Hypergeometric System of Contingency Table

By

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Abstract

In this report, we formulate a conjecture for a Gauß-Manin connection of any hyperplane arrangement. The formula relates cohomology intersection form to Gauß-Manin connection. We report that the conjecture is true for a hypergeometric system of contingency table. We also obtain a new formula for the cohomology intersection form.

§ 1. Introduction

A number of studies on Gauß-Manin connections for hypergeometric integrals have been published. An important class of such is Gauß-Manin connection of a hyperplane arrangement ([13]). The description of the Gauß-Manin derivative is given in [13, Theorem 12] in a complicated formula. When the hyperplane arrangement is generic, a formula employing cohomology intersection form is obtained in [5]. An advantage of such a formula is that it does not depend on a choice of a basis of the de Rham cohomology group. In this report, we provide a conjectural formula that generalizes the one obtained in [5] for any hyperplane arrangement (Conjecture 2.3). Moreover, we report that there is a particular class of hyperplane arrangements in which the conjecture is true: hypergeometric system of contingency table. Proofs of the results in the last section are based on studies on integral representations of GKZ systems developed in [6, 7] which will be made available in a separate paper.

§ 2. Gauß-Manin connection of a hyperplane arrangement

§ 2.1. Orlik-Solomon algebra

We recall some terminologies from [11]. Let V be a finite dimensional complex vector space and let $\alpha = (\alpha_1, \dots, \alpha_n) : V \hookrightarrow \mathbb{C}^n$ be a linear injection. We write

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$H_i = \mathrm{Ker}\alpha_i \subset V$ for the hyperplane defined by $\alpha_i \in V^*$. Let L be the intersection lattice of $\mathscr{A}$ and we set $L(\mathscr{A}) = L \setminus \{O\}$ where O is the unique maximal element of L. An element $e \in L(\mathscr{A})$ is called a dense edge if $\mathscr{A}_e = \{H \in \mathscr{A} \mid e \subset H\}$ is irreducible. The symbol $D(\mathscr{A})$ denotes the set of dense edges. Let $E := \Lambda(\mathbb{C}^n)$ be the exterior algebra of the vector space $\mathbb{C}^n$ and let $\{\mathbf{e}_i\}_{i=1}^n$ be the standard basis of $\mathbb{C}^n$. We write $E = \oplus_{i=0}^n E^i$ for the graded decomposition of E. Let us define a linear map $\partial : E \to E$ by the relations $\partial 1 = 0$, $\partial e_i = 1$ and

$$\partial(e_{i_1} \wedge \cdots \wedge e_{i_p}) = \sum_{j=1}^p (-1)^{j-1} e_{i_1} \wedge \cdots \wedge \widehat{e_{i_j}} \wedge \cdots \wedge e_{i_p} \quad (p \geq 2). \tag{2.1}$$

For an ordered tuple $S = (i_1, \ldots, i_p)$, we set $e_S := e_{i_1} \wedge \cdots \wedge e_{i_p}$. The Orlik-Solomon ideal $I = I(\mathscr{A}) \subset E$ is a graded ideal generated by $\{\partial e_S \mid \cap S \neq \varnothing \text{ and } S \text{ is dependent}\}$. Note that $\mathscr{A}$ is a central arrangement.[1] We set $A := A(\mathscr{A}) := E/I(\mathscr{A})$ and call it Orlik-Solomon algebra.

For a fixed element $H \in \mathscr{A}$, we set

$$A^p(\mathscr{A}; H) := \bigoplus_{e \in L(\mathscr{A}),\ e \not\subset H} A^p(\mathscr{A}_e) \hookrightarrow A^p(\mathscr{A}) \tag{2.2}$$

Note that the relation

$$A^p(\mathscr{A}; H) \simeq A^p(\mathbf{d}_H \mathscr{A}) \tag{2.3}$$

holds where $\mathbf{d}_H \mathscr{A}$ is the decone of $\mathscr{A}$ along H. Let us fix an index 0 and write H_0 for the corresponding hyperplane. Since $\mathscr{A}$ is a central arrangement, a relation $\partial I \subset I$ holds and we can set $\tilde{A} = \tilde{A}(\mathscr{A}) := \mathrm{Ker}\ (\partial : E \to E)$. It is readily seen that $\tilde{A}(\mathscr{A}) \subset A$ is a graded subalgebra.

