Mathematical reformulation of the Kolmogorov-Richardson energy cascade and coherent vortical structure in infinitesimal small scale vortices

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1 Introduction

Recently, by direct numerical simulations, the concrete picture of the coherent vortical structure in developed turbulence has been revealed. More precisely, DNS [8, 9, 24, 25] of turbulence at sufficiently high Reynolds numbers have reported that there exists a hierarchy of vortex stretching motions in developed turbulence. In particular, Goto-Saito-Kawahara [9] clearly observed that turbulence at sufficiently high Reynolds numbers in a periodic cube is composed of a self-similar hierarchy of antiparallel pairs of vortex tubes, and it is sustained by creation of smaller-scale vortices due to stretching in larger-scale strain fields (see also [31]). With the aid of such hierarchy of the antiparallel pairs of vortex tubes, Y-Goto-Tsuruhashi [30] reformulated the energy cascade process and derived $-5/3$ power law from the Navier-Stokes equation without directly using the Kolmogorov similarity hypothesis (for the summary of this reformulation written in Japanese, see [32]). After that, Tsuruhashi-Goto-Oka-Y [28] numerically confirmed the assumption in this formulation.

We now explain this reformulation briefly. Let us describe the Navier-Stokes equation in $\mathbb{T}^3 := (\mathbb{R}/2\pi\mathbb{Z})^3$:

$$\begin{cases} \partial_t u + u \cdot \nabla u + \nabla p = \nu \Delta u + f, \\ \nabla \cdot u = 0, \\ u(t=0) = u_0 \end{cases} \tag{1}$$

with the external force,

$$f = (-\sin x \cos y, +\cos x \sin y, 0). \tag{2}$$

Here, $u : [0,\infty) \times \mathbb{T}^3 \to \mathbb{R}^3$ and $p : [0,\infty) \times \mathbb{T}^3 \to \mathbb{R}$ denote the velocity and pressure of the fluid, respectively. Assume u is smooth, and ω be the corresponding vorticity. We may replace $\mathbb{T}^3$ to $\mathbb{R}^3$ and let us define a band-pass filter:

$$\bar{\omega}_K(t,x) := \mathcal{F}_\xi^{-1}[\chi_{A_K}(\xi)\hat{\omega}(t,\xi)](x),$$

where (in other words, coarse-graining in an annulus, Littlewood-Paley decomposition)

$$A_K := \{\xi \in \mathbb{R}^3 : K/\sqrt{\alpha} \leq |\xi| \leq \sqrt{\alpha}K\}$$

and $\alpha > 1$ is a prescribed constant expressing the ratio of adjacent scales. In what follows we take wavenumber K from $\alpha^{\mathbb{Z}}$. Also, in this paper, we assume that the spatially integrated value of any quantity is independent of time because the size of the domain is much larger than the correlation length (the integral length) of the flow, that is,

$$\frac{d}{dt}\|\bar{\omega}_K(t)\|_{L^2}^2 = 0 \quad \text{for any} \quad K \quad \text{and} \quad t \geq 0. \tag{3}$$

The specific purpose in [30] was to approximate the following vortex stretching term:

$$\int (\omega \cdot \nabla)u \cdot \bar{\omega}_K.$$

Note that $\omega = \sum_{K\in\alpha^{\mathbb{Z}}} \bar{\omega}_K$ and $u = \sum_{K\in\alpha^{\mathbb{Z}}} \bar{u}_K$. First, let us recall the new hypothesis. For the original version of Kolmogorov hypothesis, see [16], for a recasted version, see [7, Section 6].

