqrt{s}\mathbf{L}[\gamma]}}{2\sqrt{s\kappa_0}}. \tag{13}$$

Here $\Omega_{y \to x}$ denotes all the discrete paths starting from y and ending at x:

$$\Omega_{y \to x} \equiv \left\{ \gamma = \gamma(\tau) \,\middle|\, \gamma \text{ a continuous path } y \to x, \text{ and } |\gamma'(\tau)| = 1 \text{ when } \gamma(\tau) \text{ is not in } \mathscr{D} \right\},$$

where $\mathscr{D}$ is the discontinuity set of $\mu(x)$. For each $\gamma \in \Omega_{y \to x}$, there exists $0 = \tau_0 < \tau_1 < \tau_2 < \cdots < \tau_m < \tau_{m+1}$ such that

$$\begin{cases} \gamma(0) = y, \ \gamma(\tau_1) = \gamma_1 \in \mathscr{D}, \ \cdots \ \gamma(\tau_m) = \gamma_m \in \mathscr{D}, \ \gamma(\tau_{m+1}) = x, \\ \gamma((\tau_i, \tau_{i+1})) \cap \mathscr{D} = \emptyset \text{ for } i = 0, 1, \cdots, m. \end{cases} \tag{14}$$

For a path γ given by (14), the phase-length of path $\mathbf{L}[\gamma]$ is defined as

$$\mathbf{L}[\gamma] = \frac{|y - \gamma_1|}{\sqrt{\mu(y)}} + \frac{|\gamma_2 - \gamma_1|}{\sqrt{\mu(\gamma(\tau_1^+))}} + \cdots + \frac{|\gamma_m - \gamma_{m-1}|}{\sqrt{\mu(\gamma(\tau_{m-1}^+))}} + \frac{|x - \gamma_m|}{\sqrt{\mu(x)}}. \tag{15}$$

For γ given by (14), at each γ_i, $i = 1 \ldots m$, there occurs reflection and transmission, and we define

$$d_i \equiv \begin{cases} R_{--}^{\gamma_i} & \text{if path } \gamma \text{ reflects at } \gamma_i \text{ from the left,} \\ R_{++}^{\gamma_i} & \text{if path } \gamma \text{ reflects at } \gamma_i \text{ from the right,} \\ T_{-+}^{\gamma_i} & \text{if path } \gamma \text{ passes at } \gamma_i \text{ from the left,} \\ T_{+-}^{\gamma_i} & \text{if path } \gamma \text{ passes at } \gamma_i \text{ from the right.} \end{cases}$$

The measure $\mathbf{m}[\gamma]$ is then defined by

$$\mathbf{m}[\gamma] \equiv \begin{cases} 1 & \text{if } m = 0, \\ \prod_{i=1}^{m} d_i & \text{otherwise.} \end{cases} \tag{16}$$

Using the fact of Laplace transform

$$\mathcal{L}_{s \to t}^{-1} \left[\frac{e^{-\sqrt{s}a}}{2\sqrt{s}} \right] = \frac{e^{-\frac{a^2}{4t}}}{\sqrt{4\pi t}},$$

one **formally** obtains from (13) that

$$H(x,t;y,\mu)=\sum_{\gamma\in\Omega_{y\to x}}\mathbf{m}[\gamma]\frac{e^{-\frac{\mathbf{L}[\gamma]^2}{4t}}}{\sqrt{4\pi\mu(y)t}}.$$

It is shown in [8] by some combinatorics argument that when $\|\mu\|_{BV}$ is sufficiently small, the series actually converges uniformly. Therefore, one can obtain the following estimates of heat kernel and its derivative from (13):

Proposition 6 ([8]) *When step function μ satisfies that*

$$\|\mu(x)-\bar{\mu}\|_\infty\ll 1,\ \textit{and}\ \left\|\mu\right\|_{BV}\ll 1 \tag{17}$$

the heat kernel for step function coefficients satisfies the following estimates: for all $x,y\in\mathbb{R}$

$$\begin{cases} H(x,t;y;\mu)=(1+O(1)\|\mu\|_{BV})\dfrac{e^{-\frac{\left(\int_y^x\frac{dz}{\sqrt{\mu(z)}}\right)^2}{4t}}}{\sqrt{4\pi t}}, \\ \left|\partial_x H(x,t;y;\mu)\right|,\left|\partial_y H(x,t;y;\mu)\right|=O(1)\dfrac{e^{-\frac{\left(\int_y^x\frac{dz}{\sqrt{\mu(z)}}\right)^2}{Dt}}}{t}, \\ \left|\partial_t H(x,t;y;\mu)\right|,\left|\partial_{xy} H(x,t;y;\mu)\right|=O(1)\dfrac{e^{-\frac{\left(\int_y^x\frac{dz}{\sqrt{\mu(z)}}\right)^2}{Dt}}}{t^{3/2}}, \\ \left|\partial_{tx} H(x,t;y;\mu)\right|,\left|\partial_{ty} H(x,t;y;\mu)\right|=O(1)\dfrac{e^{-\frac{\left(\int_y^x\frac{dz}{\sqrt{\mu(z)}}\right)^2}{Dt}}}{t^2}. \end{cases}$$

Given two steps function μ^a and μ^b satisfying (17), consider the following identities,

$$0=\int_0^t\int_{\mathbb{R}}H(x,t-\tau;z,\mu^a)\left(\partial_\tau-\partial_z\mu^b(z)\partial_z\right)H(z,\tau;y,\mu^b)dzd\tau.$$

Using integration by parts and heat kernel estimates, one obtains that

$$|H(x,t;y;\mu^a)-H(x,t;y;\mu^b)|\le O(1)\|\mu^a-\mu^b\|_\infty t^{-1/2}e^{\frac{\left(\int_y^x\frac{dz}{\sqrt{\mu^a(z)}\vee\sqrt{\mu^b(z)}}\right)^2}{5t}}. \tag{18}$$

The comparison estimates for derivatives are much more subtle, straightforward differentiating the integral equation will induce non-integrable time singularity. One has to do delicate estimate for each path in the summation on the Laplace level, then invert it to physical variable. See [8] for details.

2.2 Time-independent BV coefficient

Now consider coefficient $\mu(x)$ is a general BV function. The idea is to construct a sequence of step functions $\mu^k(x)$ to approximate $\mu(x)$ in the following sense

$$\|\mu^k\|_{BV}\le 2\|\mu\|_{BV},\quad \|\mu^k-\mu\|_\infty<2^{-k}\to 0,\ \text{as}\ k\to\infty.$$

For each step function $\mu^k(x)$, one can construct the heat kernel $H(x,t;y;\mu^k)$. Then it is shown from (18) that $\lim_{k\to\infty} H(x,t;y;\mu^k)$ exits, which gives a heat kernel for BV coefficient $\mu(x)$. Moreover, the derivatives estimates follow from taking difference estimates.