Lemma 2.1. *The correspondence* $A(\mathbf{d}_{H_0}\mathscr{A}) \ni e_S = e_{i_1} \wedge \cdots \wedge e_{i_p} \mapsto (e_{i_1} - e_0) \wedge \cdots \wedge (e_{i_p} - e_0) \in \tilde{A}(\mathscr{A})$ *gives rise to an algebra isomorphism* $A(\mathbf{d}_{H_0}\mathscr{A}) \simeq \tilde{A}(\mathscr{A})$.

Proof. A simple computation shows that an identity $e_S = (e_{i_1} - e_0) \wedge \cdots \wedge (e_{i_p} - e_0) + e_0 \wedge \partial e_S$ holds. Thus, the map is a surjection. Injectivity is clear by definition. □

Recall that A is isomorphic to the Brieskorn algebra B generated by dlog forms $d\log\alpha_{i_1} \wedge \cdots \wedge d\log\alpha_{i_p} \in \mathrm{H}^0(V \setminus \mathscr{A}, \Omega^p_{V\setminus\mathscr{A}})$ over $\mathbb{C}$. Under this identification, it is easy to see that $\tilde{A}$ is isomorphic to a subalgebra of B generated by differential forms $\omega_S := d\log(\alpha_{i_1}/\alpha_{i_0}) \wedge \cdots \wedge d\log(\alpha_{i_p}/\alpha_{i_{p-1}})$ $(S = \{i_0, \ldots, i_p\})$. Note that any element

[1] For an arrangement $\mathscr{A}$ which is not necessarily central, the Orlik-Solomon ideal is an ideal of E generated by $\{e_S \mid \cap S = \varnothing\} \cup \{\partial e_S \mid \cap S \neq \varnothing \text{ and } S \text{ is dependent}\}$.

of $\tilde{A}$ defines an element of $\mathrm{H}^0(\mathbb{P}(V\setminus\mathscr{A}),\Omega^p_{\mathbb{P}(V\setminus\mathscr{A})})$. Hereafter, we often identify e_S with ω_S.

§ 2.2. Formal Gauß-Manin connection on combinatorial strata

Let $k<n$ be a pair of non-negative integers and let $\mathrm{Gr}_{k+1,n+1}$ denote the $(k+1)$-Grassmannian variety of $\mathbb{C}^{n+1}=\mathbb{C}^{[\![0,n]\!]}$. Here, we set $[\![0,n]\!]:=\{0,1,\dots,n\}$. We consider a realization of $\mathrm{Gr}_{k+1,n+1}$ as a moduli space of arrangements. Let $Z_{k+1,n+1}$ be the set of complex $(k+1)\times(n+1)$ matrices $z=(z_{ij})_{ij}$ which has rank $k+1$. For each element $z\in Z_{k+1,n+1}$, the i-th column vector α_i can be regarded as an element of the dual space $(\mathbb{C}^{k+1})^*$. Thus, z gives rise to a linear embedding $z:\mathbb{C}^{k+1}=\mathrm{Spec}\,\mathbb{C}[x_0,\dots,x_n]\hookrightarrow\mathbb{C}^{n+1}$ which in turn defines a matroid M_z on $[\![0,n]\!]$. Namely, a subset $B\subset[\![0,n]\!]$ is a basis of M_z if and only if $\{\alpha_i\}_{i\in B}$ is a basis of $(\mathbb{C}^{k+1})^*$. Let us consider a left action of the general linear group $\mathrm{GL}(k+1,\mathbb{C})$ on $Z_{k+1,n+1}$ given by the left multiplication of a matrix. The Grassmannian variety $\mathrm{Gr}_{k+1,n+1}$ is identified with the quotient $\mathrm{GL}(k+1,\mathbb{C})\backslash Z_{k+1,n+1}$. Clearly, any representative z of $\xi\in\mathrm{Gr}_{k+1,n+1}=\mathrm{GL}(k+1,\mathbb{C})\backslash Z_{k+1,n+1}$ defines the same matroid M_z, for which we write M_ξ. For a matroid M on $[\![0,n]\!]$, we set

$$\mathcal{R}(M):=\{\xi\in\mathrm{Gr}_{k+1,n+1}\mid M_\xi=M\}. \tag{2.4}$$

Note that M is linearly realizable if and only if $\mathcal{R}(M)\neq\varnothing$. The combinatorial stratification of $\mathrm{Gr}_{k+1,n+1}$ introduced in [4] is the following decomposition:

$$\mathrm{Gr}_{k+1,n+1}=\bigcup_{M:\text{ linearly realizable matroid on }\{0,\dots,n\}}\mathcal{R}(M). \tag{2.5}$$

