Differential operator representations of continuous homomorphisms in mixed cases

By

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Abstract

In [3], we studied continuous homomorphisms between locally convex spaces of entire functions with growth given by proximate orders and gave their differential operator representations, both in Roumieu cases and Beurling cases.

In this paper, we report our recent study on such representations of such homomorphisms in mixed cases. The result provides a unified treatment from a categorical viewpoint by virtue of the topological intersection/union theorems.

§ 1. Introduction

A linear differential operator often defines a continuous endomorphism on a function space endowed with a natural topology, and also an endomorphism of a sheaf of functions. Therefore, there arises a natural question whether every continuous endomorphism of a given space (or of a given sheaf of functions) can be represented as a differential operator.

This problem has been studied in various situations. We refer, for example, to [11, 12], [13, 5, 6], and [8], for the smooth, analytic, and ultradifferentiable cases.

For the spaces of entire functions with a growth given in terms of constant order $\rho > 0$, it is shown in [4] that any continuous endomorphism of such space is characterized as a partial differential operator of infinite order with a symbol satisfying certain growth conditions. Note that such endomorphisms play a role in the study of superoscillations (see [1, 2]). The results were generalized to the case of proximate order ([7, Corollaries

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6.5 and 6.6]). We studied continuous homomorphisms between spaces given by possibly different proximate orders and gave their differential operator representations, both in Roumieu cases and Beurling cases ([3, Theorems 4.5 and 4.7]). There, we pointed out that the roles of the proximate orders of the source and target spaces are much more visible in studying homomorphisms rather than endomorphisms.

In this paper, we report our recent study on such representations of such homomorphisms in mixed cases. The result provides a unified treatment from a categorical viewpoint by virtue of the topological intersection/union theorems.

§ 2. Orders and proximate orders

The order and the type of an entire function is defined via the comparison of $|f(z)|$ with a family of functions $\{\exp(L|z|^k)\}_{k,L>0}$.

Definition 2.1 (order and type)**.** An entire function $f(z)$ on $\mathbb{C}^n$ is said to be *of finite order* if there exist positive constants k, L, C such that

$$|f(z)| \le C\exp(L|z|^k), \text{ for any } z \in \mathbb{C}^n. \tag{2.1}$$

The *order* $\rho \in [0, +\infty]$ and the *type* $\sigma \in [0, +\infty]$ (when $0 < \rho < +\infty$) of $f(z)$ are defined by

$$\rho := \inf\{k \in (0,\infty) \mid \exists L > 0,\ \exists C,\ (2.1) \text{ holds.}\},$$
$$\sigma := \inf\{L \in (0,\infty) \mid \exists C > 0,\ (2.1) \text{ with } k = \rho \text{ holds.}\},$$

under the convention $\inf \emptyset = +\infty$.

When the order ρ of $f(z)$ is positive and finite, (i.e., $0 < \rho < +\infty$), $f(z)$ is said to be of *minimal type*, *normal type*, and *maximal type* according to the respective cases $\sigma = 0$, $0 < \sigma < +\infty$ and $\sigma = +\infty$.

Definition 2.2 (spaces A_ρ and $A_{\rho,+0}$)**.** We denote by A_ρ (resp. $A_{\rho,+0}$), the space of entire function at most of order $\rho > 0$ and normal type (resp. minimal type).

Remark 2.3. We have the following inclusions and strict inclusions:
(a) $A_{\rho,+0} \subset A_\rho$ for any $\rho > 0$.
(b) $A_{\rho_1} \subset A_{\rho_2}$ and $A_{\rho_1,+0} \subset A_{\rho_2,+0}$ if $\rho_1 \le \rho_2$.
(c) $A_{\rho_1} \subset A_{\rho_2,+0}$ if $\rho_1 < \rho_2$.
(d) $A_{\rho_1} \subsetneqq \bigcap_{\rho_2>\rho_1} A_{\rho_2,+0}$, $\bigcup_{\rho_1<\rho_2} A_{\rho_1} \subsetneqq A_{\rho_2,+0}$.
Roughly speaking, $\{A_\rho, A_{\rho,+0}\}_{\rho>0}$ forms a coarse scale of entire functions of finite order.

The notion of a proximate order was introduced in [14].

Definition 2.4 (proximate order)**.** A differentiable function $\varrho(r) \geq 0$ defined for $r \geq 0$ is said to be a *proximate order* for the *order* $\rho \geq 0$ if it satisfies

(i) $\lim_{r\to+\infty} \varrho(r) = \rho$,
(ii) $\lim_{r\to+\infty} \varrho'(r) r \ln r = 0$.