Proposition 7 ([8], Theorem 3.6) *Suppose* $\|\mu\|_{BV} \ll 1$ *and* $\inf_{z\in\mathbb{R}} \mu(z) > \underline{\mu} > 0$. *Let* μ^k *be the step functions constructed as above. Then*

$$H(x,t;y;\mu) \equiv \lim_{k\to\infty} H(x,t;y;\mu^k) \text{ exists.}$$

$H(x,t;y;\mu)$ *is a weak solution of*

$$\begin{cases} (\partial_t - \partial_x\mu(x)\partial_x)\, H(x,t;y;\mu) = 0, \quad t>0, \\ H(x,0;y;\mu) = \delta(x-y), \end{cases}$$

and satisfies similar estimates as in Proposition 6.

2.3 Time-dependent BV coefficient

Let $\rho(x,t)$ be a function satisfying

$$\begin{cases} \|\rho(\cdot)-\bar{\rho}\|_{L^1} \le \delta_*, \quad \|\rho(\cdot,t)\|_{BV} \le \delta_*, \quad \|\rho_t(\cdot,t)\|_\infty \le \delta_* \max\left(\frac{1}{\sqrt{t}},1\right), \quad 0<\delta_*\ll 1, \\ \mathscr{D} \equiv \{z \mid \rho(z,t) \text{ is not continuous at } z\} \text{ is invariant in } t. \end{cases} \tag{19}$$

We are now in the position to consider the Green's function $H(x,t;y,t_0;\rho)$ to the following equation,

$$\begin{cases} \partial_t H = \partial_x\left(\rho(x,t)\partial_x H\right),\ t>t_0, \\ H(x,t_0;y,t_0;\rho) = \delta(x-y). \end{cases} \tag{20}$$

To establish the estimate for $H(x,t;y,t_0;\rho)$, we shall represent it by an integral equation using time-independent coefficient problem. We denote $H(x,t;y,t_0;\rho)$ by $\bar{H}(x,t;y,t_0)$ for the simplicity of notation. For fixed $T>0$, set $\mu(x)\equiv\rho(x,T)$ and consider

$$0 = \int_{t_0}^t \int_{\mathbb{R}} H(x,t;z,\sigma;\mu)\left(\partial_\sigma \bar{H}(z,\sigma;y,t_0) - \partial_z\left(\rho(z,\sigma)\partial_z\bar{H}(z,\sigma;y,t_0)\right)\right) dz d\sigma.$$

By the fact that $H(x,t;z,\sigma;\mu)$ and $\mu(z)\partial_z H(x,t;z,\sigma;\mu)$ are continuous in z, one performs integration by parts to get the representation of $\bar{H}(x,t;y,t_0)$,

$$\bar{H}(x,t;y,t_0) = H(x,t;y,t_0;\mu) + \int_{t_0}^t \int_{\mathbb{R}} H_z(x,t;z,\sigma;\mu)\big(\rho(z,T)-\rho(z,\sigma)\big)\bar{H}_z(z,\sigma;y,t_0) dz d\sigma.$$

Differentiate with respect to x to yield the integral equation for $\bar{H}_x$,

$$\bar{H}_x(x,t;y,t_0) = H_x(x,t;y,t_0;\mu) + \int_{t_0}^t \int_{\mathbb{R}} H_{xz}(x,t;z,\sigma;\mu)\big(\rho(z,T)-\rho(z,\sigma)\big)\bar{H}_z(z,\sigma;y,t_0) dz d\sigma.$$

Setting $T=t$ to cancel the time singularity in H_{xz}, and solving the above integral equation, one has the estimate of $H_x(x,t;y,t_0;\rho)$. It then follows the estimate of $H(x,t;y,t_0;\rho)$. In a similar spirit, other derivative estimates of $H(x,t;y,t_0;\rho)$ can also be derived.

Proposition 8 ([8],[12]) *Suppose the conditions of ρ in* (19) *hold. Then, there exist positive constants C_* and $t_\sharp \ll 1$ such that the weak solution of* (20) *exists and satisfies the following estimates for $t \in (t_0, t_0 + t_\sharp)$*

$$\begin{cases} |H(x,t;y,t_0;\rho)| \le C_* \dfrac{e^{-\frac{(x-y)^2}{C_*(t-t_0)}}}{\sqrt{t-t_0}}, \\ |H_x(x,t;y,t_0;\rho)| + |H_y(x,t;y,\rho)| \le C_* \dfrac{e^{-\frac{(x-y)^2}{C_*(t-t_0)}}}{t-t_0}, \\ |H_{xy}(x,t;y,t_0;\rho)| + |H_t(x,t;y,t_0;\rho)| \le C_* \dfrac{e^{-\frac{(x-y)^2}{C_*(t-t_0)}}}{(t-t_0)^{\frac{3}{2}}}, \\ |H_{ty}(x,t;y,t_0;\rho)| \le C_* \dfrac{e^{-\frac{(x-y)^2}{C_*(t-t_0)}}}{(t-t_0)^2}. \end{cases}$$

In addition to the estimates of heat kernel $H(x,t;y,t_0;\rho)$ and its derivatives, we also need the comparison estimates of them for construction of local solution, which are followed by comparison estimates of time-independent problem and time-frozen techniques. For the details, see [12].

3 Local well-posedness

In this section, we will take advantage of time-dependent coefficient heat kernel $H(x,t;y,t_0;\rho)$ and Picard-type iteration scheme to construct the local solution and show its uniqueness.

3.1 Existence

Consider the following iteration scheme motivated by (3)

$$\begin{cases} V_t^{n+1} - U_x^{n+1} = 0, \\ U_t^{n+1} - \left(\dfrac{\mu U_x^{n+1}}{1+V^n} \right)_x = -p(1+V^n, 1+\Theta^n)_x, \\ \Theta_t^{n+1} - \left(\dfrac{\kappa \Theta_x^{n+1}}{c_v(1+V^n)} \right)_x = -\dfrac{p(1+V^n,1+\Theta^n)}{c_v} U_x^n + \dfrac{\mu}{c_v(1+V^n)} (U_x^n)^2, \\ (V^{n+1}, U^{n+1}, \Theta^{n+1})\Big|_{t=0} = (v_0^*, u_0^*, \theta_0^*) = (v_0 - 1, u_0, \theta_0 - 1), \\ (V^0, U^0, \Theta^0) = (0,0,0). \end{cases} \tag{21}$$

