Study of nonlinear irregular singular differential equations with Borel summable functions

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0 Introduction

A system of nonlinear differential equations

$$x^{1+\gamma}\frac{dY}{dx} = F_0(x) + A(x)Y + F(x,Y) \tag{0.1}$$

with an irregular singular point $x = 0$. There exist many works about them, Hukuhara [8], Malmquist [11], Trjitzinsky [16] Iwano [9] [10] and many other mathematicians (see Wasow [17]).They studied construction of formal solutions and showed the existence of genuine solutions under some conditions by classical analysis. The theory of multisummable functions in asymptotic analysis has been developed after their studies, hence it was not used in their researches . We study more precisely than their works, the meaning of asymptotic expansion of transformations and solutions by using Borel summable functions. The theory of multisummable functions is used for differential equations. Especially for nonlinear ordinary equations in Braaksma [1] and for some class of partial differential equations in Ōuchi [14]. In this article we apply Borel summability to study, which is a special case of multisummability.

There is a classical important result due to Malmquist [11]. Let $\{\lambda_i\}_{i=1}^{n}$ be eigenvalues of $A(0)$ and distinct. Assume $\Lambda' = \{\lambda_i; 1 \le i \le n'\}$ and $\Lambda'' = \{\lambda_i; n' < i \le n\}$. Λ' and Λ'' are separated by a straight line through the origin in the complex plane. It is shown in [11] that there exists an n'-parameter family of solutions in some sector corresponding to Λ'. It is the main purpose that we try to have another look at this result, by applying a new theory in asymptotic analysis. We construct transformations and solutions more precisely and clearly in a function class with some Gevrey type estimates. Costin,O [4], Costin,O, Costin,R.D [5] and Braaksma,Kuik [2] treated (0.1) for $\gamma = 1$ (rank 1) in a different way, by applying the

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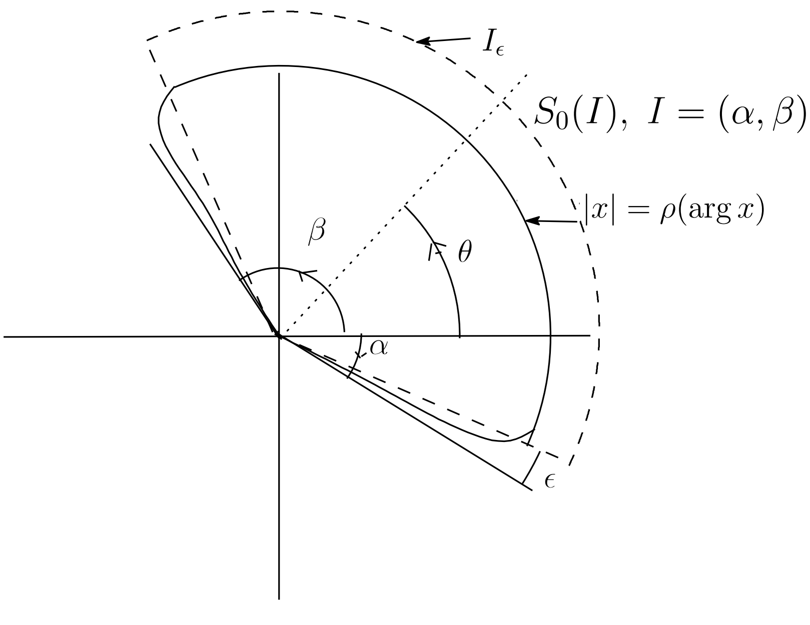
resurgence theory due to Écalle. In the present article we use only *elementary properties of Borel summable functions.* The details of this article is in Ōuchi [15].

1 Notations and definitions

$I = (\alpha, \beta)$ an interval, $I_\epsilon = (\alpha + \epsilon, \beta - \epsilon)$ $\epsilon > 0$. $\mathbb{C}^*$ is the universal covering space of $\mathbb{C} - \{0\}$. $S(I) = S(\alpha, \beta) = \{x \in \mathbb{C}^*; \arg x \in I\}$.
$S_0(I) = S_0(\alpha, \beta) = \{x \in S(I); 0 < |x| < \rho(\arg x)\}$, $\rho(t)$ is some positive continuous function on I. The same notation $S_0(\cdot)$ is used for various $\rho(\cdot)$. $\mathscr{O}(U)$ is the set of holomorphic functions on a domain U. $\mathbb{N}$ is the set of nonnegative integers. Let $k = (k_1, \cdots, k_n) \in \mathbb{N}^n$ and $Y = (y_1. \cdots, y_n) \in \mathbb{C}^n$. $k! = k_1!k_2! \cdots k_n!$, $|k| = \sum_{i=1}^n k_i$, $Y^k = y_1^{k_1} \cdots y_n^{k_n}$, $|Y| = \max_{1 \le i \le n} |y_i|$ and $(\frac{\partial}{\partial Y})^k = \prod_{i=1}^n (\frac{\partial}{\partial y_i})^{k_i}$.