Note that each stratum $\mathcal{R}(M)$ is a constructible subset of $\mathrm{Gr}_{k+1,n+1}$ in Zariski topology. However, $\mathcal{R}(M)$ can be highly non-trivial due to the universality theorem of Mnëv ([10]):

Theorem 2.2 ([10])**.** *Given any affine algebraic variety V over $\mathbb{Q}$, there exists a rank 3 realizable matroid M whose projective stratum $\mathcal{R}(M)/H$ is isomorphic to V.*

Let us take a basis set M of a linearly realizable matroid without any loop on $[\![0,n]\!]$. Let $\tilde{\mathcal{R}}(M)\subset Z_{k+1,n+1}$ be the preimage of $\mathcal{R}(M)$ by the quotient map $Z_{k+1,n+1}\to\mathrm{Gr}_{k+1,n+1}$. For a subset $T\subset[\![0,n]\!]$ of cardinality $k+1$, we set $C_T:=\{z\in Z_{k+1,n+1}\mid \det(z_{ij})_{\substack{i=0,\dots,k\\ j\in T}}=0\}$. Let the symbol $\binom{[\![0,n]\!]}{k+1}$ denote the set of subsets of $[\![0,n]\!]$ with cardinality $k+1$. It is easy to see that the identity

$$\tilde{\mathcal{R}}(M)=\bigcap_{T\in\binom{[\![0,n]\!]}{k+1}\setminus M}C_T\setminus\bigcup_{T\in M}C_T \tag{2.6}$$

holds. Moreover, setting $D_T:=C_T\cap\overline{\tilde{R}(M)}$ one has an identity

$$\overline{\tilde{\mathcal{R}}(M)}\setminus\tilde{\mathcal{R}}(M)=\bigcup_{T\in M}D_T. \tag{2.7}$$

For any $z \in \tilde{\mathcal{R}}(M)$ and $j \in [\![0, n]\!]$, we set $\ell_j(x; z) := \sum_{i=0}^{k+1} z_{ij} x_i$. Let $\tilde{R}(M)^{\text{reg}}$ denote the smooth locus of $\tilde{R}(M)$. We set $X := \{(z, x) \in \tilde{\mathcal{R}}^{\text{reg}}(M) \times \mathbb{P}^k \mid \ell_i(x; z) \neq 0\}$. Let $\pi : X \to \tilde{\mathcal{R}}(M)^{\text{reg}}$ denote the natural projection. We may assume that M defines an essential arrangement, otherwise the identity $\tilde{\mathcal{R}}(M) = \varnothing$ holds. We write $A = A(M)$ for the corresponding Orlik-Solomon algebra. For any element $B \in M$, we write $z_B = \det(z_{ij})_{\substack{i=0,\dots,k \\ j \in B}}$. For each index $i \in [\![0, n]\!]$, we associate a complex number $\lambda_i \in \mathbb{C}$ such that $\lambda_0 + \cdots + \lambda_n = 0$.

For an element $e \in D(M)$, we set $\lambda_e := \sum_{i \in e} \lambda_i$. If none of λ_e is a non-negative integer, the result of [3] (see also [12]) shows that the natural morphism

$$\tilde{A}^{k+1}/\omega_\lambda \wedge \tilde{A}^k \to \mathbb{H}^k \left(\mathbb{P}^k \setminus \mathscr{A}_M, (\Omega^\bullet_{\mathbb{P}^k \setminus \mathscr{A}_M}, \nabla_\lambda)\right) =: \mathrm{H}^k_{\mathrm{dR}}(M, \lambda) \tag{2.8}$$

is an isomorphism where $\nabla_\lambda := d_x + \omega_\lambda \wedge$. For any $z \in \tilde{\mathcal{R}}(M)$, the linear morphism