In this paper, we treat only proximate orders ϱ for positive orders, and in that case, there exists $r_0 > 0$ such that the function $r^{\varrho(r)}$ is strictly increasing on $r > r_0$, and diverges to $+\infty$.

Let $\varrho(r)$ be a proximate order for a positive order. A *normalization* $\hat{\varrho}(r)$ of $\varrho(r)$ is another proximate order satisfying the following conditions:

- there exists a constant $r_1 > 0$ such that $\hat{\varrho}(r) = \varrho(r)$ for any $r \geq r_1$,
- $r \mapsto r^{\hat{\varrho}(r)}$ is strictly increasing on $r > 0$ and maps $(0, \infty)$ onto $(0, \infty)$.

The existence of such $\hat{\varrho}$ is noted in [9, p.16, Note.].

For a given $\varrho(r)$, we fix one normalization $\hat{\varrho}(r)$. We denote the inverse function of $t = r^{\hat{\varrho}(r)}$ $(r > 0)$ by $r = \varphi(t) = \varphi_{\hat{\varrho}}(t)$ $(t > 0)$, and define a sequence $(G_q)_{q\in\mathbb{N}}$ by

$$G_q = G_{\hat{\varrho},q} := \frac{\varphi(q)^q}{(e\rho)^{q/\rho}}, \quad \text{for } q \in \mathbb{N}.$$

For any $\sigma > 0$, we consider a Banach space of entire functions

$$A_{\varrho,\sigma} := \{f \in \mathcal{O}(\mathbb{C}^n) \mid \|f\|_{\varrho,\sigma} := \sup_{z\in\mathbb{C}^n} |f(z)| \exp(-\sigma|z|^{\varrho(|z|)}) < \infty\}$$

endowed with the norm $\|\cdot\|_{\varrho,\sigma}$.

Definition 2.5 (spaces A_ϱ and $A_{\varrho,+0}$)**.** We define the spaces of entire functions *at most of normal type* and *at most of minimal type* with respect to a proximate order ϱ, by

$$A_\varrho := \varinjlim_{\sigma>0} A_{\varrho,\sigma}, \quad \text{and} \quad A_{\varrho,+0} := \varprojlim_{\sigma>0} A_{\varrho,\sigma}.$$

We can see that A_ϱ is a (DFS)-space, and that $A_{\varrho,\sigma+0}$ is an (FS)-space. They are also called the space of Roumieu type and that of Beurling type respectively.

When we fix a normalization $\hat{\varrho}(r)$ of $\varrho(r)$, the spaces A_ϱ and $A_{\hat{\varrho}}$ coincide as subspaces of $\mathcal{O}(\mathbb{C}^n)$ and they share the same locally convex topologies. The same holds for $A_{\varrho,\sigma+0}$ and $A_{\hat{\varrho},\sigma+0}$.

We define relations $\varrho_1 \preceq \varrho_2$ and $\varrho_1 \prec \varrho_2$ between two proximate orders ϱ_i $(i = 1, 2)$ for positive orders by

$$\varrho_1 \preceq \varrho_2 \iff r^{\varrho_1(r)} = O(r^{\varrho_2(r)}), \text{ as } r \to \infty,$$
$$\varrho_1 \prec \varrho_2 \iff r^{\varrho_1(r)} = o(r^{\varrho_2(r)}), \text{ as } r \to \infty.$$

We can easily see that $\varrho_1 \preceq \varrho_2$ implies $A_{\varrho_1} \subset A_{\varrho_2}$ and $A_{\varrho_1,+0} \subset A_{\varrho_2,+0}$, and that $\varrho_1 \prec \varrho_2$ implies $A_{\varrho_1} \subset A_{\varrho_2,+0}$.

Remark 2.6. We have the following inclusions and equalities:

(a) $A_{\varrho,+0} \subset A_\varrho$ for any ϱ.

(b) $A_{\varrho_1} \subset A_{\varrho_2}$ and $A_{\varrho_1,+0} \subset A_{\varrho_2,+0}$ if $\varrho_1 \preceq \varrho_2$.

(c) $A_{\varrho_1} \subset A_{\varrho_2,+0}$ if $\varrho_1 \prec \varrho_2$.

(d) $A_{\varrho_1} = \bigcap_{\varrho_2 \succ \varrho_1} A_{\varrho_2,+0}$, $\bigcup_{\varrho_1 \prec \varrho_2} A_{\varrho_1} = A_{\varrho_2,+0}$.

Roughly speaking, $\{A_\varrho, A_{\varrho,+0}\}_\varrho$ forms a "fine" scale of entire functions of finite order. (Cf. Remark 2.3).

We call Remark 2.6(d) as an algebraic intersection/union theorems, which is a part of the following theorem.

