On 1-summability of formal solution of inhomogeneous heat equation

By

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Abstract

We consider the Cauchy problem for the inhomogeneous heat equation, where inhomogeneous team is a formal power series of Gevrey order 1 with respect to t. Under the assumptions of the 1-summability of the inhomogeneous term and a global exponential growth condition with respect to x for its sum, we show the 1-summability of the formal solution of the Cauchy problem by using the integral representation of the 1-sum of the formal solution which is given in terms of the heat kernel.

§ 1. Introduction

We consider the following Cauchy problem for the complex inhomogeneous heat equation

$$(\widehat{H}) \qquad \begin{cases} \partial_t u(t,x) = \partial_x^2 u(t,x) + \hat{f}(t,x), \\ u(0,x) = \varphi(x) \in \mathcal{O}_x, \end{cases}$$

where $t, x \in \mathbb{C}$ and $\mathcal{O}_x$ denotes the set of holomorphic functions in a neighborhood of $x = 0$. The inhomogeneous term $\hat{f}(t,x) = \sum_{i\geq 0} f_i(x)t^i/i!$ is a formal power series of Gevrey order 1 with respect to t which means that all coefficients $f_i(x)$ are holomorphic in a common closed disc $\overline{D}_\sigma := \{x \in \mathbb{C}; |x| \leq \sigma\}$ for some positive σ and there exist positive constants A and B such that for all i

$$\max_{|x|\leq\sigma} |f_i(x)| \leq AB^i i!^2$$

and we denote it by $\hat{f}(t,x) \in \mathcal{O}_x[[t]]_1$.

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This Cauchy problem has a unique formal power series solution of the form

$$\hat{u}(t,x) = \sum_{n\geq 0} \frac{\varphi^{(2n)}(x)}{n!} t^n + \sum_{n\geq 1} \left(\sum_{i=0}^{n-1} f_i^{(2(n-1-i))}(x) \right) \frac{t^n}{n!} \tag{1.1}$$
$$=: \hat{u}_{hom}(t,x;\varphi) + \hat{u}_{inh}(t,x;\hat{f}),$$

which is divergent in general. Exactly, we can see that $\hat{u}(t,x)$ is the formal power series of Gevrey order 1 with respect to t.

For the divergent solution when the inhomogeneous term $\hat{f}(t,x) \equiv 0$, the problem of k-summability with $k=1$ for the formal solution $\hat{u}(t,x) = \hat{u}_{hom}(t,x;\varphi)$ was proved by Lutz, Miyake and Schäfke [5], where the definition of k-summability will be given in next section.

Theorem 1.1 ([5]). *Let $S_x(0,\pi;\varepsilon_1,\sigma) = S_x(0;\varepsilon_1) \cup S_x(\pi;\varepsilon_1) \cup \overline{D}_\sigma$, where $S_x(\theta;\varepsilon_1) := \{x \in \mathbb{C}; |\arg x - \theta| < \varepsilon_1/2\}$ and $\varepsilon_1 > 0$. Then the formal solution $\hat{u}(t,x)$ of the homogeneous Cauchy problem $(\widehat{H})$ is 1-summable in 0 direction (we denote it by $\hat{u}(t,x) \in \mathcal{O}_x\{t\}_{1,0}$) if and only if the Cauchy data $\varphi(x) \in \mathcal{O}_x$ satisfies the following conditions.*
(i) *The Cauchy data $\varphi(x)$ can be analytically continued on a region $S_x(0,\pi;\varepsilon_1,\sigma)$.*
(ii) *The Cauchy data has the exponential growth estimate of order at most 2 there, that is, $|\varphi(x)| \leq Ce^{\delta|x|^2}$ $(x \in \overline{S}_x(0,\pi;\varepsilon_1',\sigma))$ with some positive constants C and δ for any closed subsector $\overline{S}_x(0,\pi;\varepsilon_1',\sigma) \subset S_x(0,\pi;\varepsilon_1,\sigma)$.*

In this case, 1-sum of $\hat{u}_{hom}(t,x;\varphi)$ in 0 direction is obtained by

$$u^0_{hom}(t,x;\varphi) := \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{+\infty} \varphi(x+y) e^{-y^2/4t} dy \tag{1.2}$$

with $|t-\rho_0| < \rho_0$ and $|x| \leq \sigma_0$ for some positive ρ_0 and $\sigma_0 (< \sigma)$.

We remark that 1-sum $u^0_{hom}(t,x;\varphi)$ in a sector $S_t(0,\alpha,\rho)$ with some $\alpha > \pi$ and $\rho > 0$ in t-space is obtained from the analytic continuation in t-variable by rotating the integral path $\mathbb{R}$ to $e^{i\theta}\mathbb{R}$ with $|\theta| < \varepsilon_1/2$.

In the following, we write the conditions (i) and (ii) by

$$\varphi(x) \in \mathrm{Exp}^2(S_x(0,\pi;\varepsilon_1,\sigma))$$

and we call the conditions "1-summability condition" or **1-S-C** (with ε_1 and σ) for short.

We consider the case that $\hat{f}(t,x) \not\equiv 0$ and $\hat{f}(t,x) \in \mathcal{O}_x[[t]]_1$. We may assume that the Cauchy data $\varphi(x) \equiv 0$ without loss of generality. In this case, Michalik [7] gave a sufficient condition for 1-summability of the formal solution $\hat{u}(t,x) = \hat{u}_{inh}(t,x;\hat{f})$.