$$\tilde{A}^{p+1} \ni e_{i_0 \cdots i_p} \mapsto \omega_{i_0 \cdots i_p} \in \mathbb{H}^0 \left(\mathbb{P}^k \setminus \mathscr{A}_z, \Omega^p_{\mathbb{P}^k \setminus \mathscr{A}_z}\right) \tag{2.9}$$

is well-defined. Note that the following identity holds true:

$$\omega_{i_0 \cdots i_p} = d_x \log(\ell_{i_1}(x; z)/\ell_{i_0}(x; z)) \wedge \cdots \wedge d_x \log(\ell_{i_p}(x; z)/\ell_{i_{p-1}}(x; z)).$$

Let $\Omega^1(\log D_M)$ denote the vector space spanned by $\{d \log z_B\}_{B \in M}$. One can define the Gauß-Manin connection associated with $\pi : X \to \tilde{\mathcal{R}}(M)^{\text{reg}}$ in a combinatorial way. We set $\tilde{A}[\lambda] := \tilde{A}(M) \otimes_{\mathbb{C}} \mathbb{C}[\lambda]$ and $\omega_\lambda := \sum_{i=0}^n \lambda_i \omega_i \in \tilde{A}^1(M)[\lambda]$. The result of [13] shows that there is a formal Gauß-Manin derivative $\nabla^{GM}_\lambda : \tilde{A}^k[\lambda]/\omega_\lambda \wedge \tilde{A}^{k-1}[\lambda] \to \left(\tilde{A}^k[\lambda]/\omega_\lambda \wedge \tilde{A}^{k-1}[\lambda]\right) \otimes_{\mathbb{C}[\lambda]} \Omega^1(\log D_M)$ which has the following expression:

$$\nabla^{GM}_\lambda = \sum_{B \in M} f_B d \log z_B \wedge, \tag{2.10}$$

where $f_B \in \mathrm{End}_{\mathbb{C}[\lambda]} \left(\tilde{A}^k[\lambda]/\omega_\lambda \wedge \tilde{A}^{k-1}[\lambda]\right)$ is a certain endomorphism determined by the combinatorics of M. Note that $d_z \log(g \cdot z)_B = d_z \log z_B$ for any $g \in \mathrm{GL}(k+1, \mathbb{C})$.

§ 2.3. Cohomology intersection form and a universal expression of Gauß-Manin connection

We formulate a general conjecture of a general form of Gauß-Manin connection. To this end, let us briefly recall the construction of the cohomology intersection form. For details, the readers may refer to [1, 8, 9]. Let R be a ring obtained from $\mathbb{C}[\lambda]$ by inverting polynomials $\lambda_e - j$ ($e \in D(M), j \in \mathbb{Z}_{\geq 0}$). We set $\mathrm{H}^k_{\mathrm{dR}}(M; R) := \left(\tilde{A}^{k+1}/\omega_\lambda \wedge \tilde{A}^k\right) \otimes_{\mathbb{C}[\lambda]} R$. One can define a canonical bilinear pairing

$$\langle \bullet, \bullet \rangle_{ch} : \mathrm{H}^k_{\mathrm{dR}}(M; R) \otimes_R \mathrm{H}^k_{\mathrm{dR}}(M; R) \to R. \tag{2.11}$$

Given a tuple of complex numbers $\mathring{\lambda} \in \mathbb{C}^{n+1}$ such that $\mathring{\lambda}_e \notin \mathbb{Z}_{\geq 0}$ for any $e \in D(M)$, one can define an evaluation morphism

$$\mathrm{ev}(\mathring{\lambda}) : \mathrm{H}^k_{\mathrm{dR}}(M;R) \to \tilde{A}^{k+1}/\omega_{\mathring{\lambda}} \wedge \tilde{A}^k. \tag{2.12}$$

Moreover, the construction is compatible with the ususal cohomology intersection form defined over $\mathbb{C}$ ([8]):

$$\begin{CD} \mathrm{H}^k_{\mathrm{dR}}(M;R) \otimes_R \mathrm{H}^k_{\mathrm{dR}}(M;R) @>{\langle \bullet,\bullet \rangle_{ch}}>> R \\ @V{\mathrm{ev}(\mathring{\lambda})\otimes \mathrm{ev}(-\mathring{\lambda})}VV @VV{\mathrm{ev}(-\mathring{\lambda})}V \\ \mathrm{H}^k(M,\mathring{\lambda}) \otimes_{\mathbb{C}} \mathrm{H}^k(M,-\mathring{\lambda}) @>{\langle \bullet,\bullet \rangle_{ch}}>> \mathbb{C} \end{CD} \quad . \tag{2.13}$$

