On meromorphy of local zeta functions for C^∞ functions

By

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The purpose of this article is to announce some results in the paper [14].

Let us study an integral of the form

$$Z(f,\varphi)(s) := \int_{\mathbb{R}^2} |f(x,y)|^s \varphi(x,y)dxdy \qquad s \in \mathbb{C}, \tag{0.1}$$

where f, φ are real-valued C^∞ functions defined on a small open neighborhood U of the origin in $\mathbb{R}^2$ and the support of φ is contained in U. Since the integral in (0.1) locally converges on the region $\mathrm{Re}(s) > 0$, $Z(f,\varphi)$ can be regarded as a holomorphic function there, which is called a *local zeta function*. We will consider the following issue: how local zeta functions can be analytically continued to a wider region.

In the case where f is real analytic, the analytic coninuation of local zeta functions is well understood. It is shown in [4], [3] that local zeta functions can be analytically continued as meromorphic functions to the whole complex plane by using Hironaka's resolution of singularities. Furthermore, Varchenko applied the theory of toric varieties based on the Newton polyhedron of f to the analysis of local zeta functions and he gave an algorithm to determine the location and the order of their poles under some nondegeneracy condition. Moreover, the above Varchenko's result can be naturally generalized to a certain class of C^∞ functions, which contains the real analytic class. However, it is known in [21] that, in the case of specific (non-real analytic) C^∞ functions, the local zeta function has a singularity different from poles. Therefore, the following new issue is naturally raised: how widely local zeta functions can be meromorphically continued. In this article, we introduce some quantity associated with f, which is invariant under the change of coordinates, and give an answer to the above issue by using this quantity.

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Notation and symbols.

- We denote by $\mathbb{Z}_+, \mathbb{R}_+$ the subsets consisting of all nonnegative numbers in $\mathbb{Z}, \mathbb{R}$, respectively. For $s \in \mathbb{C}$, $\mathrm{Re}(s)$ expresses the real part of s.
- For $R = \mathbb{R}$ or $\mathbb{C}$, $R[[t]]$ is the ring of formal power series in t with coefficients from R. Moreover, $R[[x, y]]$ is the ring of double formal power series.
- By (0.1), $Z(f, \varphi)(s)$ is defined as an *integral*. When $Z(f, \varphi)$ can be regarded as a *function* on some region, this function is also denoted by the same symbol.

§ 1. Description of our problems

Hereafter, we usually assume that $f \in C^\infty(U)$ is non-flat and satisfies

$$f(0,0) = 0 \quad \text{and} \quad \nabla f(0,0) = (0,0). \tag{1.1}$$

Unless (1.1) is satisfied, every problem addressed in this article is easy. As for $\varphi \in C_0^\infty(U)$, we sometimes give the following conditions

$$\varphi(0,0) > 0 \quad \text{and} \quad \varphi \geq 0 \text{ on } U. \tag{1.2}$$

In order to investigate the analytic continuation of local zeta functions, we only use the half-plane of the form $\mathrm{Re}(s) > -\rho$ with $\rho > 0$. This is the reason why we observe the situation of analytic continuation through the integrability of integrals of the form (0.1). Of course, it is desirable to deal with various kinds of regions in the study of analytic continuation and this advanced issue should be investigated in the future.

§ 1.1. Newton data

Let $\overline{f}(x, y) \in \mathbb{R}[[x, y]]$ be the Taylor series of $f(x, y)$ at the origin, i.e.,

$$\overline{f}(x,y) = \sum_{(j,k)\in\mathbb{Z}_+^2} c_{jk} x^j y^k \quad \text{ with } c_{jk} = \frac{1}{j!k!} \frac{\partial^{j+k} f}{\partial x^j \partial y^k}(0,0). \tag{1.3}$$

The *Newton polygon* of f is the integral polygon

$$\Gamma_+(f) = \text{the convex hull of the set } \bigcup \{(j,k) + \mathbb{R}_+^2 : c_{jk} \neq 0\} \text{ in } \mathbb{R}_+^2$$

(i.e., the intersection of all convex sets which contain $\bigcup\{(j,k) + \mathbb{R}_+^2 : c_{jk} \neq 0\}$). The flatness of f at the origin is equivalent to the condition $\Gamma_+(f) = \emptyset$.