The last equality in (21) means that we choose the initial step to be the unperturbed constant state. Thanks to the BV coefficient heat kernel, we can apply Duhamel's principle to construct the weak solution $(V^{n+1}, U^{n+1}, \Theta^{n+1})$ to equation (21) as follows,

$$
\begin{aligned}
&U^{n+1}(x,t) = \int_{\mathbb{R}} H\left(x,t;y,0;\mu^n\right) u_0^*(y)dy + \int_0^t \int_{\mathbb{R}\setminus\mathscr{D}} H_y\left(x,t;y,s;\mu^n\right) p(1+Vnk, 1+\Theta^n)dyds,\\
&\Theta^{n+1}(x,t) = \int_{\mathbb{R}} H\left(x,t;y,0;\kappa^n\right) \theta_0^*(y)dy + \int_0^t \int_{\mathbb{R}} H\left(x,t;y,s;\kappa^n\right) \mathcal{N}_2^n(y,s)dyds,\\
&V^{n+1}(x,t) = v_0^*(x) + \int_0^t U_x^{n+1}(x,\tau)d\tau,
\end{aligned}
\tag{22}
$$

where

$$
\begin{aligned}
&\mu^n \equiv \frac{\mu}{1+V^n}, &&\mathcal{N}_1^n(x,t) \equiv -\partial_x p(1+V^n, 1+\Theta^n),\\
&\kappa^n \equiv \frac{\kappa}{c_v\left(1+V^n\right)}, &&\mathcal{N}_2^n(x,t) \equiv -\frac{p(1+V^n,1+\Theta^n)}{c_v} U_x^n + \frac{\mu}{c_v\left(1+V^n\right)}\left(U_x^n\right)^2.
\end{aligned}
\tag{23}
$$

The integral representations (22) yields a weak solution of (21).

By refined estimates of heat kernel, we can show that (V^n, U^n, Θ^n) forms a Cauchy sequence in an appropriate topology and conclude the following local existence result:

Theorem 9 ([12]) *Suppose the initial data (v_0, u_0, θ_0) satisfies the condition (5) for small δ. Then there exists a positive constant $t_\sharp$ such that, equation (3) admits a weak solution*

$$
(v,u,\theta) = (v^*+1, u^*, \theta^*+1), \quad t < t_\sharp,
$$

satisfying the following estimates

$$
\begin{cases}
\delta > 0, \quad 0 < t < t_\sharp \ll 1,\\
\max\left\{\|u(\cdot,t)\|_{L_x^1},\ \|u(\cdot,t)\|_{L_x^\infty},\ \|u_x(\cdot,t)\|_{L_x^1},\ \sqrt{t}\,\|u_x(\cdot,t)\|_{L_x^\infty}\right\} \le 2C_\sharp\delta,\\
\max\left\{\|\theta(\cdot,t)-1\|_{L_x^1},\ \|\theta(\cdot,t)-1\|_{L_x^\infty},\ \|\theta_x(\cdot,t)\|_{L_x^1},\ \sqrt{t}\,\|\theta_x(\cdot,t)\|_{L_x^\infty}\right\} \le 2C_\sharp\delta,\\
\max\left\{\|v(\cdot,t)\|_{BV},\ \|v(\cdot,t)-1\|_{L_x^1},\ \|v(\cdot,t)-1\|_{L_x^\infty},\ \sqrt{t}\,\|v_t(\cdot,t)\|_{L_x^\infty}\right\} \le 2C_\sharp\delta,\\
v^* = v_c^* + v_d^*, \quad v_d^*(x,t) = \sum_{z<x, z\in\mathscr{D}} v^*\Big|_{z^-}^{z^+} h(x-z), \quad v_c^* \text{ is continuous},\\
\left| v(\cdot,t)\Big|_{x=z^-}^{x=z^+}\right| \le 2\left| v_0^*(\cdot)\Big|_{x=z^-}^{x=z^+}\right|, \quad z\in\mathscr{D},
\end{cases}
\tag{24}
$$

for some positive constant $C_\sharp$, where $h(x)$ is the Heaviside step function. Moreover, the fluxes of u and θ

$$
\textit{flux of } u = \frac{\mu u_x}{v} - p, \qquad \textit{flux of } \theta = \frac{\kappa}{c_v v}\theta_x - \int_{-\infty}^x \left(\frac{p}{c_v}u_z - \frac{\mu}{c_v v}\left(u_z\right)^2\right)dz,
\tag{25}
$$

are both continuous with respect to x.

Indeed, we introduce the following functional of the iteration difference,

$$
\begin{aligned}
&\mathscr{F}\left[V^{n+1}-V^{n}, U^{n+1}-U^{n}, \Theta^{n+1}-\Theta^{n}\right] \qquad (26)\\
&\equiv \left|\left|\left|V^{n+1}-V^{n}\right|\right|\right|_{\infty}+\left|\left|\left|V^{n+1}-V^{n}\right|\right|\right|_{1}+\left|\left|\left|V^{n+1}-V^{n}\right|\right|\right|_{BV}\\
&\quad+\left|\left|\left|U^{n+1}-U^{n}\right|\right|\right|_{\infty}+\left|\left|\left|U^{n+1}-U^{n}\right|\right|\right|_{1}+\left|\left|\left|\frac{\sqrt{\tau}}{|\log\tau|}\left(U_x^{n+1}-U_x^{n}\right)\right|\right|\right|_{\infty}+\left|\left|\left|\frac{U_x^{n+1}-U_x^{n}}{|\log\tau|}\right|\right|\right|_{1}\\
&\quad+\left|\left|\left|\frac{\Theta^{n+1}-\Theta^{n}}{|\log\tau|}\right|\right|\right|_{\infty}+\left|\left|\left|\Theta^{n+1}-\Theta^{n}\right|\right|\right|_{1}+\left|\left|\left|\frac{\sqrt{\tau}}{|\log\tau|}\left(\Theta_x^{n+1}-\Theta_x^{n}\right)\right|\right|\right|_{\infty}+\left|\left|\left|\frac{\Theta_x^{n+1}-\Theta_x^{n}}{|\log\tau|}\right|\right|\right|_{1}.
\end{aligned}
$$

Here

$$
\left|\left|\left|\frac{\sqrt{\tau}}{|\log\tau|}\left(U_x^{n+1}-U_x^{n}\right)\right|\right|\right|_{\infty}=\sup_{0<\tau<t_\sharp}\left\|\frac{\sqrt{\tau}}{|\log\tau|}\left(U_x^{n+1}-U_x^{n}\right)(\cdot,\tau)\right\|_{L_x^\infty},
$$

and similar for other $|||\cdot|||$ norms. We show the contraction property for sufficiently small δ and $t_\sharp$,

$$
\begin{aligned}
&\mathscr{F}\left[V^{n+1}-V^{n}, U^{n+1}-U^{n}, \Theta^{n+1}-\Theta^{n}\right]\\
&\le C_\flat\left(\delta+\sqrt{t_\sharp}|\log t_\sharp|+\frac{1}{|\log t_\sharp|}\right)\mathscr{F}\left[V^{n}-V^{n-1}, U^{n}-U^{n-1}, \Theta^{n}-\Theta^{n-1}\right].
\end{aligned}
$$

3.2 Regularity

From Theorem 9, we have obtained the first order regularity with respect to x and the continuity of the fluxes for the weak solution (v,u,θ) to system (3). However, a weak solution to (3) is not necessarily a weak solution to the original system in conservative form (1) due to nonlinearity, unless some more time regularity estimates can be established.