Definition 1.1. *Let $\kappa > 0$, $I = (\alpha, \beta)$ with $\beta - \alpha > \pi/\kappa$ and $\Omega = \{Y \in \mathbb{C}^n; |Y| < R\}$. A function $f(x, Y) \in \mathscr{O}(S_0(I) \times \Omega)$ is said to be κ-Borel summable with respect to x, if there exist constants M, C and $\{a_n(y)\}_{n=0}^\infty \subset$*

$\mathscr{O}(\Omega)$ *such that for any* $N \geq 0$

$$|f(x,Y) - \sum_{n=0}^{N-1} a_n(Y)x^n| \leq MC^N|x|^N\Gamma(\frac{N}{\kappa}+1) \quad (x,Y) \in S_0(I) \times \Omega \quad (1.1)$$

holds. The totality of κ*-Borel summable functions with respect to* x *on* $S_0(I) \times \Omega$ *is denoted by* $\mathscr{O}_{\{1/\kappa\}}(S_0(I) \times \Omega)$.

We say that $f(x,Y)$ *is* κ**-Borel summable in a direction** θ, *if there exists* $\delta > \pi/2\kappa$ *such that* $f(x,Y) \in \mathscr{O}_{\{1/\kappa\}}(S_0(\theta-\delta,\theta+\delta) \times \Omega)$.

2 A system of nonlinear equations with irregular singularity

$$\begin{cases} Y = {}^t(y_1.y_2,\cdots,y_n) \\ x^{1+\gamma}\dfrac{dY}{dx} = F_0(x) + A(x)Y + F(x,Y) \end{cases} \quad (2.1)$$

γ is a positive integer, $F(x,Y) = {}^t(f_1(x,Y), f_2(x,Y),\cdots,f_n(x,y))$ with $F(x,Y) = O(|Y|^2)$ and $F(0,Y) = 0$.

$$x^{1+\gamma}\frac{dy_i}{dx} = f_{0,i}(x) + \sum_{j=1}^{n} a_{i,j}(x)y_j + f_i(x,Y) \quad i = 1,2,\cdots,n, \quad (2.2)$$

Let $I_* = (\theta_* - \delta_*, \theta_* + \delta_*)$ $(0 \leq \theta_* < 2\pi,\ \delta_* > \pi/2\gamma)$. We assume $\{a_{i,j}(x)\}_{1\leq i,j\leq n} \subset \mathscr{O}_{\{1/\gamma\}}(S_0(I_*))$ and $\{f_i(x,Y)\}_{1\leq i\leq n} \subset \mathscr{O}_{\{1/\gamma\}}(S_0(I_*) \times \Omega)$, $\Omega = \{Y \in \mathbb{C}^n : |Y| < R\}$, $f_i(x,Y) = O(|Y|^2)$. Let $\{\lambda_i\}_{1\leq i\leq n}$ be eigenvalues of $A(0)$. We assume

Condition 0. Eigenvalues are distinct

$\mathbf{\Lambda} = \{\lambda_i; i = 1,\cdots,n\}$, $\lambda_i \neq \lambda_k$ for $i \neq k$.

Set $\Lambda^\sharp = \{\lambda_i - \lambda_k;\ i \neq k\}$, $\omega_{i,k} = \arg(\lambda_i - \lambda_k)$ $(0 \leq \omega_{i,k} < 2\pi)$ for $i \neq k$ and

$$\begin{cases} \theta_{i,k,\ell} = (\omega_{i,k} + 2\pi\ell)/\gamma,\ \ell \in \mathbb{Z}, \\ \mathbf{\Theta}_1 = \{\theta_{i,k,\ell} : i,k = 1,\cdots,n, i \neq k,\ \ell \in \mathbb{Z}\}, \end{cases} \quad (2.3)$$

Firstly we assume $F_0(x) \equiv 0$. We simplify the linear part $A(x)$.