The *Newton distance* $d(f)$ of f is defined by

$$d(f) = \inf\{\alpha > 0 : (\alpha, \alpha) \in \Gamma_+(f)\}.$$

We set $d(f) = \infty$ when f is flat at the origin. Since the Newton distance depends on the coordinates system (x, y) on which f is defined, it is sometimes denoted by $d_{(x,y)}(f)$. The *height* of f is defined by

$$\delta_0(f) = \sup_{(x,y)} \{d_{(x,y)}(f)\}, \tag{1.4}$$

where the supremum is taken over all local smooth coordinate systems (x, y) at the origin. Note that the height $\delta_0(f)$ can be determined by the Taylor series $\overline{f} \in \mathbb{R}[[x, y]]$ only. From their definitions, $d(f)$ and $\delta_0(f)$ roughly indicate some kind of flatness of f at the origin (when they are larger, the flatness of f becomes stronger).

Remark 1.1. We can determine $\delta_0(f)$ for f not satisfying the conditions (1.1) from its definition. When $f(0, 0) \neq 0$, we have $\delta_0(f) = 0$. When $f(0, 0) = 0$ and $\nabla f(0, 0) \neq (0, 0)$, we have $\delta_0(f) = 1$ by using the implicit function theorem.

§ 1.2. Holomorphic extension problem

First, let us consider the following quantities:

$$\mathfrak{h}_0(f, \varphi) := \sup \left\{ \rho > 0 : \begin{array}{c} \text{The domain to which } Z(f, \varphi) \text{ can} \\ \text{be holomorphically continued} \\ \text{contains the half-plane } \mathrm{Re}(s) > -\rho \end{array} \right\}, \tag{1.5}$$

$$\mathfrak{h}_0(f) := \inf \left\{ \mathfrak{h}_0(f, \varphi) : \varphi \in C_0^\infty(U) \right\}. \tag{1.6}$$

It is obvious that $\mathfrak{h}_0(f)$ is invariant under the change of coordinates. We remark that if φ satisfies (1.2), then $\mathfrak{h}_0(f, \varphi) = \mathfrak{h}_0(f)$ holds; but otherwise, this equality does not always hold. Indeed, there exists $\varphi \in C_0^\infty(U)$ with $\varphi(0, 0) = 0$ such that $\mathfrak{h}_0(f, \varphi) > \mathfrak{h}_0(f)$ (see e.g. [5], [18]).

From the form of the integral in (0.1), the relationship between the holomorphy and the convergence of the integral implies that the quantity $\mathfrak{h}_0(f)$ is deeply related to the following famous index:

$$\mathfrak{c}_0(f) := \sup \left\{ \mu > 0 : \begin{array}{c} \text{there exists an open neighborhood } V \text{ of} \\ \text{the origin in } U \text{ such that } |f|^{-\mu} \in L^1(V) \end{array} \right\}, \tag{1.7}$$

which is called the *log canonical threshold* or the *critical integrability index*. The index $\mathfrak{c}_0(f)$ has been deeply investigated from various points of view. The equality $\mathfrak{h}_0(f) = \mathfrak{c}_0(f)$ always holds. In fact, the inequality $\mathfrak{h}_0(f) \geq \mathfrak{c}_0(f)$ is obvious; while the opposite inequality can be easily seen by Theorem 5.1 in [21]. In the real analytic case, since all the singularities of the extended $Z(f, \varphi)$ are poles on the real axis, the leading pole exists at $s = -\mathfrak{h}_0(f,\varphi)$. In the seminal work of Varchenko [27], when f is real analytic and satisfies some nondegeneracy conditions, $\mathfrak{h}_0(f)$ can be expressed as $\mathfrak{h}_0(f) = 1/d(f)$, where $d(f)$ is the Newton distance of f. An interesting work [7] treating the equality $\mathfrak{c}_0(f) = 1/d(f)$ is from another approach. We remark that these results deal with the general dimensional case. In the same paper [27], Varchenko more deeply investigates the two-dimensional case. Indeed, without any assumption, he shows that the equality

$$\mathfrak{h}_0(f) = 1/\delta_0(f) \tag{1.8}$$

holds for real analytic f. More generally, in the C^∞ case, M. Greenblatt [12] obtains a sharp result which generalizes the above two-dimensional Varchenko's result.