Theorem 1.2 ([7]). *Let $g(s,x)$ be the formal* 1*-Borel transform of $\hat{f}(t,x)$ which is defined by $g(s,x) = \sum_{i\geq 1} f_i(x)s^i/i!^2$, which is convergent at $(s,x) = (0,0)$. We assume that $g(s,x)$ can be analytically continued to a sector $S_s(0,\varepsilon)$ in s-space by some positive ε and has the exponential growth estimate of order at most* 1 *there. Moreover, we assume that $g(s,x)$ satisfies* **1-S-C** *with respect to x-variable. Then the formal solution $\hat{u}(t,x)$ of the inhomogeneous Cauchy problem $(\widehat{H})$ is* 1*-summable in* 0 *direction.*

Under the above same conditions we obtain the following result for 1-sum of $\hat{u}(t,x)$ of the inhomogeneous Cauchy problem $(\widehat{H})$ in terms of the integral representation by using heat kernel.

Theorem 1.3. *We assume that $\hat{f}(t,x) \in \mathcal{O}_x\{t\}_{1,0}$ and let $f^0(t,x)$ be* 1*-sum of $\hat{f}(t,x)$. We assume that there exists a positive constant T such that $f^0(t,x)$ is analytic in $S_T \times \overline{D}_\sigma$, where $S_T = S_t(0,\alpha,T)$ with $\alpha > \pi$. Moreover, we assume that the sum $f^0(t,x)$ satisfies* **1-S-C** *with ε_1 and σ, that is, $f^0(t,x)$ can be analytically continued to $S_x(0,\pi;\varepsilon_1,\sigma)$ with respect to x-variable and $|f^0(t,x)| \leq Ce^{\delta|x|^2}$ $(t \in \overline{S}_1)$ for $x \in \overline{S}_x(0,\pi;\varepsilon_1',\sigma)$ with some positive constants C and δ for any closed subsector $\overline{S}_1 \subset S_T$. Then*

$$u^0_{inh}(t,x;f^0) := \int_0^t \frac{1}{\sqrt{4\pi s}} \int_{-\infty}^{\infty} e^{-y^2/4s} f^0(t-s,x+y)\,dy\,ds \tag{1.3}$$

with $|t-\rho_0| < \rho_0$ and $|x| \leq \sigma_0$ for some positive constants $\rho_0(\leq T/2)$ and $\sigma_0(<\sigma)$ gives 1*-sum of $\hat{u}(t,x)$ in* 0 *direction and satisfies $(\widehat{H})$ with $f^0(t,x)$ instead of $\hat{f}(t,x)$.*

This paper consists of the following contents. In section 2, we give the definition of k-summability, and in section 3 we give related results for the 1-summability of the formal solution of inhomogeneous Cauchy problem. We prove Theorem 1.3 in section 4, and give a proof of Lemma 4.1 necessary for the proof of the theorem in section 5. In section 6, we give a remark on the integral representation of 1-sum in terms of heat kernel.

§ 2. Definition of k-summability

In this section, we give some notation and definitions in the way of Ramis or Balser (cf. Balser [1] for the details).

For $d \in \mathbb{R}, \beta > 0$ and $\rho(0 < \rho \leq \infty)$, we define a sector $S = S_t(d,\beta,\rho)$ by

$$S_t(d,\beta,\rho) := \{t \in \mathbb{C}; |d - \arg t| < \beta/2, 0 < |t| < \rho\}, \tag{2.1}$$

where d, β and ρ are called the direction, the opening angle and the radius of S, respectively. We write $S_t(d,\beta,\infty) = S_t(d;\beta)$ for short.

A closed sector $\overline{S} = \overline{S}_t(d, \beta, \rho)$ is defined by $\overline{S} = \{t \in \mathbb{C}; |d - \arg t| \le \beta/2, 0 < |t| \le \rho\}$.

For $k > 0$, we define that $\hat{v}(t,x) = \sum_{n=0}^{\infty} v_n(x) t^n \in \mathcal{O}_x[[t]]_{1/k}$ (we say $\hat{v}(t,x)$ is a formal power series of Gevrey order $1/k$) if $v_n(x)$ are holomorphic on a common closed disk $\overline{D}_\sigma$ for some $\sigma > 0$ and there exist some positive constants C and K such that for any n,

$$\max_{|x| \le \sigma} |v_n(x)| \le CK^n \Gamma\left(1 + \frac{n}{k}\right), \tag{2.2}$$

where Γ denotes the gamma function. Here when $v_n(x) \equiv v_n$ (constants) for all n, we use the notation $\mathbb{C}[[t]]_{1/k}$ instead of $\mathcal{O}_x[[t]]_{1/k}$. In the following, we use the similar notation.

Let $k > 0$, $\hat{v}(t,x) = \sum_{n=0}^{\infty} v_n(x) t^n \in \mathcal{O}_x[[t]]_{1/k}$ and $v(t,x)$ be an analytic function on $S_t(d, \beta, \rho) \times \overline{D}_\sigma$. Then we define that

$$v(t,x) \cong_k \hat{v}(t,x) \quad \text{in } S = S_t(d, \beta, \rho) \tag{2.3}$$

(we say $v(t,x)$ has the Gevrey asymptotic expansion $\hat{v}(t,x)$ of order k), if for any closed subsector $\overline{S'}$ of S, there exist some positive constants C and K such that for any $N \ge 1$, we have

$$\max_{|x| \le \sigma} \left| v(t,x) - \sum_{n=0}^{N-1} v_n(x) t^n \right| \le CK^N |t|^N \Gamma\left(1 + \frac{N}{k}\right), \quad t \in \overline{S'}. \tag{2.4}$$

For $k > 0$, $d \in \mathbb{R}$ and $\hat{v}(t,x) \in \mathcal{O}_x[[t]]_{1/k}$, we say that $\hat{v}(t,x)$ is *k-summable* in d direction, and denote it by $\hat{v}(t,x) \in \mathcal{O}_x\{t\}_{k,d}$, if there exist a sector $S = S_t(d, \beta, \rho)$ with $\beta > \pi/k$ and an analytic function $v(t,x)$ on $S \times \overline{D}_\sigma$ such that $v(t,x) \cong_k \hat{v}(t,x)$ in S.