Outline of our hypothesis. The crucial point of our hypothesis is that there exists a universal vortex stretching/compressing mechanism, independent of the energy input rate ϵ. More precisely, a pair of tubular vortices are stretching (through Biot-Savart law) several tubular vortices in the adjacent smaller scale, at the same time, the several tubular vortices are compressing the pair of tubular vortices in the adjacent larger scale. Also, supports of these tubular vortices (with compact supports) are disjoint. We conjecture that such vortex stretching/compressing in adjacent two scales are the dominant event, so, in our cascade picture, we exclude multiscale events. Note that, due to the uncertainty principle for the Fourier transform, we expect that the three-wave interaction of this adjacent scale event is rather nonlocal (c.f. [22]).

Though the study [30] still lacks the discussion on the origin of this universality, the recent study by McKeown *et al* [21] may provide with it; they investigated how the elliptical instability led to the persistence of the turbulent energy cascade through the local interactions of vortices over a hierarchy of scales.

By applying approximations which are given in [30], we can derive Kolmogorov's $-5/3$ law.

Theorem 1. *(Y-Goto-Tsuruhashi;[30]) Assume* (3) *and approximations (which are given in [30]). Taking* $\nu \to 0$*, then we have Kolmogorov's* $-5/3$ *law:*

$$E_{LP}(K) := K^{-3}\|\bar{\omega}_K\|_{L^2}^2 = c\gamma\epsilon^{2/3}K^{-5/3}K^{-\frac{1}{3}(3-D)}. \tag{4}$$

The meaning of constants c*,* γ *and* D *are explained in [30].*

As an application of this reformulation, in this paper, we employ a short time dissipative Euler flow, and clarify necessity of the Lagrangian framework for capturing the coherent vortical structure in infinitesimal small scale vortices.

Remark 1. *However, in this paper we significantly neglect "remainder" terms, which appear in initial data and the corresponding Euler solutions (thus our study here is not rigorous). For the mathematical rigorous study, we need to clarify such remainder terms and need to control them.*

2 Lagrangian framework and dissipative Euler equations

In this section we briefly review the turbulence study in the Lagrangian and Eulerian frameworks. In the Eulerian framework, two-point correlation function has been the most important analysis tool in the theory of homogeneous and isotropic turbulence. However, due to the nonlinearity, equation for the correlation function cannot be obtained rigorously in any closed form, which is well-known as the closure problem. To understand the difficulty of this closure problem, the study of DIA (direct integration approximation) is a good gateway of it (see [20] and references therein). It is understood that the failure of the earliest version of the DIA, leading to the $k^{-3/2}$ scaling (not $k^{-5/3}$) of the energy spectrum in the inertial range, was due to picking up the sweeping time scale instead of the Kolmogorov time scale. This is due to a lack of Galiliean invariance of the velocity correlation function in Eulerian coordinates. The DIA in Lagrangian coordinates was elaborated by Kraichnan [17] who succeeded in reproducing the Kolmogorov $k^{-5/3}$ spectrum. This implies that a correct approximation to the Kolmogorov spectrum should be capable of distinguishing between the time scales of the sweeping motion without distortion caused by the flow of much larger scales. Thus we naturally expect that, to construct a proper turbulence model, the Lagrangian framework may be indispensable.

Also in mathematics, the importance of this Lagrangian framework has already been recognized in the study of anomalous dissipation. The anomalous dissipation postulates that, in the limit of high Reynold number, the kinetic energy dissipation is non-zero (see [29] for example):

$$\liminf_{\nu\to 0} \nu\langle|\nabla u^\nu|^2\rangle > 0,$$

where u^ν is the corresponding velocity depending on ν, and $\langle\cdot\rangle$ usually denotes some *ensemble* or long-time, space averages. Laboratory experiments and numerical simulations of turbulence both confirm this anomalous dissipation. Drivas [5] proposed a Lagrangian measure involving the short-time dispersion of tracer particles in coarse-grained fields, and proved that pairs of Lagrangian particles initially spread faster backward-in-time than forward-in-time. In the inviscid limit of the Navier-Stokes equations, his result provides rigorous mathematical justification of the physical result of Jucha-Xu-Pumir-Bodenschatz [14] without any ensemble averaging or any assumption of isotropy or homogeneity.