Theorem 1.2 ([12]). $\mathfrak{c}_0(f)(= \mathfrak{h}_0(f)) = 1/\delta_0(f)$ *holds for every non-flat* $f \in C^\infty(U)$.

From the above result, our holomorphic extension problem is completely understood even in the C^∞ case. It is important that $\mathfrak{h}_0(f)$ is determined by information of the formal Taylor series of f only.

On the other hand, the situation of the meromorphic extension is quite different from the holomorphic one.

§ 1.3. Meromorphic extension problem

Corresponding to (1.5), (1.6) in the holomorphic continuation case, we analogously define the following quantities:

$$\mathfrak{m}_0(f,\varphi) := \sup\left\{\rho > 0 : \begin{array}{c}\text{The domain to which } Z(f,\varphi) \text{ can} \\ \text{be meromorphically continued} \\ \text{contains the half-plane } \mathrm{Re}(s) > -\rho\end{array}\right\}, \tag{1.9}$$

$$\mathfrak{m}_0(f) := \inf\left\{\mathfrak{m}_0(f,\varphi) : \varphi \in C_0^\infty(U)\right\}. \tag{1.10}$$

It is easy to see that $\mathfrak{m}_0(f)$ is invariant under the change of coordinates and that $\mathfrak{h}_0(f,\varphi) \leq \mathfrak{m}_0(f,\varphi)$ and $\mathfrak{h}_0(f) \leq \mathfrak{m}_0(f) \leq \mathfrak{m}_0(f,\varphi)$ always hold. As mentioned in the Introduction, if f is real analytic, then $\mathfrak{m}_0(f) = \infty$ always holds; while there exist specific (non-real analytic) C^∞ functions f such that $\mathfrak{m}_0(f) < \infty$. Indeed, it is shown in [21] (see also [12]) that when

$$f(x,y) = x^a y^b + x^a y^{b-q} e^{-1/|x|^p}, \tag{1.11}$$

and φ satisfies the condition (1.2), $Z(f,\varphi)$ has a non-polar singularity at $s = -1/b$, which implies $\mathfrak{m}_0(f) = 1/b$. Here, p is a positive real number and $a, b, q \in \mathbb{Z}_+$ satisfy that $a < b$, $b \geq 2$, $1 \leq q \leq b$ and q is even. Note that $d(f) = \delta_0(f) = b$ in this case. At present, properties of the singularity at $s = -1/b$ are not well understood (see Section 14.2). In order to understand how wide the meromorphically extendible region of a given local zeta function is, we consider the following problem.

Problem 1.3. For a given $f \in C^\infty(U)$, describe (or estimate) the value of $\mathfrak{m}_0(f)$ in terms of appropriate information of f.

In [21], the above problem is investigated in the case where f has the following form which is a natural generalization of (1.11).

$$f(x,y) = u(x,y)x^a y^b + (\text{a flat function}), \tag{1.12}$$

where a, b are nonnegative integers with $a \leq b$ and $u(x,y) \in C^\infty(U)$ satisfies $u(0,0) \neq 0$. It is shown in [21] that $\mathfrak{m}_0(f) \geq 1/a$. Note that $\delta_0(f) = b$ in this case.

Remark 1.4. Since $x^a y^b$ with $a, b \in \mathbb{Z}_+$ is a real analytic function, $\mathfrak{m}_0(x^a y^b) = \infty$ holds. On the other hand, $\mathfrak{m}_0(f) = 1/b$ holds if f is as in (1.11). From this observation, we see that $\mathfrak{m}_0(f)$ is not always determined by the formal Taylor series of f.

§ 2. Main results

§ 2.1. The quantity $\mu_0(f)$

Let us introduce a new important quantity $\mu_0(f)$, which will be used in the statement of our main theorem.