In the paper, we consider the direction as 0 direction only for simplicity. Therefore we use the notation $\mathcal{O}_x\{t\}_{k,0} = \mathcal{O}_x\{t\}_k$.

We remark that the function $v(t,x)$ above for a *k-summable* $\hat{v}(t,x)$ is unique if it exists. Therefore such a function $v(t,x)$ is called the *k-sum* of $\hat{v}(t,x)$ in 0 direction and we write it $v^0(t,x)$.

§ 3. Related results

Balser [2] gave the necessary and sufficient condition for 1-summability of the formal solution $\hat{u}(t,x) = \hat{u}_{inh}(t,x;\hat{f})$ of the inhomogeneous Cauchy problem $(\widehat{H})$. Here we use the notations.

$$\hat{u}(t,x) = \sum_{j \ge 0} u_{j*}(x) \frac{t^j}{j!} = \sum_{n \ge 0} \hat{u}_{*n}(t) \frac{x^n}{n!} = \sum_{j,n} u_{jn} \frac{t^j}{j!} \frac{x^n}{n!}.$$

Then Balser's result is stated as follows.

Proposition 3.1 ([2]). *The formal solution $\hat{u}(t,x)$ is 1-summable in a direction 0 if and only if $\hat{u}_{*0}(t), \hat{u}_{*1}(t)$ and $\hat{f}(t,x)$ are 1-summable in 0 direction.*

In the above Proposition, we can't know what the 1-summability of $\hat{u}_{*0}(t)$ and $\hat{u}_{*1}(t)$ mean. In the paper of Balser and Loday-Richaud [3], they tried to characterize 1-summability of $\hat{u}_{*0}(t)$ and $\hat{u}_{*1}(t)$ as a property of $\hat{f}(t,x)$ as follows.

For $D_t^{-1}\hat{f}(t,x) = \sum_{j\geq 1, n\geq 0} f_{j-1,n}\frac{t^j}{j!}\frac{x^n}{n!}$ with $D_t^{-1} = \int_0^t$, we put

$$g(t,x) := (\mathcal{L}_t\mathcal{L}_x D_t^{-1}\hat{f})(t,x) = \sum\nolimits_{j\geq 1, n} f_{j-1,n}t^j x^n,$$

where $\mathcal{L}_t$ is defined by $\mathcal{L}_t t^j = j!t^j$ and $\mathcal{L}_x$ is also same. Moreover, for

$$g(t,t^{1/2}) = \sum\nolimits_{j,\ell} f_{j-1,2\ell}t^{j+\ell} + t^{1/2}\sum\nolimits_{j,\ell} f_{j-1,2\ell+1}t^{j+\ell},$$

we put

$$\hat{G}(\tau) := ((\mathcal{B}_t^{[1]}g)(t,t^{1/2}))\,|_{t=\tau^2}$$

which is a formal series in τ-variable. Here $\mathcal{B}_t^{[1]}$ is defined by $\mathcal{B}_t^{[1]}(t^{n+i/2}) = t^{n+i/2}/n!$ for $i = 0, 1$. In this case, since we see

$$\hat{G}(t^{1/2}) = \hat{u}_{*0}(t) + t^{1/2}\hat{u}_{*1}(t),$$

they gave the following proposition.

Proposition 3.2 ([3]). *The series $\hat{u}_{*0}(t)$ and $\hat{u}_{*1}(t)$ are 1-summable in 0 if and only if the series $\hat{G}(\tau)$ associated with $\hat{f}$ is 2-summable in the directions 0 and π.*

Proposition 3.1 was extended to the heat equation with variable coefficient by Balser and Loday-Richaud [3], higher order linear partial differential equation with variable coefficients by Remy [9, 10], semilinear heat equation with variable coefficients by Remy [11], semilinear higher order equation with variable coefficients by Remy [12].

Balser [2] also gave the another necessary and sufficient condition for 1-summability of $\hat{u}(t,x) = \hat{u}_{inh}(t,x;\hat{f})$, which was refined by Michalik [6].

Proposition 3.3 ([2, 6]). *For the inhomogeneous term $\hat{f}(t,x) = \sum_{i\geq 0} f_i(x)t^i/i! \in \mathcal{O}_x[[t]]_1$, we define $g(s,x)$ and $h(\tau)$ by*

$$g(s,x) := \sum\nolimits_{i\geq 0} f_i(x)s^i/(2i)!,$$

which is convergent at $(s,x) = (0,0)$ and

$$h(\tau) := \partial_\tau \int_0^\tau g((\tau - s)^2, s)ds,$$

respectively. Then the following statements are equivalent.

i) $\hat{u}(t,x)$ *is* 1*-summable in* 0 *direction.*

ii) $h(\tau)$ *satisfies* **1-S-C**, *that is,* $h(\tau)$ *can be analytically continued to* $S_\tau(0,\pi;\varepsilon_1,\sigma)$ *and*

$$|h(\tau)| \le Ce^{\delta|\tau|^2}, \quad \tau \in \overline{S}_\tau(0,\pi;\varepsilon_1',\sigma),$$

for all $\varepsilon_1' < \varepsilon_1$.