The image of a cohomology class $[\omega] \in \mathrm{H}^k_{\mathrm{dR}}(M;R)$ by ev_λ (resp. $\mathrm{ev}_{-\lambda}$) is denoted by $[\omega]$ (resp. $[\omega^\vee]$) omitting λ. We conjecture that the following universal expression is true:

Conjecture 2.3. *The endomorphism f_B of* (2.10) *takes the form*

$$f_B(\bullet) = \lambda_B \langle \bullet, [\omega_B^\vee] \rangle_{ch} [\omega_B], \tag{2.14}$$

where $\lambda_B = \prod_{i\in B} \lambda_i$ and $\langle \bullet, \bullet \rangle_{ch}$ is the cohomology intersection form[2]*.*

The simplest case of the conjecture above is the following result.

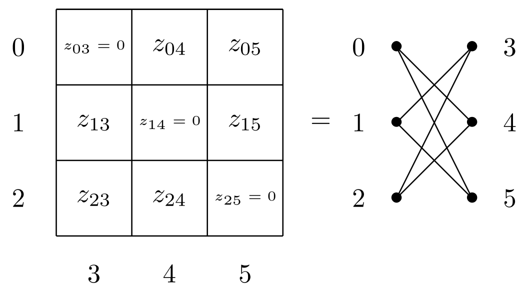
Theorem 2.4 (Theorem 3.12 of [5])**.** *Conjecture 2.3 is true for generic strata $\mathcal{R}(U_{k+1,n+1})$ where $U_{k+1,n+1}$ is the uniform matroid of rank $k+1$ on $[\![0, n+1]\!]$.*

§ 3. Hypergeometric System of Contingency Table

§ 3.1. Hyperplane arrangement and contingency table

From this subsection, we fix a pair of non-negative integers $k < n$. Let the symbol $[\![k, n]\!]$ denotes the set of consecutive integers between k and n, *i.e.*, $[\![k, n]\!] = \{k, k+1, \dots, n\}$. For a subset $\Omega \subset [\![0, k]\!] \times [\![k+1, n+1]\!]$, we associate a bipartite graph whose vertex set is $[\![0, n+1]\!]$ and whose edge set is Ω. In order to simplify the notation, we write ij for an element $(i, j) \in \Omega$.

[2]We normalize the intersection form by $(2\pi\sqrt{-1})^k$. This is a good choice when we deal with the field of definition.



A bipartite graph Ω is spanning if any $i \in [\![0, n+1]\!]$ is a vertex of Ω. In the followings, we assume that the graph of Ω is connected and spanning. For any pair of subsets $I \subset [\![0, k]\!]$, $J \subset [\![k+1, n+1]\!]$ such that $|I| = |J|$, a bijection $m : I \tilde{\to} J$ is called a matching if $(i, m(i)) \in E(G)$ for any $i \in I$. A pair (I, J) is called a matching pair if there is a matching $m : I \simeq J$. For any $(i, j) \in [\![0, k]\!] \times [\![k+1, n+1]\!] \setminus \Omega$, we set $z_{ij} = 0$. We set $z = (z_{ij})_{(i,j)\in[\![0,k]\!]\times[\![k+1,n+1]\!]}$. For any pair of subsets $I \subset [\![0, k]\!]$, $J \subset [\![k+1, n+1]\!]$ such that $|I| = |J|$, the symbol $z_{I,J}$ denotes the (I, J)-subdeterminant of z. If we treat non-zero z_{ij} as an indeterminate, $z_{I,J}$ is non-zero if and only if (I, J) is a matching pair.