Theorem 2.7 (intersection/union theorems)**.** *We have equalities as subspaces of* $\mathcal{O}(\mathbb{C}^n)$,

$$A_{\varrho_1} = \bigcap_{\varrho_2 \succ \varrho_1} A_{\varrho_2,+0}, \qquad \bigcup_{\varrho_1 \prec \varrho_2} A_{\varrho_1} = A_{\varrho_2,+0}.$$

Moreover, they are isomorphisms between locally convex spaces:

$$A_{\varrho_1} \xrightarrow{\sim} \varprojlim_{\varrho_2 \succ \varrho_1} A_{\varrho_2,+0} \qquad \varinjlim_{\varrho_1 \prec \varrho_2} A_{\varrho_1} \xrightarrow{\sim} A_{\varrho_2,+0}.$$

Remark 2.8. The names "intersection theorems" and "union theorems" were introduced for classes of ultra-differentiable functions in [10]. However, we think that the concept of (algebraic or topological) intersection/union theorems has been widely shared among many researchers for a long time. For example, in several variables, topological intersection theorems have been studied at least from 80's, and given an alternative name of "projective descriptions".

§ 3. Differential operator representations in Roumieu and Beurling cases

We define the space of formal differential operators of infinite order with coefficients in $\mathbb{C}[[z]]$ by

$$\hat{D} := \{P = \sum_{\alpha \in \mathbb{N}^n} a_\alpha(z)\partial_z^\alpha \mid a_\alpha(z) \in \mathbb{C}[[z]]\} \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}[z], \mathbb{C}[[z]]).$$

Now we study continuous homomorphisms from A_{ϱ_1} to A_{ϱ_2} and those from $A_{\varrho_1,+0}$ to $A_{\varrho_2,+0}$.

Definition 3.1 ($\boldsymbol{D}_{\varrho_1 \to \varrho_2}$ and $\boldsymbol{D}_{\varrho_1 \to \varrho_2,0}$)**.** Let ϱ_i $(i = 1, 2)$ be two proximate orders for positive orders with $\varrho_1 \preceq \varrho_2$. We take a normalization $\hat{\varrho}_1$ of ϱ_1 and a sequence $(G_{\hat{\varrho}_1,q})_q$. We denote by $\boldsymbol{D}_{\varrho_1 \to \varrho_2}$ and by $\boldsymbol{D}_{\varrho_1 \to \varrho_2,0}$, the sets of all formal differential operators

$$P = \sum_{\alpha \in \mathbb{N}^n} a_\alpha(z)\partial_z^\alpha \in \hat{D},$$

with the respective conditions: $(a_\alpha(z))_{\alpha\in\mathbb{N}^n} \subset A_{\varrho_2}$ satisfying

$$\forall\lambda > 0,\ \exists\sigma > 0,\ \exists C > 0,\ \forall\alpha\in\mathbb{N}^n,\ \|a_\alpha\|_{\varrho_2,\sigma} \le C\lambda^{|\alpha|}G_{\hat{\varrho}_1,|\alpha|}/\alpha!, \tag{3.1}$$

and $(a_\alpha(z))_{\alpha\in\mathbb{N}^n} \subset A_{\varrho_2,+0}$ satisfying

$$\forall\sigma > 0,\ \exists\lambda > 0,\ \exists C > 0,\ \forall\alpha\in\mathbb{N}^n,\ \|a_\alpha\|_{\varrho_2,\sigma} \le C\lambda^{|\alpha|}G_{\hat{\varrho}_1,|\alpha|}/\alpha!. \tag{3.2}$$

Note that the definition does not depend on the choice of a normalization. Note also that when $\varrho_1 = \varrho_2$, we denote $\boldsymbol{D}_{\varrho\to\varrho}$ by $\boldsymbol{D}_\varrho$, and $\boldsymbol{D}_{\varrho\to\varrho,0}$ by $\boldsymbol{D}_{\varrho,0}$.

Now we recall two theorems from [3]. For the case of Roumieu type, that is, for the spaces of entire functions at most of normal type, we have

Theorem 3.2 ([3, Theorem 4.5]). *Let ϱ_i $(i = 1, 2)$ be two proximate orders for positive orders satisfying $\varrho_1 \preceq \varrho_2$.*
(i) *Suppose that $P = \sum_\alpha a_\alpha(z)\partial_z^\alpha \in \boldsymbol{D}_{\varrho_1\to\varrho_2}$. For an entire function $f \in A_{\varrho_1}$,*

$$Pf := \sum_{\alpha\in\mathbb{N}^n} a_\alpha(z)\partial_z^\alpha f$$

converges in A_{ϱ_2} and $Pf \in A_{\varrho_2}$. Moreover, $f \mapsto Pf$ defines a continuous homomorphism $P : A_{\varrho_1} \longrightarrow A_{\varrho_2}$.
(ii) *Let $F : A_{\varrho_1} \longrightarrow A_{\varrho_2}$ be a continuous homomorphism. Then there is a unique $P \in \boldsymbol{D}_{\varrho_1\to\varrho_2}$ such that $Ff = Pf$ holds for any $f \in A_{\varrho_1}$.*