By exploring more estimates of heat kernel, we first show θ is Hölder continuous in time, then use it to show u_t is well-defined. Lastly, we revisit the integral equation of θ and apply time differentiability of u to prove θ_t is also defined.

Theorem 10 ([12]) *Suppose the initial data $(v_0^*, u_0^*, \theta_0^*)$ satisfy the condition (5) for small δ. Let (v,u,θ) be the corresponding local-in-time weak solution constructed in Theorem 9. The following assertions hold:*

(1) In addition to the estimates in (24), there exists a positive constant $C_\sharp$ such that when $t\in(0,t_\sharp)$, the solution satisfies

$$
\max\left\{\sqrt{t}\,\|u_t(\cdot,t)\|_{L_x^1},\ t\,\|u_t(\cdot,t)\|_{L_x^\infty},\ \sqrt{t}\,\|\theta_t(\cdot,t)\|_{L_x^1},\ t\,\|\theta_t(\cdot,t)\|_{L_x^\infty}\right\}\le 2C_\sharp\delta. \qquad (27)
$$

(2) The fluxes of u and θ (defined in (25)) are both globally Lipschitz continuous with respect to x for $t>0$.

(3) The specific volume $v(x,t)$ satisfies the following Hölder continuity in time properties

for $0 \leq s < t$*,*

$$\begin{cases} \|v(\cdot,t) - v(\cdot,s)\|_{BV} \leq O(1)\delta \dfrac{(t-s)|\log(t-s)|}{\sqrt{t}} \\ \|v(\cdot,t) - v(\cdot,s)\|_{L^\infty} \leq O(1)\delta \dfrac{t-s}{\sqrt{t}}, \\ \|v(\cdot,t) - v(\cdot,s)\|_{L^1} \leq O(1)\delta(t-s). \end{cases} \tag{28}$$

From Theorems 9 and 10, we identify the function space where constructed solution belongs to,

$$\begin{cases} v(x,t) - 1 \in C\left([0,t_\sharp); L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}) \cap BV\right), \\ u(x,t) \in L^\infty\left(0,t_\sharp; W^{1,1}(\mathbb{R}) \cap L^\infty(\mathbb{R})\right), \quad \sqrt{t}u_x(x,t) \in L^\infty\left(0,t_\sharp; L^\infty(\mathbb{R})\right), \\ \sqrt{t}u_t(x,t) \in L^\infty\left(0,t_\sharp; L^1(\mathbb{R})\right), \quad tu_t(x,t) \in L^\infty\left(0,t_\sharp; L^\infty(\mathbb{R})\right), \\ \theta(x,t) \in L^\infty\left(0,t_\sharp; W^{1,1}(\mathbb{R}) \cap L^\infty(\mathbb{R})\right), \quad \sqrt{t}\theta_x(x,t) \in L^\infty\left(0,t_\sharp; L^\infty(\mathbb{R})\right), \\ \sqrt{t}\theta_t(x,t) \in L^\infty\left(0,t_\sharp; L^1(\mathbb{R})\right), \quad t\theta_t(x,t) \in L^\infty\left(0,t_\sharp; L^\infty(\mathbb{R})\right). \end{cases} \tag{29}$$

Since it owns enough time regularity, one has the following proposition:

Proposition 11 *For the weak solution* (v,u,θ) *of* (3)*, the function* $(v,u,c_v\theta + u^2/2)$ *is a weak solution of* (1) *if* (v,u,θ) *satisfies* (29)*.*

3.3 Stability

Notice that in (29), the BV norm of v is continuous in time. Therefore, given any weak solution (v,u,θ) belong to (29) with $\|v_0\|_{BV}$ small, one can construct heat $H(x,t;y,\tau;1/v)$ in short time and represent the weak solution in terms of integral equation. Employing heat kernel estimates and its comparison estimates, we establish the following stability result.

Theorem 12 *Suppose there are two weak solutions* (v^a,u^a,θ^a) *and* (v^b,u^b,θ^b) *to the Navier-Stokes equations* (3) *both belonging to* (29)*, and their initial data both satisfy the following condition for small* δ_**,*

$$\|v_0\|_{BV} + \|u_0\|_{BV} + \|\theta_0\|_{BV} + \|v_0 - 1\|_{L^1} + \|u_0\|_{L^1} + \|\theta_0 - 1\|_{L^1} < \delta_* \ll 1,$$

Then, there exist positive constant t_* *and* $C_\flat$ *such that, the following stability hold*

$$\begin{aligned} &\mathscr{F}\left[v^a - v^b, u^a - u^b, \theta^a - \theta^b\right] \\ &\leq C_\flat\Big(\|\theta_0^a - \theta_0^b\|_{L_x^\infty} + \|\theta_0^a - \theta_0^b\|_{L_x^1} \\ &\qquad + \|u_0^a - u_0^b\|_{L_x^\infty} + \|u_0^a - u_0^b\|_{L_x^1} + \|v_0^a - v_0^b\|_{L_x^1} + \|v_0^a - v_0^b\|_{L_x^\infty} + \|v_0^a - v_0^b\|_{BV}\Big), \end{aligned}$$

where $\mathscr{F}$ *is the functional defined in* (26)*. Moreover, this immediately implies the uniqueness of the weak solution. Namely, for sufficiently small initial data, there exists a positive constant* t_* *such that, the equation* (3) *admits a unique weak solution in the sense* (29) *for* $t \in [0,t_*)$*.*

4 Green's function

In this section, we introduce the Green's function of system (1) linearized around constant state $(v, u, \theta) = (1, 0, 1)$, which is for studying time asymptotic behavior of the system (3). For the details of construction, one is referred to [12, 13].