Remark 2. *His result encourages us to construct a coherent vortical model showing this time-asymmetry of short-time Lagrangian dispersion. See Appendix.*

Let us now briefly review the mathematical study of turbulence (that is, study of the dissipative Euler flow). In 1949, Onsager [26] considered the possibility that "turbulent energy dissipation could take place just as readily without the final assistance of viscosity because the velocity field does not remain differentiable". More precisely, Onsager had conjectured that weak solutions of the Euler equation satisfying a Hölder continuity condition grater than 1/3 conserve energy. This means that, to consider developed turbulence in pure mathematics, we need to employ dissipative Euler flows and at least $C^\sigma(\sigma \leq 1/3)$ singularity is needed. After posted this conjecture, the critical function space $C^{1/3}$ have already been sophisticated. Constantin-E-Titi [3] gave a simple proof of energy conservation under the weaker and more natural assumption that the velocity belongs to the Besov space $B^\sigma_{3,\infty}$ with $\sigma > 1/3$. To the contrary, construction of the dissipative Euler flow which is not in $B^\sigma_{3,\infty}$ $(\sigma > 1/3)$ has also been studied extensively, and nowadays, the Nash-style convex integration has been becoming the central scheme of it. This scheme was first initiated by De Lellis and Székelydihi Jr. [4]. They showed the existence of a $C^{0+}_{x,t}$ weak solution of the 3D Euler equations which is non-conservative, using the Nash scheme with Beltrami building blocks. After several results appear, Isett [12] showed existence of dissipative weak solutions in the regularity class $C^{1/3-}_{x,t}$ by using the Mikado flows, as building blocks. Thus in the Nash scheme, Beltrami flows and Mikado flows (both are stationary Euler flows) are the elementary pieces in the multi-scale Euler flows. However, clearly, the antiparallel pair of vortex tubes with stretching motion is neither Beltrami flow nor Mikado flow, thus, in order to construct a concrete picture of turbulence by using this Nash scheme, we need to find another type of building blocks (which needs to be consistent with (9)) accompanied by the vortex stretching, and this would be our open question (see also [15]).

3 The main result

In this section we employ a short time dissipative Euler flow (with some appropriate approximations), and clarify necessity of the Lagrangian framework for capturing the coherent vortical structure in infinitesimal small scale vortices. The incompressible Euler equations in $\mathbb{R}^3$ (as an approximation of $\mathbb{T}^3$) are expressed as follows:

$$\partial_t u + (u\cdot\nabla)u + \nabla p = 0, \qquad \nabla\cdot u = 0.$$

Next we set the initial data (see [28, 30]). Let $a>1$ be a suitable adjacent scale parameter and γ_p be a representative eddy kinetic energy (for $p=2$), fix them. For $k\in a^{\mathbb{Z}}$, let

$$\begin{aligned}
&u_0(x) := \sum_{k\in a^{\mathbb{Z}},k\geq 1}\bar u_{0,k} + \text{remainder}, \quad \bar u_{0,k}(x) = gk^{-H}\sum_{j=1}^{ck^D} U_{j,k}(kx),\\
&U_{j,k}\in C_c^\infty(\mathbb{R}^3), \quad \|U_{j,k}\|_{L^p}^p = \gamma_p, \quad |\operatorname{supp} U_{j,k}| = \Gamma \quad \text{(absolute constant)},\\
&\operatorname{supp} U_{j,k}(k\cdot)\cap \operatorname{supp} U_{j',k'}(k'\cdot) = \emptyset \ (j'\neq j \text{ or } k\neq k'),
\end{aligned}$$

and

$$\operatorname{supp} U_{j,k}\subset \{x : |x - {}^{\exists}x^*_{j,k}| < r\}, \tag{5}$$

