An elementary approach to the exact WKB analysis of the Pearcey system with a large parameter

By

Takashi Aoki*, Takao Suzuki** and Shofu Uchida***

Abstract

This is an exposition of the article [3]. We investigate the Pearcey integral with a large parameter from the viewpoint of the exact WKB analysis of holonomic systems. We see that the Borel transform of the WKB solutions to the holonomic system for the Pearcey integral become algebraic functions.

§ 1. Introduction

This article gives an exposition of some results obtained in [3]. The aim of the paper is to investigate the Pearcey integral with a large parameter from the viewpoint of the exact WKB analysis in an elementary way. Our main subject is the integral

$$u = \int \exp\left(\eta(z^4 + x_2 z^2 + x_1 z)\right) dz. \tag{1.1}$$

Here η is a positive large parameter. In [1], the first author proposed an exact WKB theoretic approach to this integral and this approach has been intensively developed by Hirose and his co-authors [4, 5, 6, 7, 8, 9]. In [3], we investigate the integral (1.1) from a slightly different point of view from that of those articles, that is, we use the holonomic system in the variable (x_1, x_2, η) that characterizes the linear subspace spanned by (1.1). Using the system and its Borel transform, we find that the Borel transform of the WKB solutions to the system can be expressed as linear combinations of the branches of an

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*Kindai University, Kowakae 3-4-1, Higashi-Osaka, 577-8502, Japan.
e-mail: aoki@math.kindai.ac.jp
**Kindai University, Kowakae 3-4-1, Higashi-Osaka, 577-8502, Japan.
e-mail: suzuki@math.kindai.ac.jp
***Kindai University, Kowakae 3-4-1, Higashi-Osaka, 577-8502, Japan.
e-mail: uchida@math.kindai.ac.jp

algebraic function defined by a quartic equation. This fact gives global information of the Borel transforms and hence that of the Borel sum of the WKB solutions only by using elementary properties of algebraic equations. Some of the results are also announced in [2].

§ 2. The Pearcey system and its Borel transform

Let P_i $(i = 1, 2, 3, 4)$ be partial differential operators defined by

$$\begin{aligned} P_1 &= 4\partial_1\partial_2 + 2\eta x_2\partial_1 + \eta^2 x_1, \\ P_2 &= 4\partial_2^2 + \eta x_1\partial_1 + 2\eta x_2\partial_2 + \eta, \\ P_3 &= \eta\partial_2 - \partial_1^2, \\ P_4 &= 3x_1\partial_1 + 2x_2\partial_2 - 4\eta\partial_\eta - 1. \end{aligned}$$

We can see that $P_i u = 0$ holds for $i = 1, 2, 3, 4$. The first three equations are given in [10] and the fourth equation comes from the weighted homogeneity of (1.1) in (x_1, x_2, η). Let D denote the Weyl algebra in the variable (x_1, x_2, η). Let I be the left ideal in D generated by P_i $(i = 1, 2, 3, 4)$. We denote by M the left D-module defined by I:

$$M = D/I.$$

Lemma 2.1. *M is a holonomic system of rank 3.*

In the integral (1.1), we take an infinite path connecting two valleys of the integrand. There are three independent such paths and the linear subspace spanned by (1.1) has the dimension 3. Hence M characterizes this linear subspace. We call M the Pearcey system with the large parameter.

Let us consider the system of partial differential equations consisting of first three equations $P_i\psi = 0$ $(i = 1, 2, 3)$ of M. This system is holonomic in the variable (x_1, x_2) and it equivalent to the system

$$\begin{cases} Q_1\psi = 0, \\ Q_2\psi = 0. \end{cases} \tag{2.1}$$

Here we set

$$Q_1 = 4\partial_1^3 + 2\eta^2 x_2\partial_1 + \eta^3 x_1, \quad Q_2 = P_3.$$

In fact, we have

$$\begin{aligned} P_1 &= \eta^{-1}\left(Q_1 + 4\partial_1 Q_2\right), \\ P_2 &= \eta^{-2}\partial_1 Q_1 + \left(4\eta^{-2}Q_2 + 8\eta^{-2}\partial_1^2 + 2x_2\right)Q_2, \\ Q_1 &= \eta P_1 - 4\partial_1 P_3. \end{aligned}$$

Using this system and setting $S^{(1)} = \partial_1\psi/\psi$, $S^{(2)} = \partial_2\psi/\psi$, we find a system of nonlinear partial differential equations for $S^{(1)}, S^{(2)}$:

$$\begin{cases} 4\left(S^{(1)}\right)^3 + 2\eta^2 x_2 S^{(1)} + \eta^3 x_1 + 12 S^{(1)}\partial_1 S^{(1)} + 4\partial_1^2 S^{(1)} = 0, \\ \eta S^{(2)} - \partial_1 S^{(1)} - \left(S^{(1)}\right)^2 = 0. \end{cases} \tag{2.2}$$