In order to preserve the conservative form of equation (1), we define the following state variables,

$$E = e + \frac{1}{2}u^2, \quad U = (v, u, E), \quad p(v, e(E, u)) \equiv \frac{E - \frac{1}{2}u^2}{v}, \tag{30}$$

and thus

$$e_u = -u, \quad e_E = 1.$$

Then, the system (1) is rewritten as the following conservation form with unknowns defined in (30),

$$\begin{cases} v_t - u_x = 0 \\ u_t + p_v v_x + p_e e_u u_x + p_e e_E E_x = \left(\dfrac{\mu u_x}{v}\right)_x \\ E_t + u p_v v_x + (p + u p_e e_u) u_x + u p_e e_E E_x = \left(\dfrac{\kappa \theta_e e_u + \mu u}{v} u_x + \dfrac{\kappa \theta_e e_E}{v} E_x\right)_x . \end{cases} \tag{31}$$

We can also write the system into a vector form as follows,

$$U_t + F(U)_x = (B(U)U_x)_x \Longleftrightarrow U_t + F'(U)U_x = (B(U)U_x)_x.$$

where U, F, $F'(U)$ and B are defined as below,

$$U = \begin{pmatrix} v \\ u \\ E \end{pmatrix}, \quad F(U) = \begin{pmatrix} -u \\ p \\ pu \end{pmatrix}, \quad F'(U) = \begin{pmatrix} 0 & -1 & 0 \\ p_v & -p_e u & p_e \\ p_v u & p - p_e u^2 & p_e u \end{pmatrix}, \quad B(U) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{\mu}{v} & 0 \\ 0 & \left(\frac{\mu}{v} - \frac{\kappa \theta_e}{v}\right) u & \frac{\kappa \theta_e}{v} \end{pmatrix}.$$

Now we consider the linearization of equations (31) around a constant state $\bar{U}$. Let $U = \bar{U} + V$. We have

$$V_t + F'(\bar{U})V_x - B(\bar{U})V_{xx} = [N_1(V; \bar{U}) + N_2(V; \bar{U})]_x, \tag{32}$$

where N_1 and N_2 are nonlinear terms coming from the hyperbolic and parabolic parts respectively,

$$N_1(V; \bar{U}) = -[F(\bar{U} + V) - F(\bar{U}) - F'(\bar{U})V], \quad N_2(V; \bar{U}) = B(U) - B(\bar{U}). \tag{33}$$

The Green's function $\mathbb{G}(x, t; \bar{U})$ for the linearized equation (32) is the solution to the following system,

$$\begin{cases} \partial_t \mathbb{G}(x, t; \bar{U}) = \left(-F'(\bar{U})\partial_x + B(\bar{U})\partial_{xx}\right) \mathbb{G}(x, t; \bar{U}), \\ \mathbb{G}(x, 0; \bar{U}) = \delta(x) I, \end{cases} \tag{34}$$

where I is the 3×3 identity matrix and $\delta(x)$ is the Dirac-delta function. The Green's function $\mathbb{G}(x, t)$ can be decomposed into singular and regular parts, $\mathbb{G}^*(x, t)$ and $\mathbb{G}^\dagger(x, t)$

respectively. Roughly speaking, the singular part is the short wave approximation of $\mathbb{G}$, which extracts the leading singularity from $\mathbb{G}$; while the regular part dominates the large time behavior of $\mathbb{G}$.

Proposition 13 ([13]) *Let $\mathbb{G}(x,t)$ be the Green's function of the linearized equation of (1) around the constant equilibrium state $(v,u,\theta)=(1,0,1)$. Then, the Green's function $\mathbb{G}(x,t)$ has the following estimates for $t\le 1$,*

$$\Bigg|\mathbb{G}(x-y,t)-e^{-\frac{K}{\mu}t}\delta(x-y)\begin{pmatrix}1&0&0\\0&0&0\\0&0&0\end{pmatrix}-\frac{e^{\beta_2^* t}}{\sqrt{4\pi\mu t}}e^{-\frac{(x-y)^2}{4\mu t}}\begin{pmatrix}0&0&0\\0&1&0\\0&0&0\end{pmatrix}$$
$$-\frac{e^{\beta_3^* t}}{\sqrt{4\pi\frac{\kappa}{c_v}t}}e^{-\frac{(x-y)^2}{4\frac{\kappa}{c_v}t}}\begin{pmatrix}0&0&0\\0&0&0\\0&0&1\end{pmatrix}\Bigg|\le O(1)e^{-\sigma_0^* t-\sigma_0|x-y|}+O(1)te^{-\sigma_0|x-y|},\qquad t\le 1,$$

$$\Bigg|\mathbb{G}_x(x-y,t)-e^{-\frac{K}{\mu}t}\delta'(x-y)\begin{pmatrix}1&0&0\\0&0&0\\0&0&0\end{pmatrix}+e^{-\frac{K}{\mu}t}\delta(x-y)\begin{pmatrix}0&\frac{1}{\mu}&0\\\frac{K}{\mu}&0&0\\0&0&0\end{pmatrix}$$
$$-\partial_x\left(\frac{e^{\beta_2^* t}}{\sqrt{4\pi\mu t}}e^{-\frac{(x-y)^2}{4\mu t}}\right)\begin{pmatrix}0&0&0\\0&1&0\\0&0&0\end{pmatrix}+\frac{e^{\beta_2^* t}}{\sqrt{4\pi\mu t}}e^{-\frac{(x-y)^2}{4\mu t}}\begin{pmatrix}0&-\frac{1}{\mu}&0\\-\frac{K}{\mu}&0&\frac{K}{c_v\mu-\kappa}\\0&\frac{c_vK}{c_v\mu-\kappa}&0\end{pmatrix}$$
$$-\partial_x\left(\frac{e^{\beta_3^* t}}{\sqrt{4\pi\frac{\kappa}{c_v}t}}e^{-\frac{(x-y)^2}{4\frac{\kappa}{c_v}t}}\right)\begin{pmatrix}0&0&0\\0&0&0\\0&0&1\end{pmatrix}+\frac{e^{\beta_3^* t}}{\sqrt{4\pi\frac{\kappa}{c_v}t}}e^{-\frac{(x-y)^2}{4\frac{\kappa}{c_v}t}}\begin{pmatrix}0&0&0\\0&0&-\frac{K}{c_v\mu-\kappa}\\0&-\frac{c_vK}{c_v\mu-\kappa}&0\end{pmatrix}\Bigg|$$
$$\le O(1)e^{-\sigma_0^* t-\sigma_0|x-y|}+O(1)te^{-\sigma_0|x-y|},\qquad t\le 1,$$