Proposition 3.3 was extended to $1/p$-fractional equations by Michalik [7] and moment partial differential equations by Michalik [8].

§ 4. A proof of Theorem 1.3

First, from the analytic continuation of the integral representation of $u^0(t,x) = u_{inh}(t,x;f^0)$, and the assumption that $f^0(t,x)$ satisfies 1-**S-C**, we can see that $u^0(t,x)$ is analytic in a sector $S_t(0,\alpha,\rho) \times \overline{D}_{\sigma_0}$ with $\alpha > \pi$ for some sufficiently small positive constants ρ and σ_0.

Next, we shall show that $u^0(t,x)$ has the Gevrey asymptotic expansion $\hat{u}(t,x)$ of order 1, which completes a proof of Theorem 1.3(cf. [4]).

We put

$$\hat{u}(t,x) = \sum_{n\ge1}\left(\sum_{i=1}^{n} f_{i-1}^{(2(n-i))}(x)\right)\frac{t^n}{n!} =: \sum_{n\ge1} u_n(x)t^n \in \mathcal{O}_x[[t]]_1.$$

For all $M \in \mathbb{N}$, we put

$$\begin{aligned}\hat{f}(t,x) &= \sum_{i\ge1} f_{i-1}(x)\frac{t^{i-1}}{(i-1)!} \;=\; \left\{\sum_{i=1}^{M} + \sum_{i>M}\right\} f_{i-1}(x)\frac{t^{i-1}}{(i-1)!}\\ &=: \hat{f}_M(t,x) + \hat{f}^M(t,x).\end{aligned}$$

Moreover, we put

$$F_M(t,x) := f^0(t,x) - \hat{f}_M(t,x). \tag{4.1}$$

From the assumptions that $\hat{f}(t,x) = \sum_{i\ge0} f_i(x)t^i/i! \in \mathcal{O}_x\{t\}_1$ and $f^0(t,x)$ satisfies 1-**S-C**, we can prove the following lemma, which will be proved in the next section.

Lemma 4.1.

(4.2) $|f_i(x)| \le C_1K_1^i i!^2 e^{\delta|x|^2}, \quad x \in \overline{S}_x(0,\pi;\varepsilon_1',\sigma),$

(4.3) $|f_i^{(n)}(x)| \le C_2K_2^{i+n} i!^2 n! e^{\tilde{\delta}|x|^2}, \quad x \in \overline{S}_x(0,\pi;\varepsilon_1'',\sigma') \quad (\varepsilon_1'' < \varepsilon_1', \sigma' < \sigma/2),$

(4.4) $|F_M(t,x)| \le C_3K_3^M M!|t|^M e^{\delta|x|^2} \quad x \in \overline{S}_x(0,\pi;\varepsilon_1',\sigma),\ t \in \overline{S}_1$

with some positive constants C_j, K_j $(j=1,2,3)$ *and* $\delta, \tilde{\delta}$ *for all* i, n, M. *Here* $\overline{S}_1$ *is any closed subsector of the region* $\{t \in \mathbb{C}; |t-\rho_0| < \rho_0\}$.

We have the following Taylor formula

$$f_{i-1}(x+y) = \sum_{j=0}^{L-1} \frac{f_{i-1}^{(j)}(x)}{j!} y^j + \int_0^y \frac{(y-\eta)^{L-1}}{(L-1)!} f_{i-1}^{(L)}(x+\eta)d\eta \tag{4.5}$$

for all L

By substituting into the integral representation of $u^0(t,x)$ the expression (4.1) with $M = [\frac{N-1}{2}]$ and the Taylor formula (4.5) with $L = N-2i$ for $1 \le i \le [\frac{N-1}{2}]$, we have

$$\begin{aligned}
u^0(t,x) &= \int_0^t \frac{1}{\sqrt{4\pi s}} \int_{-\infty}^{\infty} e^{-y^2/4s} f^0(t-s, x+y)\, dy\, ds \\
&= \int_0^t \frac{1}{\sqrt{4\pi s}} \int_{-\infty}^{\infty} e^{-y^2/4s} \left\{ \hat{f}_M(t-s,x+y) + F_M(t-s,x+y) \right\} dy\, ds \\
&= \sum_{1 \le i \le [(N-1)/2]} \int_0^t \frac{1}{\sqrt{4\pi s}} \sum_{j=0}^{N-2i-1} \frac{f_{i-1}^{(j)}(x)}{j!} \int_{-\infty}^{+\infty} e^{-\frac{y^2}{4s}} y^j\, dy \frac{(t-s)^{i-1}}{(i-1)!} ds \\
&+ \sum_{1 \le i \le [(N-1)/2]} \int_0^t \frac{1}{\sqrt{4\pi s}} \int_{-\infty}^{+\infty} e^{-\frac{y^2}{4s}} \int_0^y \frac{(y-\eta)^{N-2i-1}}{(N-2i-1)!} f_{i-1}^{(N-2i)}(x+\eta) d\eta\, dy \frac{(t-s)^{i-1}}{(i-1)!} ds \\
&\quad + \int_0^t \frac{1}{\sqrt{4\pi s}} \int_{-\infty}^{\infty} e^{-y^2/4s} F_M(t-s,x+y) dy\, ds \\
&=: I_N(t,x) + R_N^1(t,x) + R_N^2(t,x).
\end{aligned}$$

Then from the following lemma, we can get the desired result.