We set $Z = Z(\Omega) = \mathbb{C}^{\Omega} \setminus \prod_{(I,J):\text{ matching pair}}\{z_{I,J} = 0\}$, $\ell_i(x; z) := x_i$ $(i \in [\![0, k]\!])$ and $\ell_j(x; z) := \sum_{i=0}^{k} z_{ij} x_i$ $(j \in [\![k+1, n+1]\!])$. The Gauß-Manin connection of a contingency table is associated with a family of arrangements $\pi : \mathbb{P}^k \times Z \setminus \{(x, z) \mid \prod_{i=0}^{n+1} \ell_i(x; z) = 0\} \to Z$. Let us state it more clearly. We identify $z \in Z$ with a $[\![0, k]\!] \times [\![0, n+1]\!]$ matrix

$$\tilde{z} := \left(\begin{array}{ccc|ccc} 1 & & & z_{0k+1} & \cdots & z_{0n+1} \\ & \ddots & & \vdots & \ddots & \vdots \\ & & 1 & z_{kk+1} & \cdots & z_{kn+1} \end{array}\right) \tag{3.1}$$

Clearly, any $\tilde{z}$ defines the same matroid M as long as $z \in Z$ and therefore, we obtain an open embedding $Z \ni z \mapsto [\tilde{z}] \in \tilde{\mathcal{R}}(M)^{\mathrm{reg}}$. This embedding naturally induces a Gauß-Manin connection on Z whose fiber is given by $\tilde{A}^k/\omega_\lambda \wedge \tilde{A}^{k-1}$.

Theorem 3.1. *Conjecture 2.3 is true on Z. More concretely, the formal Gauß-Manin connection takes the form*

$$\nabla_\lambda^{GM} = \sum_{(I,J):\text{matching pair}} \lambda_{I;J} \langle \bullet, [\omega_{I,J}^\vee] \rangle_{ch} [\omega_{I,J}] d\log z_{I,J} \wedge . \tag{3.2}$$

Here, we set $\lambda_{I,J} := \prod_{i\in[\![0,k]\!]\setminus I} \lambda_i \prod_{j\in J} \lambda_j$, $\omega_{I,J} := \omega_{([\![0,k]\!]\setminus I)\cup J}$.

§ 3.2. matching formula of the cohomology intersection form

Given a subset $S \subset [\![0, n+1]\!]$ such that $|S| = k+1$, we set $S_0 := S \cap [\![0, k]\!]$, $S_1 = S \cap [\![k+1, n+1]\!]$. In view of the identity $|S_1| = k+1-|S_0|$, setting $I :=$

$[\![0,k]\!] \setminus S_0$ and $J := S_1$, we have $|I| = |J|$. For $I = \{i_0 < \cdots < i_l\} \subset [\![0,k]\!]$, we set $I^c = [\![0,k]\!] \setminus I = \{i'_0 < \cdots < i'_m\}$. We set $\mathrm{sgn}(I^c, I) := \frac{\mathbf{e}_{I^c} \wedge \mathbf{e}_I}{\mathbf{e}_0 \wedge \cdots \wedge \mathbf{e}_k} = \frac{\mathbf{e}_{i'_0} \wedge \cdots \wedge \mathbf{e}_{i'_m} \wedge \mathbf{e}_{i_0} \wedge \cdots \wedge \mathbf{e}_{i_l}}{\mathbf{e}_0 \wedge \cdots \wedge \mathbf{e}_k}$. A simple computation shows the identity

$$\omega_{I,J} = \omega_S = \mathrm{sgn}(I^c, I) z_{I,J} \frac{\prod_{i \in I} \ell_i}{\prod_{j \in J} \ell_j} \omega_0. \tag{3.3}$$

Let us fix a generic weight vector w on variables $(z_{ij})_{ij \in \Omega}$. A matching m corresponds to a monomial $z^m := \prod_{i \in I} z_{im(i)}$. Note that $z_{I,J} = \sum_{m:matching} \mathrm{sgn}(m) z^m$. Here sgn is the signature of a matching regarded as a permutation. We always align elements of I, J in increasing order. For a given matching pair (I, J), a matching $m : I \simeq J$ such that $\mathrm{in}_<(z_{I,J}) = \pm z^m$ is called the w-minimal matching. When I, J comes from a subset $S \subset [\![0, n+1]\!]$ of cardinality $k+1$, we write m_S for the w-minimal matching. We set $M(S) := \{(i, m_S(i))\}_{i \in I}$. Note that $M(S) = \varnothing$ if $I = \varnothing$.