Proposition 2.1. (Diagonalization of the linear part) *Let* $\theta_* \notin \mathbf{\Theta}_1$. *Then there exists a matrix* $P(x)$ *with elements in* $\mathscr{O}_{\{1/\gamma\}}(S_0(I_*))$ *and* $P(0) =$

Id such that $Y = P(x)Z$ *transforms* $x^{1+\gamma}\frac{dY}{dx} = A(x)Y$ *to*

$$x^{1+\gamma}\frac{dZ}{dx} = \Lambda(x)Z, \tag{2.4}$$

where $\Lambda(x) = diag.(\lambda_1(x), \lambda_2(x), \cdots, \lambda_n(x))$ *is a diagonal matrix and* $\lambda_i(x)$ *is a polynomial with degree* $\leq \gamma$ *and* $\lambda_i(0) = \lambda_i$

Hence we begin to study

$$\begin{cases} Y =^t(y_1.y_2, \cdots, y_n) \\ x^{1+\gamma}\dfrac{dY}{dx} = \Lambda(x)Y + F(x, Y) \\ \Lambda(x) = diag.(\lambda_1(x), \cdots, \lambda_n(x)) \\ F(x, Y) =^t(f_1(x, Y), \cdots, f_n(x, Y)), \end{cases} \tag{Eq-Y}$$

where $\{\lambda_i(x)\}_{1\leq i\leq n}$ are polynomials with degree $\leq \gamma$ and $\lambda_i(0) = \lambda_i$, $F(x, Y) = O(|Y|^2)$ and $F(0, Y) = 0$. Let $\emptyset \neq \mathbf{\Lambda'} = \{\lambda_i; i = 1, \cdots, n'\} \subset \mathbf{\Lambda}$. We give 2 conditions on $\mathbf{\Lambda'}$.

Condition 1

There exist $0 \leq \theta_{\mathbf{\Lambda'}} < 2\pi$ and $0 < \delta_{\mathbf{\Lambda'}} < \pi/2$ such that $\mathbf{\Lambda'} \subset \Sigma = \{\eta \neq 0; |\arg \eta - \theta_{\mathbf{\Lambda'}}| < \delta_{\mathbf{\Lambda'}}\}$.

Condition 2

$$\sum_{j=1}^{n'} \lambda_j m_j - \lambda_i \neq 0 \;\; for \;\; |m| \geq 2 \;\; and \;\; 1 \leq i \leq n. \tag{2.5}$$

$|m| = \sum_{j=1}^{n'} m_j$, $m = (m_1, \cdots, m_{n'}) \in \mathbb{N}^{n'}$

Remark 2.2. *We note that if* $\mathbf{\Lambda'} = \{\lambda_1 \neq 0\}$, *Condition 1 is obvious and* Condition 2 $\Longleftrightarrow \lambda_1 m_1 - \lambda_i \neq 0 \;\; for \;\; m_1 \geq 2 \;\; and \;\; 2 \leq i \leq n.$

Let

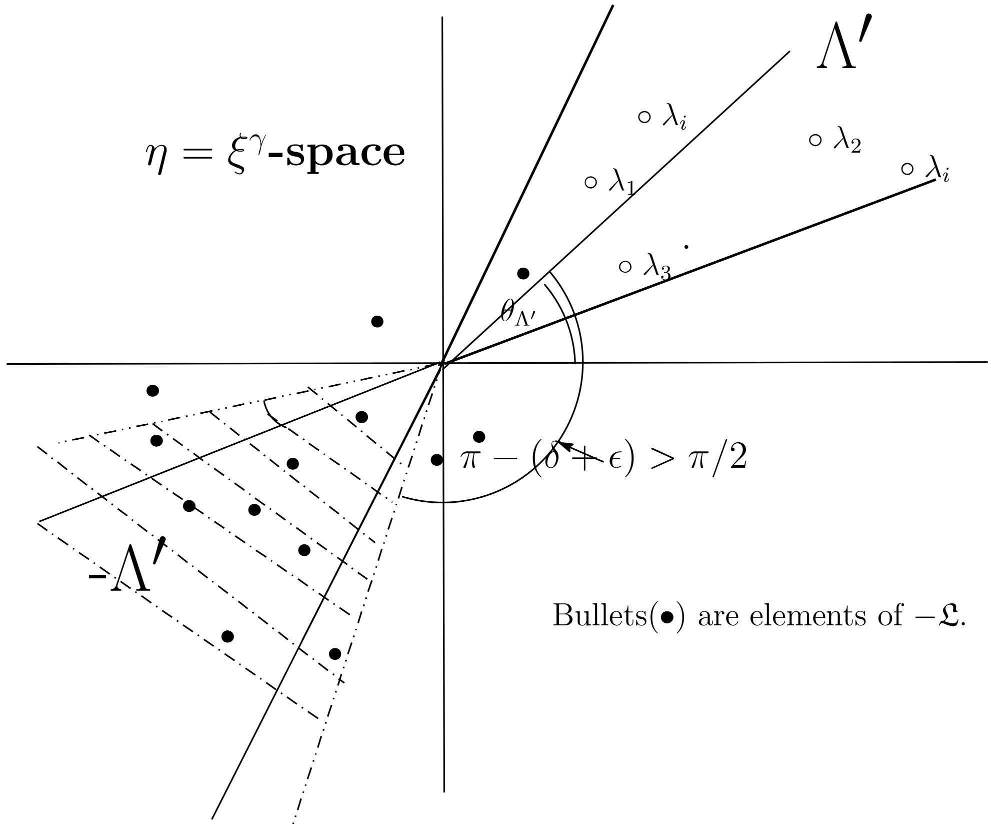
$$\mathfrak{L} = \bigcup_{i=1}^{n} \Big\{ \sum_{j=1}^{n'} \lambda_j m_j - \lambda_i; \; |m| \geq 2 \Big\} \tag{2.6}$$

and $L(\theta) = \{r \geq 0; re^{i\theta}\}$ be a half line in a direction θ.