Formal solutions

$$S^{(1)} = \sum_{j=-1}^{\infty} \eta^{-j} S_j^{(1)}, \quad S^{(2)} = \sum_{j=-1}^{\infty} \eta^{-j} S_j^{(2)}$$

to (2.2) can be constructed as follows. We put the above expressions into (2.2) and compare the coefficients of the like powers of η^{-j}. Then we find that the leading term $S_{-1}^{(1)}$ of $S^{(1)}$ should satisfy the cubic equation

$$(S_{-1}^{(1)})^3 + 2x_2 S_{-1}^{(1)} + x_2 = 0$$

and if we choose a root ζ of the cubic equation

$$\zeta^3 + 2x_2\zeta + x_1 \tag{2.3}$$

as $S_{-1}^{(1)}$, we can determine successively other terms by

$$\begin{cases} S_0^{(1)} = -\frac{1}{2}\partial_1 \log\left(6(S_{-1}^{(1)})^2 + x_2\right), \\ S_j^{(1)} = -\frac{2}{6(S_{-1}^{(1)})^2 + x_2}\left(\sum_{\substack{j_1+j_2+j_3=j-2 \\ -1\le j_1,j_2,j_3<j}} S_{j_1}^{(1)} S_{j_2}^{(1)} S_{j_3}^{(1)} \right. \\ \qquad\qquad \left. +3\sum_{\substack{j_1+j_2=j-2 \\ -1\le j_1,j_2<j}} S_{j_1}^{(1)}\partial_1 S_{j_2}^{(1)} + \partial_1^2 S_{j-2}^{(1)}\right) \quad (j\ge 1), \end{cases} \tag{2.4}$$

$$\begin{cases} S_{-1}^{(2)} = \left(S_{-1}^{(1)}\right)^2, \\ S_j^{(2)} = \sum_{m=0}^{j+1} S_{m-1}^{(1)} S_{j-m}^{(1)} + \partial_1 S_{j-1}^{(1)} \quad (j\ge 0). \end{cases} \tag{2.5}$$

We set $\omega = S^{(1)}dx_1 + S^{(2)}dx_2$. Then we have $d\omega = 0$. Hence a formal solution

$$\psi = \exp\left(\int \omega\right)$$

to (2.1) is obtained and it is called a WKB solution in [1, 4, 5]. We note that this construction treats η as a parameter. Next we consider the equation

$$P_4\psi = 0.$$

That is, we consider η as a variable. Then the normalization of the integral $\int \omega$ is uniquely determined up to additive pure constants by the following lemma.

Lemma 2.2. *Let ζ_ℓ $(\ell = 1,2,3)$ be the roots of* (2.3) *(for a fixed general (x_1, x_2)) and set $S_{-1}^{(1),\ell} = \zeta_\ell$. Accordingly, we have three formal solutions $(S^{(1),\ell}, S^{(2),\ell})$ $(\ell = 1,2,3)$ to* (2.2)*. Set $\omega^{(\ell)} = S^{(1),\ell}dx_1 + S^{(2),\ell}dx_2$ and write*

$$\omega^{(\ell)} = \sum_{j=-1}^{\infty} \eta^{-j}\omega_j^{(\ell)}.$$

Consider formal solutions $\psi_\ell = \eta^{-1/2}\exp\left(\int \omega^{(\ell)}\right)$ to (2.1) *$(\ell = 1,2,3)$. Then ψ_ℓ satisfies $P_4\psi_\ell = 0$ if and only if the normalization of the integrals are taken so that*

$$\int \omega_0^{(\ell)} = -\frac{1}{2}\log\left(6\left(S_{-1}^{(1),\ell}\right)^2 + x_2\right) + C, \tag{2.6}$$

$$\int \omega_j^{(\ell)} = -\frac{1}{4j}\left(3x_1 S_j^{(1),\ell} + 2x_2 S_j^{(2),\ell}\right) \quad (j \geq -1, j \neq 0) \tag{2.7}$$

hold. Here C is an arbitrary constant.