Let $\overline{f}(x,y) \in \mathbb{R}[[x,y]]$ be the formal Taylor series of a non-flat C^∞ function $f(x,y)$ at the origin. It is known (c.f. [28], Corollary 2.4.2, p.32) that $\overline{f}(x,y)$ can be expressed as in the following factorization in terms of the formal Puiseux series

$$\overline{f}(t^N, y) = \overline{u}(t^N, y) t^{Nm_0} \prod_{j=1}^{r} (y - \overline{\phi}_j(t))^{m_j}, \tag{2.1}$$

where N is a positive integer, m_0 is a nonnegative integer, m_j are positive integers, $\overline{u}(x,y) \in \mathbb{C}[[x,y]]$ has a non-zero constant term and $\overline{\phi}_j(t) \in \mathbb{C}[[t]]$ are distinct (i.e., $\overline{\phi}_j(t) \neq \overline{\phi}_k(t)$ if $j \neq k$). Let $\mathcal{R}(f)$ be the subset of $\{0, 1, \ldots, r\}$ defined by

$$j \in \mathcal{R}(f) \iff j = 0 \text{ or } \overline{\phi}_j(t) \in \mathbb{R}[[t]]. \tag{2.2}$$

The case $r = 0$ is possible; when $\overline{f}(x,y)$ is expressed as $\overline{u}(x,y)x^{m_0}$, we set $\mathcal{R}(f) = \{0\}$. The quantity $\mu_0(f)$ is defined by

$$\mu_0(f) = \max\{m_j : j \in \mathcal{R}(f)\}. \tag{2.3}$$

Remark 2.1. (1) It is obvious from the definition that the quantity $\mu_0(f)$ is determined by the formal Taylor series of f only, as well as the height $\delta_0(f)$ in (1.4).

(2) We define $\mu_0(f)$ for a C^∞ function f not satisfying the conditions (1.1) as follows. When $f(0,0) \neq 0$, $\mathcal{R}(f) = \{0\}$ with $m_0 = 0$, which gives $\mu_0(f) = 0$. When $f(0,0) = 0$ and $\nabla f(0,0) \neq 0$, $\mathcal{R}(f) = \{0\}$ with $m_0 = 1$ or $\mathcal{R}(f) = \{0,1\}$ with $m_0 = 0$ and $m_1 = 1$, which gives $\mu_0(f) = 1$, by the implicit function theorem.

(3) The quantity $\mu_0(f)$ is invariant under the change of coordinates.

(4) When f is real analytic and $\mu_0(f) \geq 1$, $\mu_0(f)$ is equal to the maximal order of vanishing of f along the set $\{(x,y) \in \mathbb{R}^2 : |x|^2 + |y|^2 = \gamma\}$ with sufficiently small $\gamma > 0$.

(5) If a real analytic function f satisfies $f(x,y) > 0$ away from the origin, then $\mu_0(f) = 0$ holds. But, in the C^∞ case, the above implication is not true. For example, consider the C^∞ function $f(x,y) = y^{2k} + e^{-1/x^2}$ with $k \in \mathbb{N}$. In this case, $\mu_0(f) = 2k$.

§ 2.2. Main theorem

Now let us state a theorem, which gives an answer to Problem 1.3. Indeed, we show that the meromorphically extendible region can be described by using the quantity $\mu_0(f)$.

Theorem 2.2. *Let f be a non-flat C^∞ function defined in a neighborhood of the origin in $\mathbb{R}^2$. Then we have*
(i) If $\mu_0(f) = 0, 1$, then $\mathfrak{m}_0(f) = \infty$ holds;
(ii) If $\mu_0(f) \geq 2$, then $\mathfrak{m}_0(f) \geq 1/\mu_0(f)$ holds.
Furthermore, when $\mu_0(f) < \delta_0(f)$, the poles of the extended local zeta function on $\mathrm{Re}(s) > -1/\mu_0(f)$ *exist in the finitely many arithmetic progressions that are constructed from negative rational numbers.*

Remark 2.3. (1) The assumption of the theorem does not need the condition (1.1).

(2) Recalling Theorem 2.1 given by Greenblatt [12], we can see $\mu_0(f) \leq \delta_0(f)$ for $f \in C^\infty(U)$ by using the above theorem with the obvious inequality $\mathfrak{m}_0(f) \leq \mathfrak{h}_0(f)$. The inequality $\mu_0(f) \leq \delta_0(f)$ itself can be directly shown.

(3) Since the equality $\mathfrak{m}_0(f) = 1/\mu_0(f)$ holds for f in (1.11), the estimate in (ii) is optimal in the uniform sense for f. From the obvious inclusion $C^\omega(U) \subset C^\infty(U)$, there are many C^∞ functions f such that $\mu_0(f) \geq 2$ and $\mathfrak{m}_0(f) = \infty$ (in particular, $\mathfrak{m}_0(f) > 1/\mu_0(f)$).