Lemma 2.3. *There exist $\widetilde{\theta}$ and $\widetilde{\epsilon} > 0$ such that $L(\theta) \cap (-\overline{\mathfrak{L}}) = \emptyset$ for $\widetilde{J} = (\tilde{\theta} - \tilde{\epsilon}, \tilde{\theta} + \tilde{\epsilon})$ and a constant $C_{\tilde{I}} > 0$ such that for $\eta \in S(\tilde{J})$*

$$|\gamma\eta + \sum_{j=1}^{n'} \lambda_j m_j - \lambda_i| \geq C_{\tilde{I}}(|\eta| + |m|) \quad |m| \geq 2. \tag{2.7}$$

Remark 2.4. (1)*Let $\theta_* = \widetilde{\theta}/\gamma$. Then $L(\gamma\theta_*) \cap (-\overline{\mathfrak{L}}) = \emptyset$.*
(2) *We can choose $\widetilde{\theta} = \theta_{\mathbf{\Lambda}'}$, by changing $\widetilde{\delta}_{\mathbf{\Lambda}'}$ if necessary.*



We take θ_* such that .

Condition 3

$\widehat{I} = (\theta_* - \epsilon_*, \theta_* + \epsilon_*)$ ($\epsilon_* > 0$) such that $\widehat{I} \cap \mathbf{\Theta}_1 = \emptyset$ and $L(\gamma\theta) \cap (-\overline{\mathfrak{L}}) = \emptyset$ for $\theta \in \widehat{I}$.

Let $\widehat{J}$ be that in Lemma 2.3. Since $\mathbf{\Lambda}^\sharp$ is finite, we can take $\widehat{J}$ such that

$S(\widehat{J}) \cap \mathbf{\Lambda}^\sharp = \emptyset$. Hence $S(\widehat{J}) \cap \big((-\overline{\mathfrak{L}}) \cup \mathbf{\Lambda}^\sharp\big) = \emptyset$, Let $\theta_* = \widehat{\theta}_0/\gamma$, $\epsilon_* = \widehat{\epsilon}_0/\gamma$ and $\widehat{I} = (\theta_* - \epsilon_*, \theta_* + \epsilon_*)$. Then $\gamma\widehat{I} = \widehat{J}$ and if $\xi \in S(\widehat{I})$, $\xi^\gamma \in S(\widehat{J})$. This $\widehat{I}$ satisfies Condition 3.

Let $\delta_* = \pi/2\gamma + \epsilon_* > \pi/2\gamma$ and $I = (\theta_* - \delta_*, \theta_* + \delta_*)$. Then we give one of the main results

Theorem 2.5. *There exists* $\Phi(x, Z) = (\phi_1(x, Z), \cdots, \phi_n(x, Z))$ *such that for any small* $\epsilon > 0$ *there exists* $r_\epsilon > 0$, $\phi_i(x, Z) \in \mathscr{O}_{\{1/\gamma\}}(S_{\{0\}}(I_\epsilon) \times \{Z \in \mathbb{C}^{n'}; |z| < r_\epsilon\})$ *and the followings hold.*

(1) $\phi_i(x, Z) = z_i + O(|Z|^2)$ *for* $1 \le i \le n'$ *and* $\phi_i(x, Z) = O(|Z|^2)$ *for* $i > n'$.

(2) Let $\mathcal{S}$ *be an open set in* $S_{\{0\}}(I_\epsilon)$ *and* $Z(x) = (z_1(x), \cdots, z_{n'}(x))$ $(x \in \mathcal{S}, |Z(x)| < r_\epsilon)$ *be a solution of*

$$x^{1+\gamma}\frac{dz_i}{dx} = \lambda_i(x) z_i, \ i = 1, 2, \cdots, n'. \tag{2.8}$$

Then $Y(x) = \Phi(x, Z(x))$, $y_i(x) = \phi_i(x, z_1(x), \cdots, z_{n'}(x))$ $(1 \le i \le n)$, *satisfies* (Eq-Y) *in* $\mathcal{S}$.