where r is independent of j and k. $g, H>0$ are unknown which will be determined later, and $\{U_{j,k}\}_{j,K}$ are representative velocity of coherent eddies. Moreover we assume that $U_{j,k}$ satisfies the following symmetry condition (this is due to the fact that vortices are likely to be symmetric velocity fields): for any unit vector $\tau\in\mathbb{R}^3$, there are positive functions $\rho_+,\rho_-\in C_c^\infty(\mathbb{R}^3)$ satisfying $\rho_+(x)=\rho_-(-x)$, $\operatorname{supp}\rho_+\cap\operatorname{supp}\rho_- = \emptyset$ and $\operatorname{supp}\rho_+\cup\operatorname{supp}\rho_- = \operatorname{supp} U_{j,k}$ such that

$$U_{j,k}(x)\cdot\tau = \rho_+(x) - \rho_-(x). \tag{6}$$

Let $\tilde\rho_\pm(x) := \chi_{\{x:\rho_\pm(x)<1\}}(x)\rho_\pm(x)$. By using the symmetry condition (6), we immediately have

$$\begin{aligned}
\|\tau + U_{j,k}\|^p_{L^p(\operatorname{supp} U_{j,k})} &\geq \|1 + U_{j,k}\cdot\tau\|^p_{L^p(\operatorname{supp} U_{j,k})}\\
&\geq \|1+\tilde\rho_+\|^p_{L^p(\operatorname{supp}\rho_+)} + \|1-\tilde\rho_-\|^p_{L^p(\operatorname{supp}\rho_-)}\\
&\geq 2|\operatorname{supp}\rho_+| = |\operatorname{supp} U_{j,k}| = \Gamma
\end{aligned} \tag{7}$$

for any unit vector $\tau\in\mathbb{R}^3$. Here we used the fact that $(1+x)^p + (1-x)^p \geq 2$ for $0<x<1$. We express intermittency by $D<3$ (c.f. beta model) and by

$$\operatorname{dist}(U_{j,k}(k\cdot), U_{j',k}(k\cdot)) \gg k^{-1} \quad (j\neq j').$$

By the construction of this initial data, $\operatorname{supp}(\lim_{k\to\infty}\bar u_{0,k})$ is a Cantor type set (if it converges), so, it has measure zero, and uncountable. Note that, by Subsection 1.3.3

in [13], anomalous dissipation never occur if the singularity set is countable (see also [18, 19, 27]). We can regard each eddies in each scale as smooth functions surviving in each turnover time$\sim g^{-1}k^{H-1}$ (some positive time), so, we can reasonably define the smooth Lagrangian flow η as follows:

$$\begin{aligned}
&\eta : \bigcup_{k\in a^{\mathbb{Z}}, j\in ck^D} \operatorname{supp} U_{j,k}(k\cdot) \to \mathbb{T}^3,\\
&\frac{d}{dt}\eta(t,x) = u(t,\eta(t,x)) = u\circ\eta \quad (t<T\sim g^{-1}k^{H-1}), \quad \eta(0,x)=x.
\end{aligned} \tag{8}$$

Physically, in turnover time, each eddies should not deform too much (i.e. sweeping motion without distortion), and we need to express it mathematically. Let $\mathcal{D}$ be a diffeomorphism group such that

$$\mathcal{D} := \Big\{ \eta : \mathbb{T}^3 \to \mathbb{T}^3 : x \mapsto \eta(x) = (\eta_1,\eta_2,\eta_3) \text{ is bijective,}$$
$$\eta_1,\eta_2,\eta_3,\eta_1^{-1},\eta_2^{-1},\eta_3^{-1} \text{ are smooth} \Big\},$$

and let D_μ be the corresponding volume preserving diffeomorphism subgroup such that

$$\mathcal{D}_\mu := \{\eta\in\mathcal{D} : \det D\eta = 1\},$$

where

$$D\eta := \begin{pmatrix} \partial_1\eta_1 & \partial_2\eta_1 & \partial_3\eta_1\\ \partial_1\eta_2 & \partial_2\eta_2 & \partial_3\eta_2\\ \partial_1\eta_3 & \partial_2\eta_3 & \partial_3\eta_3 \end{pmatrix}.$$