Let $\{\mathbf{e}(i)\}_{i=0}^{n+1}$ be the standard basis of $\mathbb{Z}^{[\![0,n+1]\!]}$, i.e., all the entries of $\mathbf{e}_i$ but the i-th one is zero and the i-th entry is 1. We set $\mathbf{a}(i,j) = \mathbf{e}(i) + \mathbf{e}(j)$ $(i,j) \in \Omega$. For any spanning tree $\sigma \subset \Omega$, the equation

$$\sum_{i=0}^{k} \sum_{j=k+1}^{n+1} v_{ij}(\mathbf{e}_i + \mathbf{e}_j) = \sum_{i=0}^{k} \lambda_i \mathbf{e}_i - \sum_{j=k+1}^{n+1} \lambda_j \mathbf{e}_j \tag{3.4}$$

has a unique solution $v = v(\sigma; \lambda)$ under the condition $v_{ij} = 0$ $((i,j) \notin \sigma)$. If we regard $A = \{\mathbf{a}(i,j)\}_{ij \in \Omega}$ as a lattice configuration of $\mathbb{Z}^{[\![0,n+1]\!]}$, any simplex σ in the convex hull of A is a spanning tree and vice versa.

For a given vector $w = (w_{ij}) \in \mathbb{R}^\Omega$, one defines a w-weight of a monomial $z^e = \prod_{ij \in \Omega} z^{e_{ij}}$ by $\sum_{ij \in \Omega} w_{ij} e_{ij}$. For a given polynomial f in z, let the symbol $\mathrm{in}_w(f)$ denote the w-initial term of f, that is the sum of monomials with minimal w-weight. Finally, for a generic vector $w \in \mathbb{R}^\Omega$, let T_w denote the regular triangulation of A determined by w ([2, Chapter 5]). The following formula expresses the cohomology intersection number in terms of combinatorics of a regular triangulation.

Theorem 3.2. *Let $S_1, S_2 \subset [\![0, n+1]\!]$ be subsets such that $|S_1| = |S_2| = k+1$. Fix a generic weight vector w and set $M_a := M(S_a)$ $(a = 1, 2)$, $T(S_1; S_2) = \{\sigma \in T_w \mid$*

$M_1 \cup M_2 \subset \sigma\}$. *Then, the following identity holds:*

$$\begin{aligned}
&\langle [\omega_{S_1}], [\omega_{S_2}] \rangle_{ch} \\
&= \operatorname{sgn}(I_1^c, I_1) \operatorname{sgn}(I_2^c, I_2) \operatorname{sgn}(m_{S_1}) \operatorname{sgn}(m_{S_2}) \frac{(-\lambda_{k+1}) \cdots (-\lambda_{n+1})}{\left(\prod_{j \in J_1}(-\lambda_j)\right)\left(\prod_{j \in J_2}(-\lambda_j)\right)} \times \\
&\sum_{\sigma \in T(S_1;S_2)} \frac{\prod_{(i,j) \in M_1 \cap M_2} v_{i,j}(\sigma;\lambda)}{\prod_{(i,j) \in \sigma \setminus (M_1 \cup M_2)} v_{i,j}(\sigma;\lambda)} \qquad (3.5) \\
&= \operatorname{sgn}(I_1^c, I_1) \operatorname{sgn}(I_2^c, I_2) \operatorname{sgn}(m_{S_1}) \operatorname{sgn}(m_{S_2}) \frac{(-\lambda_{k+1}) \cdots (-\lambda_{n+1})}{\left(\prod_{j \in J_1}(-\lambda_j)\right)\left(\prod_{j \in J_2}(-\lambda_j)\right)} \times \\
&\sum_{\sigma \in T(S_1;S_2)} \frac{\prod_{(i,j) \in M_1} v_{i,j}(\sigma;\lambda) \prod_{(i,j) \in M_2} v_{i,j}(\sigma;\lambda)}{\prod_{(i,j) \in \sigma} v_{i,j}(\sigma;\lambda)}. \qquad (3.6)
\end{aligned}$$

Here $v_{i,j}(\sigma;\lambda)$ *is the* ij*-th entry of* $v(\sigma;\lambda)$. *In particular, the intersection number* $\langle [\omega_{S_1}], [\omega_{S_2}] \rangle_{ch}$ *is zero if* $T(S_1;S_2) = \varnothing$.

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