Remark 2.6. *Theorem 2.5 means existence of solutions of* (Eq-Y) *with exponential series called transseries of equations,*

$$z_i(x) = A_i \exp\Big(\int^x \frac{\Lambda_i(\tau)}{\tau^{\gamma+1}} d\tau\Big) \ \ (1 \le i \le n'),$$
$$y_i = z_i(x) + \sum_{|p| \ge 2} C_p^i(x) Z(x)^p \ \ (1 \le i \le n'), \quad y_i = \sum_{|p| \ge 2} C_p^i(x) Z(x)^p, \ \ (i > n').$$

3 Costruction of $\Phi(x, Z)$ of Theorem 2.5

The system of equations to study is

$$\begin{cases} x^{1+\gamma}\dfrac{dY}{dx} =\Lambda(x)Y + F(x, Y), \ Y = {}^t(y_1 . y_2, \cdots, y_n) \\ \Lambda(x) = diag.(\Lambda_1(x), \cdots, \Lambda_n(x)), \\ F(x, Y) = {}^t(f_1(x, Y), \cdots, f_n(x, Y)), \end{cases} \tag{Eq-Y}$$

where $\{\Lambda_i(x)\}_{1\le i\le n}$ are polynomials with degree $\le \gamma$ and $\Lambda_i(0) = \lambda_i$. Our assumptions are

$$\begin{cases} \{f_i(x,Y)\}_{1\le i\le n}\} \subset \mathscr{O}_{\{1/\gamma\}}(S_0(I)\times\{|Y|<R\}), \\ I = (\theta_* - \delta_*, \theta_* + \delta_*), \widehat{I} = (\theta_* - \epsilon_*, \theta_* + \epsilon_*),\ \delta_* = \pi/2\gamma + \epsilon_*, \\ \widehat{I} \cap \mathbf{\Theta}_1 = \emptyset,\ L(\gamma\theta) \cap (-\overline{\mathfrak{L}}) = \emptyset\ for\ \theta \in \widehat{I} \end{cases} \tag{3.1}$$

with $f_i(0,Y) = 0,\ f_i(x,Y) = O(|Y|^2)$ and (2.7) holds.

3.1 A system of nonlinear partial differential equations

We introduce the following system of nonlinear partial differential equations which is similar to that appeared in [13] to construct $\Phi(x,Z)$ in Theorem 2.5,

$$\begin{cases} \Phi(x,Z) = (\phi_1(x,Z), \phi_2(x,Z), \cdots, \phi_n(x,Z)) \\ x^{1+\gamma}\dfrac{\partial\phi_i}{\partial x} + \displaystyle\sum_{j=1}^{n'} \lambda_j(x) z_j \dfrac{\partial\phi_i}{\partial z_j} - \lambda_i(x)\phi_i = f_i(x,\Phi)\ \ 1\le i\le n \\ (x,Z) = (x, z_1, \cdots, z_{n'}) \in \mathbb{C}\times\mathbb{C}^{n'} \end{cases} \tag{3.2}$$

Let $Z(x) = (z_1(x), \cdots, z_{n'}(x))$ be a solution of

$$x^{1+\gamma}\frac{dz_i}{dx} = \lambda_i(x) z_i,\ i = 1, 2, \cdots, n'.$$

If we find a nice solution $\Phi(x,Z)$ of (3.2), then $Y(x) = \Phi(x, Z(x))$ $\big(y_i(x) = \phi_i(x, z_1(x), \cdots, \phi_{n'}(x))\big)$ will satisfy (Eq-Y)

$$\begin{aligned} x^{1+\gamma}y'(x) &= x^{1+\gamma}\big(\partial_x\phi_i(x, Z(x)) + \sum_{j=1}^{n'} \partial_{z_j}\phi_i(x, Z(x)z_j'(x))\big) \\ &= x^{1+\gamma}\partial_x\phi_i(x, Z(x) + \big(\sum_{j=1}^{n'} \lambda_j(x)\partial_{z_j}\phi_i(x, Z(x))\big) \\ &= \lambda_i(x)\phi_i + f_i(x,\Phi) = \lambda_i(x) y_i + f_i(x, Y(x)). \end{aligned}$$

Let $\Psi(Z) = (\psi_1(Z), \psi_2(Z), \cdots, \psi_n(Z)) = (z_1, z_2, \cdots, z_{n'}, 0, \cdots, 0)$ and $\Phi(x,Z) = U(x,Z) + \Psi(Z)$, $(\phi_i(x,Z) = u_i(x,Z) + \psi_i(Z))$. Then

$$\big(x^{1+\gamma}\frac{\partial}{\partial x} + \sum_{j=1}^{n'} \Lambda_j(x) z_j \frac{\partial}{\partial z_j} - \Lambda_i(x)\big)u_i, = f_i(x, U + \Psi(Z)). \tag{Eq-U}$$