Lemma 4.2. *For* $(t,x) \in \overline{S}_1 \times \overline{D}_{\sigma_0}$ *with any closed subsector* $\overline{S}_1 \subset \{t \in \mathbb{C}; |t-\rho_0| < \rho_0\}$, *we have*

$$I_N(t,x) = \sum_{n=1}^{[(N-1)/2]} \left(\sum_{i=1}^n f_{i-1}^{(2(n-i))}(x) \right) \frac{t^n}{n!}, \tag{4.6}$$

$$\max_{|x| \le \sigma_0} |R_N^1(t,x)| \le C_1 K_1^N |t|^{N/2} \Gamma((N+1)/2), \tag{4.7}$$

$$\max_{|x| \le \sigma_0} |R_N^2(t,x)| \le C_2 K_2^{[(N-1)/2]} |t|^{[(N-1)/2]+1} \Gamma\left(\left[\frac{N-1}{2} \right] + 1 \right) \tag{4.8}$$

with some positive constants C_1, C_2, K_1 *and* K_2 *for any* N.

In fact, when $N = 2m$, since $[(N-1)/2] = m-1$, we have

$$\begin{aligned}
I_N(t,x) &= \sum_{n=1}^{m-1} u_n(x)t^n, \\
|R_N^1(t,x)| &\le C_1 K_1^{2m} |t|^m \Gamma(m+1/2) \le \tilde{C}_1 \tilde{K}_1^m |t|^m \Gamma(m+1), \\
|R_N^2(t,x)| &\le C_2 K_2^{m-1} |t|^m \Gamma(m) \le \tilde{C}_2 K_2^m |t|^m \Gamma(m+1)
\end{aligned}$$

with $C_1 < \tilde{C}_1, C_2 < \tilde{C}_2$ and $K_1 < \tilde{K}_1$.

When $N = 2m+1$, since $[(N-1)/2] = m$, we have

$$\begin{aligned}
I_N(t,x) &= \sum_{n=1}^{m} u_n(x)t^n = \sum_{n=1}^{m-1} u_n(x)t^n + u_m(x)t^m, \\
|R_N^1(t,x)| &\le C_1 K_1^{2m+1} |t|^{m+1/2} \Gamma(m+1) \le \tilde{C}_1 \tilde{K}_1^m |t|^m \Gamma(m+1), \quad (\because |t| < 2\rho_0) \\
|R_N^2(t,x)| &\le C_2 K_2^{m} |t|^{m+1} \Gamma(m+1) \le \tilde{C}_2 K_2^m |t|^m \Gamma(m+1).
\end{aligned}$$

Here we remark that for $|x| \le \sigma$

$$\begin{aligned}
|u_m(x)t^m| &\le \left| \left(\sum_{i=1}^{m} f_{i-1}^{(2(m-i))}(x) \right) \frac{t^m}{m!} \right| \\
&\le \sum C_2 K_2^{(i-1)+2(m-i)} (i-1)!^2 (2(m-i))! e^{\tilde{\delta}|x|^2} \frac{|t|^m}{m!} \\
&\le \tilde{C}_2 \tilde{K}_2^m m! |t|^m.
\end{aligned}$$

From the above observations, we see that $u^0(t,x)$ has the Gevrey asymptotic expansion $\hat{u}(t,x)$ of order 1.

Before giving a proof of Lemma 4.2, we give a formula for $a > 0$ and $b \ge 0$

$$\int_0^\infty e^{-y^2/a} y^b dy = \frac{1}{2} a^{\frac{b+1}{2}} \Gamma\left(\frac{b+1}{2}\right).$$

Let us show Lemma 4.2.

First, we show the equality (4.6). We put $I_j(s) := \int_{-\infty}^{\infty} e^{-y^2/4s} y^j dy$. When j is odd, $I_j(s) = 0$. When $j = 2n$, we have

$$I_{2n}(s) = 2\int_0^\infty e^{-y^2/4s} y^{2n} dy = (4s)^{n+1/2}\Gamma(n+1/2).$$

Therefore by noticing $(2n)! = 2^{2n}\Gamma(n+1/2)n!/\sqrt{\pi}$, we have

$$\begin{aligned} I_N(t,x) &= \sum_{i=1}^{[(N-1)/2]} \sum_{n=0}^{[(N-1)/2]-i} \frac{f_{i-1}^{(2n)}(x)}{(2n)!} \int_0^t \frac{(t-s)^{i-1}}{(i-1)!} s^n ds \frac{4^n\Gamma(n+1/2)}{\sqrt{\pi}} \\ &= \sum_{i=1}^{[(N-1)/2]} \sum_{n=0}^{[(N-1)/2]-i} f_{i-1}^{(2n)}(x) \int_0^t \frac{(t-s)^{i-1}}{(i-1)!} \frac{s^n}{n!} ds \\ &= \sum_{i=1}^{[(N-1)/2]} \sum_{n=0}^{[(N-1)/2]-i} f_{i-1}^{(2n)}(x) \frac{t^{n+i}}{(n+i)!} \\ &= \sum_{i=1}^{[(N-1)/2]} \sum_{n=i}^{[(N-1)/2]} f_{i-1}^{(2(n-i))}(x) \frac{t^n}{n!} \\ &= \sum_{n=1}^{[(N-1)/2]} \sum_{i=1}^{n} f_{i-1}^{(2(n-i))}(x) \frac{t^n}{n!}. \end{aligned}$$

Next, by using the inequality (4.4), we show the inequality (4.8).