We call ψ_ℓ with the normalization of integrals (2.6), (2.7) the WKB solutions to M. The Borel transform of ψ_ℓ is denoted by $\psi_{\ell,B}$ for $\ell = 1,2,3$. Let $P_{k,B}$ denote the formal Borel transform of P_k $(k = 1,2,3,4)$. By the definition, we have

$$\begin{aligned}
P_{1,B} &= 4\partial_1\partial_2 + 2x_2\partial_y\partial_1 + x_1\partial_y^2, \\
P_{2,B} &= 4\partial_2^2 + x_1\partial_y\partial_1 + 2x_2\partial_y\partial_2 + \partial_y, \\
P_{3,B} &= \partial_y\partial_2 - \partial_1^2, \\
P_{4,B} &= 3x_1\partial_1 + 2x_2\partial_2 - 4\partial_y(-y) - 1 \\
&\quad (= 3x_1\partial_1 + 2x_2\partial_2 + 4y\partial_y + 3)
\end{aligned}$$

and $P_{k,B}\psi_{\ell,B} = 0$ for $k = 1,2,3,4; \ell = 1,2,3$. Here y denotes the variable of the Borel plane.

Let D_B denote the Weyl algebra of the variable (x_1, x_2, y) and I_B the left D_B-ideal generated by $P_{k,B}$ $(k = 1,2,3,4)$. We set $M_B = D_B/I_B$.

Theorem 2.3. *M_B is a holonomic system of rank 3.*

Thus M_B characterizes the subspace of analytic functions spanned by $\psi_{\ell,B}$ $(\ell = 1,2,3)$.

Remark. As is mentioned above, the system $P_i\psi = 0$ $(i = 1,2,3)$ is equivalent to the system $Q_i\psi = 0$ $(i = 1,2)$ in the variable (x_1, x_2). But we do not consider the system

M' defined by the ideal generated by Q_i $(i = 1, 2)$ and P_4 instead of M. The holonomic rank of M' is also 3, however, in addition to 3 dimensional analytic solution space, it has 3 dimensional redundant solutions expressed in terms of the delta function:

$$\psi = c_0 x_2^{-3/2}\delta(\eta) + c_1 x_1 x_2^{-3}\delta(\eta) + c_2\left(x_1^2 x_2^{-9/2}\delta(\eta) + \frac{4}{7}x_2^{-7/2}\delta'(\eta)\right),$$

where c_1, c_2, c_3 are arbitrary constants. Hence the Borel transform M'_B of M', that is, the system

$$Q_{i,B}\varphi = 0 \ (i = 1, 2),\ \ P_{4,B}\varphi = 0$$

has rank 6 (This was pointed out by Hirose.) whereas it is holonomic.

§ 3. Borel transform of WKB solutions

We set $y = -(t^4 + x_2 t^2 + x_1 t)$ in the right-hand side of (1.1). Then we have

$$u = \int \exp(-\eta y) g(x_1, x_2, \eta) dy.$$

Here g is defined by

$$g(x_1, x_2, y) = \left.\frac{1}{4t^3 + 2x_2 t + x_1}\right|_{t=t(x_1,x_2)}$$

with t being a root of the quartic equation $t^4 + x_2 t^2 + x_1 t = -y$. The path of integration is suitably modified.

Lemma 3.1. *The function g defined as above satisfies the quartic equation*

$$\begin{aligned}(4x_1^2 x_2(36y - x_2^2) + 16y(x_2^2 - 4y)^2 - 27x_1^4)\, g^4 \\ + 2(-8x_2 y + 2x_2^3 + 9x_1^2)\, g^2 - 8x_1 g + 1 = 0\end{aligned}$$

and it is a solution to the holonomic system M_B.

For general (x_1, x_2, y), there are four roots g_k $(k = 1, 2, 3, 4)$ of the quartic equation, which satisfy $g_1 + g_2 + g_3 + g_4 = 0$. Looking at the singularity of g_k, we find that any three of g_k's are linearly independent. Thus we have

Theorem 3.2. *The Borel transform $\psi_{j,B}$ of the WKB solution ψ_ℓ $(\ell = 1, 2, 3)$ can be written as a linear combination of any three of g_k's. In particular, $\psi_{j,B}$'s are algebraic and hence the WKB solutions are resurgent.*

We see that ψ_ℓ are Borel summable for general (x_1, x_2) as a corollary (cf. [11]). If $|x_2|$ is sufficiently small, we can specify the branch of g_k, $\psi_{\ell,B}$ so that

$$\psi_{k,B} = (-1)^k \frac{i}{\sqrt{\pi}}(g_k - g_4) \quad (k = 1, 2, 3)$$

holds. This relation implies that $\psi_{k,B}$ is described as a constant multiple of the discontinuity of a suitably specified root of the quartic equation at a singularity. Using the quartic equation, we can take the analytic continuation of $\psi_{k,B}$ with respect to the variable y and hence we may obtain the connection formulas for the WKB solutions ψ_k elementarily. See [3] for details.

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