There exist $\{g_{i,k,\ell}(x) \in \mathscr{O}_{\{1/\gamma\}}(S(I))\}$ such that

$$f_i(x, U + \Psi(Z)) == \sum_{\substack{k\in\mathbb{N}^{n'},\ell\in\mathbb{N}^n,\\ |k|+|\ell|\geq 2,\ell\neq 0}} g_{i,k,\ell}(x) Z^k U^\ell + f_i(x, \Psi(Z))$$

with $g_{i,k,\ell}(0) = 0$ ($\because f_i(0, Y) = 0$) and $|f_i(x, \Psi(Z))| \leq M|x||Z|^2$. Let

$$\begin{aligned} &L = x^{1+\gamma}\frac{\partial}{\partial x} + \sum_{j=1}^{n'} \lambda_j z_j \frac{\partial}{\partial z_j} - \lambda_i, \quad \lambda_i = \Lambda_i(0) \\ &\Lambda_i^*(x) = \Lambda_i(x) - \lambda_i, \ \ h_i(x, Z) = f_i(x, \Psi(Z)) = \sum_{|p|\geq 2} h_{i,p}(x) Z^p. \end{aligned} \tag{3.3}$$

Then we get

$$Lu_i = -\,(\sum_{j=1}^{n'} \Lambda_j^*(x) z_j \frac{\partial}{\partial z_j} - \Lambda_i^*(x)) u_i + \sum_{\substack{k\in\mathbb{N}^{n'},\ell\in\mathbb{N}^n,\\ |k|+|\ell|\geq 2,\ell\neq 0}} g_{i,k,\ell}(x) Z^k U^\ell + h_i(x, Z). \tag{3.4}$$

We introduce an auxiliary parameter ε

$$Lu_i = -\,\varepsilon(\sum_{j=1}^{n'} \Lambda_j^*(x) z_j \frac{\partial}{\partial z_j} - \Lambda_i^*(x)) u_i + \varepsilon \sum_{\substack{k\in\mathbb{N}^{n'},\ell\in\mathbb{N}^n,\\ |k|+|\ell|\geq 2,\ell\neq 0}} g_{i,k,\ell}(x) Z^k U^\ell + \varepsilon h_i(x, Z). \tag{Eq-U$_\varepsilon$}$$

If $\varepsilon = 1$, (Eq-U$_\varepsilon$) coincides with (Eq-U). Let

$$\begin{cases} U(x, Z, \varepsilon) = (u_1(x, Z, \varepsilon), \cdots, u_n(x, Z, \varepsilon)) \\ u_i(x, Z, \varepsilon) = \displaystyle\sum_{\substack{(p,q)\in\mathbb{N}^{n'}\times\mathbb{N} \\ |p|\geq 2,\ q\geq 1}} C_{i,p,q}(x) Z^p \varepsilon^q, \quad 1 \leq i \leq n. \end{cases} \tag{3.5}$$

By substituting $u_i(x,Z,\varepsilon)$ into (Eq-U$_\varepsilon$), we have

$$(x^{1+\gamma}\frac{d}{dx}+\sum_{j=1}^{n'}p_j\lambda_j-\lambda_i)C_{i,p,q}(x)=-\big(\sum_{j=1}^{n'}p_j\Lambda_j^*(x)-\Lambda_i^*(x)\big)C_{i,p,q-1}(x)$$

$$+\sum_{\substack{k\in\mathbb{N}^{n'},\ell\in\mathbb{N}^n\\ |k|+|\ell|\geq 2,\ell\neq 0}} g_{i,k,\ell}(x)\Big(\sum_{\substack{\sum_{i=1}^n\big(\sum_{j=1}^{\ell_i}p^{i,j}\big)+k=p\\ \sum_{i=1}^n\big(\sum_{j=1}^{\ell_i}q^{i,j}\big)+1=q}}\prod_{j=1}^{\ell_1}C_{1,p^{1,j},q^{1,j}}(x)$$

$$\times\prod_{j=1}^{\ell_2}C_{2,p^{2,j},q^{2,j}}(x)\cdots\cdots\prod_{j=1}^{\ell_n}C_{n,p^{n,j},q^{n,j}}(x)\Big)+\delta_{q,1}h_{i,p}(x), \tag{Eq-C}$$

3.2 Construction of $C_{i,p,q}(x)$ and convolution equations

We solve (Eq-C) by Laplace integral. Let

$$C_{i,p,q}(x)=\int_0^{\infty e^{i\theta}}e^{-(\frac{\xi}{x})^\gamma}\widehat{C}_{i,p,q}(\xi)d\xi^\gamma \quad \theta\in\widehat{I}_\epsilon \;\; d\xi^\gamma=\gamma\xi^{\gamma-1}d\xi \tag{3.6}$$