We have

$$\begin{aligned} r_N^2(t,s,x) &:= \left| \int_{-\infty}^{\infty} e^{-y^2/4s} F_M(t-s,x+y)\,dy \right| \quad (M = [\tfrac{N-1}{2}]) \\ &\le 2CK^{[(N-1)/2]}\Gamma\left(\left[\tfrac{N-1}{2}\right]+1\right)|t-s|^{[(N-1)/2]} e^{2\delta|x|^2} \int_0^\infty e^{-c_1y^2/4|s|} e^{2\delta y^2} dy \end{aligned}$$

for some positive c_1, where we use the inequality $|x+y|^2 \le 2(|x|^2+|y|^2)$. Here since there exists $c_2 > 0$ such that $c_1 - 8\delta|s| \ge c_2$ for sufficiently small $|s|$, we have

$$\int_0^\infty e^{-c_1y^2/4|s|} e^{2\delta y^2} dy \le \int_0^\infty e^{-c_2y^2/4|s|} dy = \sqrt{\frac{4|s|}{c_2}} \frac{\sqrt{\pi}}{2}.$$

Therefore we have

$$\begin{aligned} \max_{|x|\le\sigma_0} |R_N^2(t,x)| &\le \tilde{C}K^{[(N-1)/2]}\Gamma\left(\left[\tfrac{N-1}{2}\right]+1\right) \left| \int_0^t |t-s|^{[(N-1)/2]} ds \right| \\ &\le \tilde{C}K^{[(N-1)/2]}\Gamma\left(\left[\tfrac{N-1}{2}\right]+1\right) |t|^{[(N-1)/2]+1}. \end{aligned}$$

Finally, by using the inequality (4.3), we show the inequality (4.7).

From the inequality (4.3), we have

$$\begin{aligned} |r_{N1}^1(y,x)| &:= \left| \int_0^y \frac{(y-\eta)^{N-2i-1}}{(N-2i-1)!} f_{i-1}^{(N-2i)}(x+\eta) d\eta \right| \\ &\le CK^{N-i-1}(i-1)!^2 e^{2\delta(|x|^2+|y|^2)} |y|^{N-2i}. \end{aligned}$$

Therefore since

$$\left|\int_{-\infty}^{\infty} e^{-y^2/4s} r^1_{N1}(y,x)dy\right| \le C'K^{N-i}(i-1)!^2 \int_0^{\infty} e^{-c_2 y^2/4|s|} y^{N-2i} dy$$
$$\le C''K'^{N-i}(i-1)!^2 |s|^{(N+1)/2-i}\Gamma((N+1)/2-i)$$

with some positive constants C', C'' and K' for $|x| \le \sigma_0$, we have

$$\max_{|x|\le\sigma_0} |R^1_N(t,x)| \le \sum_i C''K'^{N-i}(i-1)!\Gamma((N+1)/2-i)\left|\int_0^t |s|^{N/2-i}(t-s)^{i-1}ds\right|$$
$$\le C''K'^N \sum_i (i-1)!\Gamma((N+1)/2-i) \times |t|^{N/2}$$
$$\le C_1 K_1^N |t|^{N/2}\Gamma((N+1)/2).$$

§ 5. Proof of Lemma 4.1

We give a proof of Lemma 4.1.

First, we consider the formal 1-Borel transform of $\hat{f}(t,x) = \sum_{i\ge 0} f_i(x)t^i/i!$

$$g(s,x) := (\hat{\mathcal{B}}_1 \hat{f})(s,x) = \sum_{i\ge 0} f_i(x)\frac{s^i}{i!^2},$$

which is convergent in $|s| < r$ and $|x| \le \sigma$ for some $r > 0$. Then from the assumptions that $\hat{f}(t,x) \in \mathcal{O}_x\{t\}_1$ and that $f^0(t,x)$ satisfies **1-S-C**, we can show that $g(s,x)$ is analytic in the region $(D_r \cup S_s(0;\varepsilon)) \times S_x(0,\pi;\varepsilon_1,\sigma)$ for sufficiently small $\varepsilon > 0$ and has the estimate

$$|g(s,x)| \le Ce^{\delta_1|s|+\delta_2|x|^2} \tag{5.1}$$

with some positive constants C, δ_1 and δ_2 for $s \in (\overline{D}_{r'} \times \overline{S}_2)$ and $x \in \overline{S}_x(0,\pi;\varepsilon_1',\sigma)$, where $r' < r$ and $\overline{S}_2$ is any closed subsector $S_s(0;\varepsilon)$.

Indeed, from the assumption that $\hat{f}(t,x) \in \mathcal{O}_x\{t\}_1$, we see that $g(s,x)$ is analytic in $(D_r \cup S_s(0;\varepsilon)) \times \overline{D}_\sigma$ and has the exponential growth estimate of order at most 1 with respect to s, that is, $\max_{|x|\le\sigma} |g(s,x)| \le C_1 e^{\delta_1|s|}$ for $s \in \overline{S}_2$.