$$
\begin{aligned}
&\Bigg|\mathbb{G}_{xx}(x-y,t) - e^{-\frac{K}{\mu}t}\delta''(x-y)\begin{pmatrix}1&0&0\\0&0&0\\0&0&0\end{pmatrix} + e^{-\frac{K}{\mu}t}\delta'(x-y)\begin{pmatrix}0&\frac{1}{\mu}&0\\\frac{K}{\mu}&0&0\\0&0&0\end{pmatrix}\\
&- e^{-\frac{K}{\mu}t}\delta(x-y)\left(\begin{pmatrix}\frac{K}{\mu^2}&0&\frac{K}{\kappa\mu}\\0&-\frac{K}{\mu^2}&0\\\frac{c_vK^2}{\kappa\mu}&0&0\end{pmatrix} - t\begin{pmatrix}-\frac{\left(\kappa K^2-\mu K^3\right)}{\kappa\mu^3}&0&0\\0&0&0\\0&0&0\end{pmatrix}\right)\\
&- \partial_{xx}\left(\frac{e^{\beta_2^* t}}{\sqrt{4\pi\mu t}}e^{-\frac{(x-y)^2}{4\mu t}}\right)\begin{pmatrix}0&0&0\\0&1&0\\0&0&0\end{pmatrix} + \partial_x\left(\frac{e^{\beta_2^* t}}{\sqrt{4\pi\mu t}}e^{-\frac{(x-y)^2}{4\mu t}}\right)\begin{pmatrix}0&-\frac{1}{\mu}&0\\-\frac{K}{\mu}&0&\frac{K}{c_v\mu-\kappa}\\0&\frac{c_vK}{c_v\mu-\kappa}&0\end{pmatrix}\\
&- \frac{e^{\beta_2^* t}}{\sqrt{4\pi\mu t}}e^{-\frac{(x-y)^2}{4\mu t}}\begin{pmatrix}-\frac{K}{\mu^2}&0&\frac{K}{c_v\mu^2-\kappa\mu}\\0&\frac{c_vK^2}{(c_v\mu-\kappa)^2}+\frac{K}{\mu^2}&0\\\frac{c_vK^2}{c_v\mu^2-\kappa\mu}&0&-\frac{c_vK^2}{(c_v\mu-\kappa)^2}\end{pmatrix}\\
&- \partial_{xx}\left(\frac{e^{\beta_3^* t}}{\sqrt{4\pi\frac{\kappa}{c_v}t}}e^{-\frac{(x-y)^2}{4\frac{\kappa}{c_v}t}}\right)\begin{pmatrix}0&0&0\\0&0&0\\0&0&1\end{pmatrix} + \partial_x\left(\frac{e^{\beta_3^* t}}{\sqrt{4\pi\frac{\kappa}{c_v}t}}e^{-\frac{(x-y)^2}{4\frac{\kappa}{c_v}t}}\right)\begin{pmatrix}0&0&0\\0&0&-\frac{K}{c_v\mu-\kappa}\\0&-\frac{c_vK}{c_v\mu-\kappa}&0\end{pmatrix}\\
&+ \frac{e^{\beta_3^* t}}{\sqrt{4\pi\frac{\kappa}{c_v}t}}e^{-\frac{(x-y)^2}{4\frac{\kappa}{c_v}t}}\begin{pmatrix}0&0&\frac{c_vK}{\kappa(c_v\mu-\kappa)}\\0&\frac{c_vK^2}{(c_v\mu-\kappa)^2}&0\\\frac{c_v^2K^2}{\kappa(c_v\mu-\kappa)}&0&-\frac{c_vK^2}{(c_v\mu-\kappa)^2}\end{pmatrix}\Bigg| \le O(1)e^{-\sigma_0^* t-\sigma_0|x-y|} + O(1)te^{-\sigma_0|x-y|}, \qquad t\le 1.
\end{aligned}
$$

On the other hand, for large time $t \ge 1$, the Green's function has the following estimates,

$$
\begin{aligned}
&\Bigg|\mathbb{G}(x-y,t) - e^{-\frac{K}{\mu}t}\delta(x-y)\begin{pmatrix}1&0&0\\0&0&0\\0&0&0\end{pmatrix}\\
&- \sum_{j=1}^{3}\frac{e^{-\frac{(x-y+\beta_j t)^2}{4\alpha_j t}}}{2\sqrt{\pi\alpha_j t}}M_j^0 - \sum_{j=1}^{3}\partial_x\left(\frac{e^{-\frac{(x-y+\beta_j t)^2}{4\alpha_j t}}}{2\sqrt{\pi\alpha_j t}}\right)M_j^1\Bigg|\\
&\qquad \le \sum_{j=1}^{3}\frac{O(1)e^{-\frac{(x-y+\beta_j t)^2}{4Ct}}}{t}M_j^0 + \sum_{j=1}^{3}\frac{O(1)e^{-\frac{(x-y+\beta_j t)^2}{4Ct}}}{t^{\frac{3}{2}}} + O(1)e^{-\sigma_0^* t-\sigma_0|x-y|}, \qquad 1\le t,
\end{aligned}
$$

$$
\begin{aligned}
&\Bigg|\mathbb{G}_x(x-y,t)-e^{-\frac{K}{\mu}t}\delta'(x-y)\begin{pmatrix}1&0&0\\0&0&0\\0&0&0\end{pmatrix}+e^{-\frac{K}{\mu}t}\delta(x-y)\begin{pmatrix}0&\frac{1}{\mu}&0\\\frac{K}{\mu}&0&0\\0&0&0\end{pmatrix}\\
&-\sum_{j=1}^{3}\partial_x\left(\frac{e^{-\frac{(x-y+\beta_j t)^2}{4\alpha_j t}}}{2\sqrt{\pi\alpha_j t}}\right)M_j^0-\sum_{j=1}^{3}\partial_x^2\left(\frac{e^{-\frac{(x-y+\beta_j t)^2}{4\alpha_j t}}}{2\sqrt{\pi\alpha_j t}}\right)M_j^1\Bigg|\\
&\qquad\le\sum_{j=1}^{3}\frac{O(1)e^{-\frac{(x-y+\beta_j t)^2}{4Ct}}}{t^{\frac{3}{2}}}M_j^0+\sum_{j=1}^{3}\frac{O(1)e^{-\frac{(x-y+\beta_j t)^2}{4Ct}}}{t^2}+O(1)e^{-\sigma_0^* t-\sigma_0|x-y|},\qquad 1\le t,
\end{aligned}
$$

$$
\begin{aligned}
&\Bigg|\mathbb{G}_{xx}(x-y,t)-e^{-\frac{K}{\mu}t}\delta''(x-y)\begin{pmatrix}1&0&0\\0&0&0\\0&0&0\end{pmatrix}+e^{-\frac{K}{\mu}t}\delta'(x-y)\begin{pmatrix}0&\frac{1}{\mu}&0\\\frac{K}{\mu}&0&0\\0&0&0\end{pmatrix}\\
&-e^{-\frac{K}{\mu}t}\delta(x-y)\left(\begin{pmatrix}\frac{K}{\mu^2}&0&\frac{K}{\kappa\mu}\\0&-\frac{K}{\mu^2}&0\\\frac{c_vK^2}{\kappa\mu}&0&0\end{pmatrix}-t\begin{pmatrix}-\frac{\left(\kappa K^2-\mu K^3\right)}{\kappa\mu^3}&0&0\\0&0&0\\0&0&0\end{pmatrix}\right)\\
&-\sum_{j=1}^{3}\partial_x^2\left(\frac{e^{-\frac{(x-y+\beta_j t)^2}{4\alpha_j t}}}{2\sqrt{\pi\alpha_j t}}\right)M_j^0-\sum_{j=1}^{3}\partial_x^3\left(\frac{e^{-\frac{(x-y+\beta_j t)^2}{4\alpha_j t}}}{2\sqrt{\pi\alpha_j t}}\right)M_j^1\Bigg|\\
&\qquad\le\sum_{j=1}^{3}\frac{O(1)e^{-\frac{(x-y+\beta_j t)^2}{4Ct}}}{t^2}M_j^0+\sum_{j=1}^{3}\frac{O(1)e^{-\frac{(x-y+\beta_j t)^2}{4Ct}}}{t^{\frac{5}{2}}}+O(1)e^{-\sigma_0^* t-\sigma_0|x-y|},\qquad 1\le t,
\end{aligned}
$$

where the parameters α_i, β_i, β_i^* are some explicit constants, M_j^l are some explicit constant matrices.