Let us note relations about product of functions and their convolution and use a notation $W_1(\xi)\underset{\gamma}{*}W_2(\xi)\underset{\gamma}{*}\cdots\underset{\gamma}{*}W_N(\xi)=\underbrace{\prod_{i=1}^N W_i(\xi)}_{*\gamma}$. We get a system of convolution equations to $\widehat{C}_{i,p,q}(\xi)$.

$$(\gamma\xi^\gamma+\sum_{j=1}^{n'}p_j\lambda_j-\lambda_i)\widehat{C}_{i,p,q}(\xi)=-\big(\sum_{j=1}^{n'}p_j\widehat{\Lambda_j^*}(\xi)-\widehat{\Lambda_i^*}(\xi)\big)\underset{\gamma}{*}\widehat{C}_{i,p,q-1}(\xi)$$

$$+\sum_{\substack{k\in\mathbb{N}^{n'},\ell\in\mathbb{N}^n\\ |k|+|\ell|\geq 2,\ell\neq 0}}\widehat{g}_{i,k,\ell}(\xi)\underset{\gamma}{*}\Big(\sum_{\substack{\sum_{i=1}^n(\sum_{j=1}^{\ell_i}p^{i,j})+k=p\\ \sum_{i=1}^n(\sum_{j=1}^{\ell_i}q^{i,j})+1=q}} \tag{3.7}$$

$$\underbrace{\prod_{j=1}^{\ell_1}\widehat{C}_{1,p^{1,j},q^{1,j}}(\xi)\prod_{j=1}^{\ell_2}\widehat{C}_{2,p^{2,j},q^{2,j}}(\xi)\cdots\prod_{j=1}^{\ell_n}\widehat{C}_{n,p^{n,j},q^{n,j}}(\xi)}_{*\gamma}\Big)+\delta_{q,1}\widehat{h}_{i,p}(\xi).$$

$\widehat{C}_{i,p,q}(\xi)(|p| \geq 2, q \geq 1)$ are successively determined and they are holomorphic in $(\{0| < |\xi| < r\} \cup S(\widehat{I}_\epsilon)) \times \{|z| < R\}$. Moreover $\xi^{\gamma-1}\widehat{C}_{i,p,q}(\xi)(|p| \geq 2, q \geq 1)$ is holomrphic at $\xi = 0$. We have

Proposition 3.1. (Estimate of $\widehat{C}_{i,p,q}(\xi)$) *There exist positive constants $r, M_{i,p,q}$ and c such that*

$$|\widehat{C}_{i,p,q}(\xi)| \leq \frac{M_{i,p,q}|\xi|^{q-\gamma}e^{c|\xi|^\gamma}}{\Gamma(q/\gamma)} \quad \xi \in \{0 < |\xi| < r\} \cup S(\widehat{I}_\epsilon) \tag{3.8}$$

and the series $\sum_{\substack{p\in\mathbb{N}^{n'},|p|\geq 2\\ q\in\mathbb{N},q\geq 1}} M_{i,p,q}T^p s^q$ converges in a neighborhood of $(T, s) = (0, 0) \in \mathbb{C}^{n'} \times \mathbb{C}$.

We apply the method of implicit functions to obtain the estimate. This method is often used in [7]. It follows from Proposition 3.1 that there exist A, B and $c' > 0$ such that $M_{i,p,q} \leq A^{|p|}B^q$ and

$$\begin{aligned}
\sum_{|p|\geq 2,q\geq 1} |\widehat{C}_{i,p,q}(\xi)Z^p\varepsilon^q| &\leq \sum_{|p|\geq 2,q\geq 1} \frac{M_{i,p,q}|\xi|^{q-\gamma}|Z^p\varepsilon^q|}{\Gamma(q/\gamma)} e^{c|\xi|^\gamma} \\
\leq \sum_{|p|\geq 2,q\geq 1} \frac{A^{|p|}B^q|\xi|^{q-\gamma}|Z^p\varepsilon^q|}{\Gamma(q/\gamma)} e^{c|\xi|^\gamma} &\leq (\sum_{|p|\geq 2} A^{|p|}|Z^p|)\frac{|\xi|^{1-\gamma}e^{(c+c'\varepsilon)|\xi|^\gamma}}{\Gamma(1/\gamma)},
\end{aligned} \tag{3.9}$$

which converges for any ε and $\{Z \in \mathbb{C}^{n'}; |z_i| < A^{-1},\ 1 \leq i \leq n'\}$. $\{\widehat{C}_{i,p,q}(\xi)\}$ $(1 \leq n)$ satisfy (3.7). Let $\varepsilon = 1$, $\widehat{C}_{i,p}(\xi, 1) = \sum_{q\geq 1} \widehat{C}_{i,p,q}(\xi)$ and

$$C_{i,p}(x) = \int_{L(\theta)} e^{-(\frac{\xi}{x})^\gamma} \widehat{C}_{i,p}(\xi, 1)d\xi^\gamma \quad \theta \in \widehat{I}_\epsilon.. \tag{3.10}$$