For $\eta\in\mathcal{D}_\mu$, let us define the adjoint representation Ad as follows (see [23] for example):

$$\mathrm{Ad}_\eta u(x) := (D\eta u)\left(\eta^{-1}x\right).$$

Let us define the corresponding vorticity as $\omega := -\nabla\times\Delta^{-1}u$ and $\omega_0 := -\nabla\times\Delta^{-1}u_0$. Then the Euler equations can be rephrased as follows:

$$\begin{aligned}
&\omega = \mathrm{Ad}_\eta\omega_0, \quad \omega_0 := \sum_{k\in a^{\mathbb{Z}}, k\geq 1} \bar\omega_{0,k}\\
&\bar\omega_{0,k}(x) = gk^{-H+1}\sum_{j=1}^{ck^D} W_{j,k}(kx) + \text{remainder}, \quad W_{j,k} = \nabla\times U_{j,k}.
\end{aligned}$$

Then the corresponding solution can be precisely expressed as follows:

$$\bar\omega_k(t,x) = gk^{-H+1}\sum_j \mathrm{Ad}_{\Phi_{j,k}(t)} W_{j,k}(kx) + \text{remainder}, \tag{9}$$

where $\{\Phi_{j,k}(t)\}_{j,k}$ is a family of time evolution of volume preserving diffeomorphisms such that $\Phi_{j,k}(0) = Id$. We mathematically assume $\Phi_{j,k} = \Psi_{j,k}(t) + \text{remainder}$, where $\Psi_{j,k}$ is a combination of translation and rigid rotation, which expresses "sweeping motion without distortion". Since the main terms of $\{\Phi_{j,k}(t)\}$ are rigid rotation+translation, Ad and $-\nabla \times \Delta^{-1}$ are commutative. Thus we have

$$\bar{u}_k(t,x) := gk^{-H} \sum_j \mathrm{Ad}_{\Psi_{j,k}} U_{j,k}(kx), \quad u(t,x) = \sum_k \bar{u}_k(t,x) + \text{remainder}.$$

Significant approximation: In what follows, we neglect "remainder" terms.

Again, we express intermittency by

$$\mathrm{dist}(\mathrm{Ad}_{\Psi_{j,k}} U_{j,k}, \mathrm{Ad}_{\Psi_{j',k}} U_{j',k}) \gg k^{-1} \quad (j \neq j'). \tag{10}$$

Let us recall [30]. In mathematics, it is reasonable to assume that, there is an energy dissipation rate $\epsilon > 0$ such that

$$gk^{-H+1} \|\bar{u}_k(t,\cdot)\|_{L^2}^2 = gk^{-H+1} \|\bar{u}_{0,k}\|_{L^2}^2 = \gamma_2 \epsilon, \quad t < T \sim \epsilon^{-1/3} k^{H-1} \tag{11}$$

for any k. By this (11), we obtain the relations: $-3H - 2 + D = 0$ and $g = \epsilon^{1/3}$. Then we have the key formula:

$$k^{(-3+p)H+1} \|\bar{u}_k(t,\cdot)\|_{L^p}^p = \gamma_p \epsilon^{p/3} \quad (\forall k). \tag{12}$$

In this paper we assume the following time interval:

$$T = 2r\epsilon^{-1/3} k^{H-1}. \tag{13}$$

Note that, for $p = 3$, the dimension of (12) is the same as the Besov scale $B^{1/3}_{3,\infty}$ (c.f. [3, 6]). Now we examine whether or not perturbation develops instantaneously within turnover time. Let us define

$$G_k u(t,x) := \tau_k + u(t, x - \tau_k t),$$

where $\tau_k = \epsilon^{\frac{1}{3}} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} k^{-H}$. Then this $G_k u$ also satisfies the Euler equations. We now restrict this G_k to the eddy kinetic energy regions, that is,

$$G_k \bar{u}_k(t,x) := \begin{cases} \tau_k + \bar{u}_k(t, x - \tau_k t), & x \in \mathrm{supp}\, \bar{u}_{0,k} =: D_k, \\ 0, & \text{otherwise.} \end{cases}$$