In what follows, we write such conclusions (i) and (ii) as

$$\boldsymbol{D}_{\varrho_1\to\varrho_2} \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{LCS}}(A_{\varrho_1}, A_{\varrho_2}).$$

As a corollary, we recover $\boldsymbol{D}_\varrho \xrightarrow{\sim} \mathrm{End}_{\mathrm{LCS}}(A_\varrho)$, ([7, Corollary 6.5]).

For the case of Beurling type, that is, for the spaces of entire function at most of minimal type, we have

Theorem 3.3 ([3, Theorem 4.7]). *Let ϱ_i $(i = 1, 2)$ be two proximate orders for positive orders satisfying $\varrho_1 \preceq \varrho_2$. Then, $\boldsymbol{D}_{\varrho_1\to\varrho_2,0} \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{LCS}}(A_{\varrho_1,+0}, A_{\varrho_2,+0})$.*

Again, as a corollary, we recover $\boldsymbol{D}_{\varrho,0} \xrightarrow{\sim} \mathrm{End}_{\mathrm{LCS}}(A_{\varrho,+0})$, ([7, Corollary 6.6]).

§ 4. Differential operator representations in mixed cases

Now we study continuous homomorphisms from A_{ϱ_1} to $A_{\varrho_2,+0}$ and those from $A_{\varrho_1,+0}$ to A_{ϱ_2}.

Definition 4.1 ($\boldsymbol{D}_{\{\varrho_1\}\to(\varrho_2)}$ and $\boldsymbol{D}_{(\varrho_1)\to\{\varrho_2\}}$). Let ϱ_i $(i = 1, 2)$ be two proximate orders for positive orders. We take a normalization $\hat{\varrho}_1$ of ϱ_1 and a sequence $(G_{\hat{\varrho}_1,q})_q$. We denote by $\boldsymbol{D}_{\{\varrho_1\}\to(\varrho_2)}$ for $\varrho_1 \prec \varrho_2$ (resp. by $\boldsymbol{D}_{(\varrho_1)\to\{\varrho_2\}}$ for $\varrho_1 \preceq \varrho_2$), the set of

$$P = \sum_{\alpha\in\mathbb{N}^n} a_\alpha(z)\partial_z^\alpha \in \hat{D},$$

with the respective conditions: $(a_\alpha(z))_{\alpha\in\mathbb{N}^n} \subset A_{\varrho_2,+0}$ satisfying

$$\forall\lambda > 0,\ \forall\sigma > 0,\ \exists C > 0,\ \forall\alpha \in \mathbb{N}^n,\ \|a_\alpha\|_{\varrho_2,\sigma} \le C\lambda^{|\alpha|}G_{\hat{\varrho}_1,|\alpha|}/\alpha!, \tag{4.1}$$

and $(a_\alpha(z))_{\alpha\in\mathbb{N}^n} \subset A_{\varrho_2}$ satisfying

$$\exists\sigma > 0,\ \exists\lambda > 0,\ \exists C > 0,\ \forall\alpha \in \mathbb{N}^n,\ \|a_\alpha\|_{\varrho_2,\sigma} \le C\lambda^{|\alpha|}G_{\hat{\varrho}_1,|\alpha|}/\alpha!. \tag{4.2}$$

Theorem 4.2. *Let ϱ_i $(i = 1, 2)$ be two proximate orders for positive orders. We have*

$$\boldsymbol{D}_{\{\varrho_1\}\to(\varrho_2)} \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{LCS}}(A_{\varrho_1}, A_{\varrho_2,+0}), \quad \text{if } \varrho_1 \prec \varrho_2,$$
$$\boldsymbol{D}_{(\varrho_1)\to\{\varrho_2\}} \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{LCS}}(A_{\varrho_1,+0}, A_{\varrho_2}), \quad \text{if } \varrho_1 \preceq \varrho_2.$$

We can prove it in a parallel manner as the cases $\boldsymbol{D}_{\varrho_1\to\varrho_2}$ and $\boldsymbol{D}_{\varrho_1\to\varrho_2,0}$. However, we can take an alternative strategy that the four cases can be reduced to the case of the Roumieu to Beurling case, if we take Theorem 2.7 into account.

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