5 Global well-posedness

In this section, we give sketch of proof for global well-posedness by using "effective Green's function" and a priori estimate.

In order to construct a new effective integral representation of v, u and θ, we introduce an effective Green's function G similar as in [8]. Define a smooth non-increasing cutoff function as follows,

$$
\mathcal{X}(t)\in C^\infty(\mathbb{R}_+),\quad \mathcal{X}'(t)\le 0,\quad \|\mathcal{X}'\|_{L^\infty(\mathbb{R}_+)}\le 2,\quad \mathcal{X}(t)=\begin{cases}1, & \text{for } t\in(0,1],\\ 0, & \text{for } t>2.\end{cases}\tag{35}
$$

Then, we choose a small positive constant ν_0 (which will be determined later) such that, the heat kernel $H(x,t;y,\tau;\frac{1}{v})$ and the local weak solution $(v(x,\tau),u(x,\tau),E(x,\tau))$ for (1) both exist when $\tau\in(t-2\nu_0,t]$. We interpolate the heat kernel for short time and

Green's function for large time via cutoff function (35), thus introduce the effective Green's functions as below,

$$\begin{cases} G_{22}(x,t;y,\tau) = \mathcal{X}\left(\dfrac{t-\tau}{\nu_0}\right) H\left(x,t;y,\tau;\dfrac{\mu}{v}\right) + \left(1-\mathcal{X}\left(\dfrac{t-\tau}{\nu_0}\right)\right)\mathbb{G}_{22}(x-y;t-\tau), \\ G_{33}(x,t;y,\tau) = \mathcal{X}\left(\dfrac{t-\tau}{\nu_0}\right) H\left(x,t;y,\tau;\dfrac{\kappa}{c_v v}\right) + \left(1-\mathcal{X}\left(\dfrac{t-\tau}{\nu_0}\right)\right)\mathbb{G}_{33}(x-y;t-\tau). \end{cases} \tag{36}$$

Now we can represent the solution (v,u,E) in terms of the effective Green's function, which is given by $(\mathbb{G}_{ij})$ in Proposition 13 with replacing $\mathbb{G}_{22}$ by G_{22}, $\mathbb{G}_{33}$ by G_{33}. It captures both the local-in-time regularity and global-in-time space-time structure of the solution.

For example, to obtain the effective integral representation of u, we multiply the vector $(\mathbb{G}_{21}(x-y,t-\tau), G_{22}(x,t;y,\tau), \mathbb{G}_{23}(x-y,t-\tau))$ to the system (1), apply integration by parts, and split the time integral into three parts $[0,t-2\nu_0]$, $[t-2\nu_0,t-\nu_0]$ and $[t-\nu_0,t]$ to have

$$\begin{aligned} u(x,t) = &\int_{\mathbb{R}} \mathbb{G}_{21}(x-y,t)(v(y,0)-1)dy + \int_{\mathbb{R}} G_{22}(x,t;y,0)u(y,0)dy \\ &+ \int_{\mathbb{R}} \mathbb{G}_{23}(x-y,t)\left(E(y,0)-c_v\right)dy + \sum_{i=1}^{3}\mathcal{R}_i^u, \end{aligned}$$

where the inhomogeneous remainders $\mathcal{R}_i$ are space-time double integral corresponding to time intervals $[0,t-2\nu_0]$, $[t-2\nu_0,t-\nu_0]$ and $[t-\nu_0,t]$ respectively. The interested readers are referred to [13] for the expressions and computational details. The representations of v and θ can be derived similarly.

According to Theorems 10 and 12, if the initial data is controlled by a sufficiently small constant δ as in (5), there exists a unique weak solution (v,u,θ) to (3), or equivalently, (v,u,E) to (1), for $t < t_\sharp$. Moreover, the solutions are kept small in the sense of (24) and (27).

We define a stopping time as below,

$$\begin{aligned} &T = \sup_{t\geq 0}\left\{ t \;\middle|\; \mathcal{G}(\tau) < \delta, \quad \text{for } 0<\tau<t \right\}, \qquad (37) \\ &\mathcal{G}(\tau) \equiv \|\sqrt{\tau+1}(v(\cdot,\tau)-1)\|_\infty + \|\sqrt{\tau+1}u(\cdot,\tau)\|_\infty + \|\sqrt{\tau+1}(\theta(\cdot,\tau)-1)\|_\infty \\ &\qquad + \|v(\cdot,\tau)-1\|_{L^1} + \|u(\cdot,\tau)\|_{L^1} + \|\theta(\cdot,\tau)-1\|_{L^1} \\ &\qquad + \|v(\cdot,\tau)-1\|_{BV} + \|u(\cdot,\tau)\|_{BV} + \|\theta(\cdot,\tau)-1\|_{BV} \\ &\qquad + \|\sqrt{\tau}u_x(\cdot,\tau)\|_{L^\infty} + \|\sqrt{\tau}\theta_x(\cdot,\tau)\|_{L^\infty}. \end{aligned}$$

By Theorem 10, there exists a positive constant δ_* (smaller than δ) such that, if the initial data satisfy

$$\|v_0-1\|_{BV} + \|u_0\|_{BV} + \|\theta_0-1\|_{BV} + \|v_0-1\|_{L^1} + \|u_0\|_{L^1} + \|\theta_0-1\|_{L^1} < \delta^*, \tag{38}$$

then the stopping time $T > t_\sharp$. Here $t_\sharp$ is the existence time associated with δ in Theorem 10.

Based on the integral representations of the solution and their derivatives, we can prove the following a priori estimate, which then yields a sharper estimates of the solution. This is the key lemma for the proof of global existence. We refer the interested reader to [13] for the details of proof.