The corresponding Lagrangian flow η^k can be defined as

$$\partial_t \eta^k(t,x) = (G_k \bar{u}_k)(t, \eta^k(t,x)), \quad x \in D_k,$$

and it satisfies

$$\partial_t(\eta^k(t,x) - \tau_k t) = u(t, \eta^k(t,x) - \tau_k t), \quad x \in D_k.$$

Thus by uniqueness of ODE (compare with (8)), we have

$$\eta^k(t,x) - \tau_k t = \eta(t,x), \quad x \in D_k. \tag{14}$$

The initial perturbation can be estimated as follows:

$$\epsilon^{-\frac{p}{3}} k^{(-3+p)H+1} \|G_k \bar{u}_{0,k}(\cdot) - \bar{u}_{0,k}(\cdot)\|^p_{L^p} = \epsilon^{-\frac{p}{3}} k^{(-3+p)H+1} \|\tau_k\|^p_{L^p(D_k)} \leq c\Gamma$$

for $1 \leq p < \infty$. Then, as an application of Theorem 2.1 in [11] (see also [1]), we immediately have the following theorem.

Theorem 2. *(Comparison with Eulerian and Lagrangian.) We have*

$$\begin{aligned} \textit{(Eulerian)} \quad & \epsilon^{-\frac{p}{3}} k^{(-3+p)H+1} \|G_k \bar{u}_k(T,\cdot) - \bar{u}_k(T,\cdot)\|^p_{L^p} \\ & \geq c(\Gamma + \gamma_p), \\ \textit{(Lagrangian)} \quad & \epsilon^{-\frac{p}{3}} k^{(-3+p)H+1} \|G_k \bar{u}_k(T, \eta^k(T,\cdot)) - \bar{u}_k(T, \eta(T,\cdot))\|^p_{L^p} \\ & = \epsilon^{-\frac{p}{3}} k^{(-3+p)H+1} \|\tau_k\|^p_{L^p(D_k)} \leq c\Gamma \end{aligned}$$

for any $k > 1$ (c.f. (2.3) in [17]). These estimates suggest that, in order to capture the coherent vortical structures in infinitesimal small scale vortices, the Eulerian framework is not enough.

Remark 3. *It is well-known that the Euler equations in $B^{1/3}_{3,\infty}$ is already illposed. See Cheskidov-Shvidkoy [2]. They proved the illposedness by using 2D shear flow structure. On the other hand, our construction is just using Galilean boost, so, our initial data is nothing to do with theirs.*

Proof. The Lagrangian case is trivial, since, by change of variables: $\eta(T,x) \to x$ with (14), we have

$$\begin{aligned} &\|G_k \bar{u}_k(T, \eta^k(T,\cdot)) - \bar{u}_k(T, \eta(T,\cdot))\|^p_{L^p} = \|G_k \bar{u}_k(T, \cdot + \tau_k T) - \bar{u}_k(T,\cdot)\|^p_{L^p} \\ =&\|\bar{u}_k(T,\cdot) + \tau_k - \bar{u}_k(T,\cdot)\|^p_{L^p(D_k)} = \|\tau_k\|^p_{L^p(D_k)}. \end{aligned}$$

Now we show the lower bound of the Eulerian case. By (5) and (13), we have

$$\operatorname{supp} \mathrm{Ad}_{\Phi_{j,k}} U_{j,k}(k\cdot) \bigcap \operatorname{supp} G_k \mathrm{Ad}_{\Phi_{j,k}} U_{j,k}(k,\cdot) = \emptyset.$$