(4) At present, very few properties of non-polar singularities of local zeta functions are known. These issues are investigated in [21], [23], [15].

§ 3. Idea of the proof of Theorem 2.2

The proof of Theorem is roughly divided into the two parts: geometric part and analytic part.

§ 3.1. The geometric part

Understanding the geometry of the variety defined by $f(x,y) = 0$ is crusial in the analytic continuation of local zeta functions. When the set defined by $f(x,y) = 0$ is restricted to the real space $\mathbb{R}^2$, this restricted set sometimes has very few information and is not always useful for precise analysis. In the case where f is real analytic, the defining region of f can be naturally extended to the complex region in $\mathbb{C}^2$. The zero variety in $\mathbb{C}^2$ of the extended f is so-called a *holomorphic plane curve*, which has been very widely studied. Actually, many fruitful results about these curves improve the investigation of local zeta functions associated with real analytic functions. For example, the theory of toric varieties based on the geometry of Newton polyhedra gives quantitative results about poles of local zeta functions ([27], [9], [10], [8], [24], [5], [17], [18], etc.). On the other hand, when a C^∞ function f is extended to the complex space, the conjugate variables must be considered in general, which makes it difficult to understand geometric properties of the zero variety of f in $\mathbb{C}^2$. Therefore, we give up handling this variety itself and instead look for an essentially important subset in it, which is easier to deal with. With the aid of the factorization formula for C^∞ functions of V. S. Rychkov [25], an important *curve* in the zero locus of f in $\mathbb{C}^2$ is defined, which will be called the *decisive curve* defined by f, and this curve has sufficient information for our analysis. The decisive curve defined by f consist of branches in $\mathbb{C}^2$ parametrized by using the Puiseux series of one real variable. Although the singularity of this curve might not be completely resolved by using algebraic transforms only, this curve can be locally expressed as in *almost* normal crossings form via finite compositions of ordinary blowings up. To be more exact, there exist a two-dimensional C^∞ real manifold Y and a proper map $\pi : Y \to \mathbb{R}^2$ such that $f \circ \pi$ can be locally expressed at any point on the zero locus of the map π as

$$(f \circ \pi)(x,y) = u(x,y)x^a \left(y^m + \varepsilon_1(x)y^{m-1} + \cdots + \varepsilon_m(x)\right), \tag{3.1}$$

where a, m are nonnegative integers and u, ε_k are real-valued C^∞ functions satisfying that $u(0,0) \neq 0$ and ε_k are flat at the origin. Note that in the real analytic case, since ε_k must be zero functions, $f \circ \pi$ can be locally expressed in ordinary normal crossings form, which implies that each local zeta function can be meromorphically extended to the whole complex plane by using an elementary method ([1], see also Section 11).

§ 3.2. The analytic part

Throughout the above geometric process, it is sufficient to deal with local zeta functions associated with functions of the form (3.1), which is considered as a model in the C^∞ case. In the case of (3.1), the analytic continuation of local zeta functions can be effectively investigated by using real analysis methods; the most important tool is a van der Corput-type lemma. The original van der Corput's lemma gives an estimate for one-dimensional oscillatory integrals, which is explained in [26]. This lemma has been rewritten in various forms according to the purposes. Our analysis needs one of the versions used in [6] (see also [12]). As a result, we show that the meromorphically extendible region of local zeta functions associated with (3.1) contains the region $\mathrm{Re}(s) > -1/m$. The above mentioned analysis has been essentially performed in the paper [22].

After the above explained investigation, we give an answer to the meromorphic extension issue for local zeta functions in the C^∞ case. For this purpose, we introduce a quantity $\mu_0(f)$ for a given C^∞ function f. In general, the double formal power series has the factorization formula by using the Puiseux series. Through the above explained resolution process, the multiplicities of real roots in this factorization formula essentially appear in the index m in the expression (3.1). The maximum of the multiplicities of real roots in the factorization formula is denoted by $\mu_0(f)$. Then we can see that the meromorphically extendible region always contains the region $\mathrm{Re}(s) > -1/\mu_0(f)$. This result is optimal in the uniform sense. Note that the quantity $\mu_0(f)$ is an invariant of f, i.e., it is independent of the choice of coordinates.

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