Let us consider 1-Borel transform of $f^0(t,x)$

$$\tilde{g}(s,x) := \frac{-1}{2\pi i}\int_\gamma f^0(t,x)e^{\frac{s}{t}}\frac{dt}{t}$$

for $s \in S_s(0;\varepsilon)$ and $x \in \overline{D}_\sigma$, where the path γ denotes the path from the origin along $\arg t = (\varepsilon+\pi)/2$ to some point t_1 with a positive ε, then along the circle $|t| = |t_1|$ to the ray $\arg t = -(\varepsilon+\pi)/2$, and back to the origin along this ray such that $\gamma \subset$

$S_T = S_t(0, \alpha, T)$ with $\alpha > \pi + \varepsilon$. Then from the assumptions that $\hat{f}(t, x) \in \mathcal{O}_x\{t\}_1$ and that $f^0(t, x)$ is 1-sum of $\hat{f}(t, x)$, we have $\tilde{g}(s, x) = g(s, x)$ in a neighborhood at $(s, x) = (0, 0)$. Moreover, since $f^0(t, x)$ satisfies **1-S-C**, we see that $\tilde{g}(s, x)$ is analytic in $(D_r \cup S_s(0; \varepsilon)) \times S_x(0, \pi; \varepsilon_1, \sigma)$, and has the estimate

$$|\tilde{g}(s, x)| = |g(s, x)| \le C e^{\delta_1 |s| + \delta_2 |x|^2}$$

by some positive constants C, δ_1 and δ_2 for $s \in (\overline{D}_{r'} \times \overline{S}_2)$ and $x \in \overline{S}_x(0, \pi; \varepsilon_1', \sigma)$.

Next, by using the inequality (5.1), we have

$$|f_i(x)/i!| = |\partial_s^i g(0, x)| = \left| \frac{i!}{2\pi\sqrt{-1}} \oint_{|\tau|=r'} \frac{g(\tau, x)}{\tau^{i+1}} d\tau \right| \le \frac{C_1 i!}{r'^i} e^{\delta_2 |x|^2}$$

with some positive constant C_1 for $x \in \overline{S}_x(0, \pi; \varepsilon_1', \sigma)$, which gives a proof of (4.2).

By using the inequality (4.2), we have

$$|f_i^{(n)}(x)| = \left| \frac{n!}{2\pi i} \oint_{|\xi - x| = c(x)} \frac{f_i(\xi)}{(\xi - x)^{n+1}} d\xi \right| \le n! C_1 K_1^i i!^2 e^{\tilde{\delta}_2 |x|^2} / (c(x))^n$$

for $x \in \overline{S}_x(0, \pi; \varepsilon_1'', \sigma')$ with $\varepsilon_1'' < \varepsilon_1'$, $\sigma' < \sigma/2$ and $\tilde{\delta}_2 > 0$, where $c(x) = \sigma'$ if $|x| \le \sigma'$ and $c(x) = c_0 |x|$ for some $c_0 > 0$ if $|x| \ge \sigma'$ and $x \in \overline{S}_x(0, \pi; \varepsilon_1'', \sigma')$. Here, if $|x| \le \sigma'$, then $1/(c(x))^n = 1/\sigma'^n$. If $|x| \ge \sigma'$, then $1/(c(x))^n = 1/(c_0|x|)^n \le 1/(c_0 \sigma')^n$. Therefore we have

$$|f_i^{(n)}(x)| \le C_2 K_2^{i+n} n! i!^2 e^{\tilde{\delta}_2 |x|^2}$$

for $x \in \overline{S}_x(0, \pi; \varepsilon_1'', \sigma')$, which gives a proof of (4.3).

Finally, we give a proof of (4.4). We put

$$G_M(s, x) := g(s, x) - \sum_{i=0}^{M-1} f_i(x) \frac{s^i}{i!^2}$$

for $s \in (D_r \cup S_s(0; \varepsilon)) \times S_x(0, \pi; \varepsilon_1, \sigma)$. Then we have $G_M(s, x) = D_s^{-M} \partial_s^M g(s, x)$, where $D_s^{-1} = \int_0^s$. By using the inequality (5.1), we have

$$|\partial_s^M g(s, x)| = \left| \frac{M!}{2\pi i} \oint_{|\tau - s| = c(s)} \frac{g(\tau, x)}{(\tau - s)^{M+1}} d\tau \right| \le C_1 K_1^M M! e^{\tilde{\delta}_1 |s| + \delta_2 |x|^2}$$

by some positive constants C_1, K_1 and $\tilde{\delta}_1$ for $(s, x) \in (\overline{D}_{r'} \cup \overline{S}_2) \times \overline{S}_x(0, \pi; \varepsilon_1', \sigma)$ with any closed subsector $\overline{S}_2 \subset S_s(0; \varepsilon)$. Here $c(s) = r' (< r/2)$ if $|s| \le r'$, and $c(s) = c_1 |s|$ for some $c_1 > 0$ if $|s| \ge r'$ and $s \in \overline{S}_2$.

Therefore we have for $s \in \overline{S}_2$ and $x \in \overline{S}_x(0, \pi; \varepsilon_1', \sigma)$

$$\begin{aligned} |G_M(s, x)| &= \left| \int_0^s \frac{(s - p)^{M-1}}{(M-1)!} \partial_s^M g(p, x) dp \right| \\ &\le \int_0^1 \frac{|s|^M (1 - q)^{M-1}}{(M-1)!} C_1 K_1^M M! e^{\tilde{\delta}_1 p |s| + \delta_2 |x|^2} dq = C_1 K_1^M |s|^M e^{\tilde{\delta}_1 |s| + \delta_2 |x|^2}. \end{aligned}$$