Also, by (10), we have

$$\operatorname{supp} \mathrm{Ad}_{\Phi_{j,k}} U_{j,k}(k\cdot) \bigcap \operatorname{supp} \mathrm{Ad}_{\Phi_{j',k}} U_{j',k}(k,\cdot) = \emptyset,$$
$$\operatorname{supp} \mathrm{Ad}_{\Phi_{j,k}} U_{j,k}(k\cdot) \bigcap \operatorname{supp} G_k \mathrm{Ad}_{\Phi_{j',k}} U_{j',k}(k,\cdot) = \emptyset$$

for $j \neq j'$. Thus,

$$\|G_k \bar{u}_k(T,\cdot) - \bar{u}_k(T,\cdot)\|_{L^p}^p = \|G_k \bar{u}_k(T,\cdot)\|_{L^p}^p + \|\bar{u}_k(T,\cdot)\|_{L^p}^p.$$

By this equality and (7), we immediately have the desired lower bound. □

4 Appendix

In this appendix, we reformulate time-asymmetry of short-time Lagrangian dispersion in terms of coherent vortical structure. This reformulation is essentially based on Goto-Vassilicos [10], and the key hypothesis of this reformulation is that particle pairs aggregate in stepwise. Let us explain it more precisely. Let $\xi = a^{-1}(< 1)$ be the adjacent length scale of vortices, and let ξ_0 be the initial length scale. If two particles belong to a same vortex of the length scale $\xi^n\xi_0$, their distance remains in the order of $\xi^n\xi_0$. Let $T(\xi^n\xi_0)$ be the characteristic time for pair aggregation from $\xi^n\xi_0$ to $\xi^{n+1}\xi_0$, and assume this characteristic time is comparable to the turnover time $\sim \epsilon^{-\frac{1}{3}}\xi^{n(1-H)}$. Also we assume that the particle pair aggregation evolves in a series of some physical process (which is the crucial hypothesis, but not well clarified so far) as follows:

$$\xi_0 \overset{T(\xi_0)}{\longrightarrow} \xi\xi_0 \overset{T(\xi\xi_0)}{\longrightarrow} \xi^2\xi_0 \overset{T(\xi^2\xi_0)}{\longrightarrow} \cdots .$$

The time t can be estimated as follows:

$$t = \sum_{j=0}^{n-1} T(\xi^j\xi_0) = \epsilon^{-\frac{1}{3}}\xi_0 \frac{1-\xi^{(1-H)n}}{1-\xi^{1-H}},$$

and then we switch it to the backward-in-time t^*:

$$t^* := \epsilon^{-1/3}\xi_0 \frac{1}{1-\xi^{1-H}} - t = \epsilon^{-1/3}\xi_0 \frac{\xi^{n(1-H)}}{1-\xi^{1-H}} = C\epsilon^{-1/3}\xi^{n(1-H)},$$

where $C := \epsilon^{-\frac{1}{3}}\xi_0 \frac{1}{1-\xi^{1-H}}$. In this case we can figure out the hierarchy exponent n as follows:

$$n = \frac{1}{1-H}\log(C^{-1}\epsilon^{\frac{1}{3}}t^*) = \log(C^{-1}\epsilon^{\frac{1}{3}}t^*)^{\frac{1}{1-H}}.$$

The estimate of the square aggregation (if it occurs for some two particles) is ξ^{2n}, and the corresponding probability with intermittency exponent is $\xi^{n(3-D)}$. Thus, with the aid of the exponent relation $3H + 2 - D = 0$, the mean square aggregation can be estimated as follows:

$$\xi^{2n}\xi^{n(3-D)} = (C^{-1}\epsilon^{\frac{1}{3}}t_*)^{\frac{2}{1-H}}(C^{-1}\epsilon^{\frac{1}{3}}t_*)^{\frac{3-D}{1-H}} = C^{-3}\epsilon t_*^3.$$

This desired estimate is consistent with [5].

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