Lemma 14 (A priori estimate, [13]) *Let (v, u, E), $t_\sharp$ and δ be the local solution and corresponding parameters constructed in Theorem 10. We further suppose the following properties hold for the solution,*

$$\begin{cases} \|v_0 - 1\|_{BV} + \|u_0\|_{BV} + \|\theta_0 - 1\|_{BV} + \|v_0 - 1\|_{L^1} + \|u_0\|_{L^1} + \|\theta_0 - 1\|_{L^1} < \delta^*, \\ \mathcal{G}(\tau) < \delta, \quad \textit{for } \forall \tau < t, \\ t_\sharp \geq 4\nu_0, \end{cases} \tag{39}$$

where $\mathcal{G}(\tau)$ is defined in (37), δ^* is as in (38) and ν_0 is given in (36). Then, $u(x,t)$ has the following estimates for $t \geq t_\sharp$,

$$\begin{cases} \|u(\cdot,t)\|_{L^1} \leq C(\nu_0)\delta^* + O(1)\left(\delta^2 + \sqrt{\nu_0}\delta^2 + \sqrt{\nu_0}\delta + \nu_0\delta + \delta^2\right), \\ \|\sqrt{1+t}u(\cdot,t)\|_{L^\infty} \leq C(\nu_0)\delta^* + O(1)\left(\sqrt{\nu_0}\delta + \delta^2\right), \\ \|u_x(\cdot,t)\|_{L^1} \leq C(\nu_0)\delta^* + O(1)\dfrac{|\log(\nu_0)|}{\sqrt{\nu_0}}\delta^2 + O(1)\sqrt{\nu_0}\delta, \\ \|\sqrt{t}u_x(\cdot,t)\|_{L^\infty} \leq C(\nu_0)\delta^* + O(1)\dfrac{|\log(\nu_0)|}{\sqrt{\nu_0}}\delta^2 + O(1)\sqrt{\nu_0}\delta. \end{cases}$$

$\theta(x,t)$ has the following estimates for $t \geq t_\sharp$,

$$\begin{cases} \|\theta(\cdot,t) - 1\|_{L^1} \leq O(1)\left(C(\nu_0)\delta^* + \sqrt{\nu_0}\delta + \delta^2\right) + O(1)\left(C(\nu_0)\delta^* + \sqrt{\nu_0}\delta + \delta^2\right)^2, \\ \|\sqrt{1+t}(\theta(\cdot,t) - 1)\|_{L^\infty} \leq O(1)\left(C(\nu_0)\delta^* + \sqrt{\nu_0}\delta + \delta^2\right) + O(1)\left(C(\nu_0)\delta^* + \sqrt{\nu_0}\delta + \delta^2\right)^2, \\ \|\theta_x(\cdot,t)\|_{L^1} \leq C(\nu_0)\delta^* + O(1)\dfrac{|\log(\nu_0)|}{\sqrt{\nu_0}}\delta^2 + O(1)\sqrt{\nu_0}\delta, \\ \|\sqrt{t}\theta_x(\cdot,t)\|_{L^\infty} \leq C(\nu_0)\delta^* + O(1)\dfrac{|\log(\nu_0)|}{\sqrt{\nu_0}}\delta^2 + O(1)\sqrt{\nu_0}\delta. \end{cases}$$

And $v(x,t)$ has the following estimates for $t \geq t_\sharp$,

$$\begin{cases} \|v(\cdot,t) - 1\|_{L^1} \leq C(\nu_0)\delta^* + O(1)\delta^2, \\ \|\sqrt{1+t}(v(\cdot,t) - 1)\|_{L^\infty} \leq C(\nu_0)\delta^* + O(1)\delta^2, \\ \|v(\cdot,t)\|_{BV} \leq C(\nu_0)\delta^* + O(1)\dfrac{\delta^2}{\sqrt{\nu_0}}. \end{cases}$$

The global existence then follows from a standard continuity argument and choosing δ_*, ν_0 properly.

Theorem 15 (Global existence) *Suppose initial data (v_0, u_0, θ_0) of Navier-Stokes equation (3) satisfy*

$$\|v_0 - 1\|_{L^1_x} + \|v_0\|_{BV} + \|u_0\|_{L^1_x} + \|u_0\|_{BV} + \|\theta_0 - 1\|_{L^1_x} + \|\theta_0\|_{BV} \leq \delta^*, \tag{40}$$

for δ^* sufficiently small. Then the solution constructed in Theorems 10 and 12 exists globally in time, and there exists positive constant $\mathscr{C}$ such that, the solution satisfies

$$\left\|\sqrt{t+1}(v(\cdot,t)-1)\right\|_{L_x^\infty} + \left\|\sqrt{t+1}u(\cdot,t)\right\|_{L_x^\infty} + \left\|\sqrt{t+1}(\theta(\cdot,t)-1)\right\|_{L_x^\infty} + \left\|\sqrt{t}u_x(\cdot,t)\right\|_{L_x^\infty} + \left\|\sqrt{t}\theta_x(\cdot,t)\right\|_{L_x^\infty} \le \mathscr{C}\delta^* \quad \text{for } t \in (0,+\infty).$$

6 Outlook

The most important ingredient in this work and [8] is the construction of BV coefficient heat kernel, because it accurately captures the quasi-linear nature of the compressible Navier-Stokes system. Combining BV heat kernel and Green's function, it is even possible to establish the space-time pointwise estimate of the solution for rough initial data. Actually, we have the following result for isentropic gas

Theorem 16 ([14]) *Suppose that the initial data (v_0, u_0) of (4) satisfies*

$$\left\|e^{|x|}(v_0-1)\right\|_\infty + \left\|e^{|x|}u_0\right\|_\infty + \left\|v_0-1\right\|_{BV} + \|u_0\|_{BV} \le \varepsilon. \tag{41}$$

Then, there exist positive constants D_0 and ε_0 such that for $\varepsilon \in (0, \varepsilon_0)$ the solution (v,u) satisfies for $t > 0$

$$\begin{aligned}&|v(x,t)-1|\,, |u(x,t)| \\ &\le D_0\varepsilon\Big[\frac{e^{-\frac{(x-\beta t)^2}{2D_0(t+1)}} + e^{-\frac{(x+\beta t)^2}{2D_0(t+1)}}}{\sqrt{t+1}} + \frac{\mathbf{1}_{[-\beta t,\beta t]}(x)}{\sqrt{\beta t - x + \sqrt{t+1}}\sqrt{\beta t + x + \sqrt{t+1}}} + e^{-(|x|+t)/D_0}\Big],\end{aligned}$$

where $\beta = \sqrt{-p'(1)}$ is the sound speed, and $\mathbf{1}_{[a,b]}(x)$ is the characteristic function of the interval $[a,b]$.

This approach has potential to be applied to other problems with rough data, such as initial-boundary value problem, perturbation around non-constant state (shock, rarefaction). Moreover, it would be more interesting and challenging to see how to develop the theory for multi-dimensional problem.

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