We remark that $F_M(t,x)$ is given by the analytic continuation of 1-Laplace transform of $G_M(s,x)$

$$F_M(t,x)=(\mathcal{L}_1G_M)(t,x):=\frac{1}{t}\int_0^{\infty(0)}e^{-\frac{s}{t}}G_M(s,x)ds$$

for $|t-\rho_0|<\rho_0$ and $x\in S_x(0,\pi;\varepsilon_1,\sigma)$ for sufficiently small $\rho_0>0$. Here the path $\int_0^{\infty(d)}$ takes from 0 to ∞ along $\arg s=d$. Therefore for $t\in\overline{S}_1\subset\{t\in\mathbb{C};|t-\rho_0|<\rho_0\}$ and $x\in\overline{S}_x(0,\pi;\varepsilon_1',\sigma)$ we have

$$\begin{aligned}|F_M(t,x)|&=\left|\frac{1}{t}\int_0^{\infty(0)}e^{-\frac{s}{t}}G_M(s,x)ds\right|=\left|\int_0^{\infty(-\arg t)}e^{-u}G_M(ut,x)du\right|\\&\le\int_0^{\infty}e^{-cv}C_1K_1^M|vt|^Me^{\tilde{\tilde{\delta}}_1|vt|+\delta_2|x|^2}dv\quad(c,\tilde{\tilde{\delta}}_1>0)\\&\le C_1K_1^M|t|^Me^{\delta_2|x|^2}\int_0^{\infty}e^{-\tilde{c}v}v^Mdv\quad(\tilde{c}=c-2\tilde{\tilde{\delta}}_1\rho_0>0)\\&\le\tilde{C}_1\tilde{K}_1^M|t|^Me^{\delta_2|x|^2}M!\quad(\tilde{C}_1,\tilde{K}_1>0).\end{aligned}$$

§ 6. Remark on the integral representation of 1-sum

We could not obtain the integral representation of 1-sum in terms of the heat kernel under the assumption of the inhomogeneous term $\hat{f}(t,x)\in\mathcal{O}_x[[t]]_1$. However, we can get the integral representation of 1-sum if the assumption is relaxed.

We consider the Cauchy problem

$$(H)\qquad\begin{cases}\partial_tu(t,x)=\partial_x^2u(t,x)+f(t,x),\\u(0,x)=0,\end{cases}$$

where we assume that the inhomogeneous term $f(t,x)=\sum_{i\ge0}f_i(x)t^i/i!$ is convergent in t-variable and entire in x-variable, and has the following estimate

$$\max_{|t|\le\rho}|f(t,x)|\le Ce^{\delta|x|},\quad x\in\mathbb{C}\tag{6.1}$$

by some positive constants ρ,C and δ. Then the formal solution $\hat{u}(t,x)$, which is given by

$$\hat{u}(t,x)=\sum_{n\ge1}\left(\sum_{i=1}^{n}f_{i-1}^{(2(n-i))}(x)\right)\frac{t^n}{n!}=:\sum_{n\ge1}u_n(x)t^n,$$

is a convergent series in t-variable. (This holds if $f(t,x)$ has the exponential growth estimate of order at most 2 instead of the condition (6.1).) In fact, from the condition (6.1), we have the following inequalities.

$$\begin{gathered}|f_i(x)|\le C_1K_1^ii!e^{\delta|x|}\quad(x\in\mathbb{C}),\\|f_i^{(\ell)}(x)|\le C_2K_2^{i+\ell}i!e^{\delta|x|}\quad(x\in\mathbb{C})\end{gathered}$$

by some positive constants C_1, C_2, K_1 and K_2. Therefore since $|u_n(x)| \le C_3 K_3^n e^{\delta|x|}$ by some C_3 and K_3, the formal solution $\hat{u}(t,x)$ is convergent in t-variable.

In the following, we write the formal solution $\hat{u}(t,x)$ by $u(t,x)$.

For the solution $u(t,x)$, which is a convergent series in $|t| \le \rho$ and $x \in \mathbb{C}$, we put

$$\begin{aligned} u(t,x) &= \sum_{n\ge 1} \left(\sum_{i=1}^{n} f_{i-1}^{(2(n-i))}(x) \right) \frac{t^n}{n!} = \sum_{i\ge 1} \left(\sum_{n\ge i} f_{i-1}^{(2(n-i))}(x) \frac{t^n}{n!} \right) \\ &= \sum_{i\ge 1} D_t^{-i} \left(\sum_{n\ge 0} f_{i-1}^{(2n)}(x) \frac{t^n}{n!} \right) =: \sum_{i\ge 1} D_t^{-i} U_i(t,x), \end{aligned} \tag{6.2}$$

where $D_t^{-1} = \int_0^t$. Here, for each i, $U_i(t,x)$ is a convergent series (in fact, entire) in t-variable and satisfies the following Cauchy problem.

$$\begin{cases} \partial_t U(t,x) = \partial_x^2 U(t,x), \\ U(0,x) = f_{i-1}(x). \end{cases} \tag{H_i}$$

Therefore by restricting in the region $O_t := \{t \in \mathbb{C}; |t-\rho_0| < \rho_0\}$ for sufficiently small ρ_0, we have the following integral representation of $U_i(t,x)$ in terms of the heat kernel.

$$U_i(t,x) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-\frac{y^2}{4t}} f_{i-1}(x+y)dy.$$

Then when $t \in O_t$, we have

$$\begin{aligned} u(t,x) &= \sum_{i\ge 1} D_t^{-i} U_i(t,x) = \sum_{i\ge 1} \int_0^t \frac{(t-s)^i}{(i-1)!} U_i(s,x)ds \\ &= \int_0^t \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-\frac{y^2}{4s}} \sum_{i\ge 1} f_{i-1}(x+y) \frac{(t-s)^i}{(i-1)!} dyds \\ &= \int_0^t \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-\frac{y^2}{4s}} f(t-s, x+y)dyds. \end{aligned}$$

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