Then $u_i(x, Z) = \sum_{p\in\mathbb{N}^{n'},|p|\geq 2} C_{i,p}(x)Z^p$ and we get $\phi_i(x, Z) = \psi_i(z)+u_i(x, Z)$ $(1 \leq i \leq n)$ in Theorem 2.5

4 Equation with $F_0(x) \not\equiv 0$

Let us study the case $F_0(x) \not\equiv 0$,

$$\begin{cases} Y &= {}^t(y_1.y_2, \cdots, y_n) \\ x^{1+\gamma}\dfrac{dY}{dx} &= F_0(x) + A(x)Y + F(x, Y), \end{cases} \tag{4.1}$$

$A(x) = (a_{i,j}(x))_{1\le i,j\le n}$. $F_0(x)$ and $a_{i,j}(x)$ are holomorphic in a neighborhood of $x = 0$ and $F_0(0) = 0$. $F(x, Y)$ is holomorphic in a neighborhood of $(x, Y) = (0, 0)$ and $F(x, Y) = O(|Y|^2)$. $\{\lambda_i\}_{1\le i\le n}$ are eigenvalues of $A(0)$. $\lambda_i \neq 0$ and distinct. Let $\omega_i = \arg \lambda_i,\ 0 \le \omega_i < 2\pi$. and

$$\mathbf{\Theta}_0 = \{(\omega_i + 2\pi\ell)/\gamma,\ 1 \le i \le n, \ell \in \mathbb{Z}\}. \tag{4.2}$$

$\mathbf{\Theta}_0 \cup \mathbf{\Theta}_1$ is called the set of singular directions. There exists a unique formal power series solution $\widetilde{K}(x)$ with $\widetilde{K}(0) = 0$ of (4.1). Its Borel summability follows from Braaksma [1].

Proposition 4.1. *Let $\theta_* \notin \mathbf{\Theta}_0$. Then there exists $K(x)$, which is γ-Borel summable in the direction θ_* with $K(x) \sim \widetilde{K}(x)$.*

Let us transform (4.1) to the case we can apply Theorem 2.5.
(1) Let $Y = xW + K(x)$ and $\theta_* \notin \mathbf{\Theta}_0 \cup \mathbf{\Theta}_1$. Then

$$\begin{aligned} x^{1+\gamma}\frac{dW}{dx} =& (A(x) - x^\gamma I)W + x^{-1}\big(F(x, xW + K(x)) - F(x, K(x))\big) \\ =& \sum_{j=1}^{n}(a_{i,j}(x) - \delta_{i,j}x^\gamma)w_j + \sum_{j=1}^{n}\frac{\partial}{\partial w_j}f_i(x, K(x))w_j + g_i(x, W), \end{aligned}$$

and we get

$$\begin{cases} x^{1+\gamma}\dfrac{dW}{dx} = A'(x)W + G(x, W) \quad W = {}^t(w_1.w_2, \cdots, w_n) \\ A'(x) = \big(a_{i,j}(x) - \delta_{i,j}x^\gamma + \dfrac{\partial}{\partial w_j}f_i(x, K(x))\big),\ A'(0) = A(0) \\ G(x, W) = {}^t(g_1(x, W), \cdots, g_n(x, W)) \end{cases} \tag{4.3}$$

(2) Next we transform $W = P(x)U$ by an invertible Linear transformation $P(x)$ with elements in $\mathscr{O}_{\{1/\gamma\}}(S_0(I))$ ($I = (\theta_* - \delta_*, , \theta_* + \delta_*)$, $\delta_* > \pi/2\gamma$) and have

$$\begin{cases} U =& {}^t(u_1.u_2, \cdots, u_n) \\ x^{1+\gamma}\dfrac{dU}{dx} =& B(x)U + H(x, U) \\ B(x) =& diag.\ (b_1(x), b_2(x), \cdots, b_n(x))). \\ H(x, U) =& {}^t(h_1(x, U), h_2(x, U), \cdots, h_n(x, U)), \end{cases} \tag{4.4}$$

$b_i(x)$ is a polynomial with degree $\le \gamma$ and $b_i(0) = \lambda_i$, $h_i(x, U) \in \mathscr{O}_{\{1/\gamma\}}(S_0(I)\times \Omega)$ with $h_i(0, U) = 0$ and $h_i(x, U) = O(|U|^2)$. $B(x)$ depends on $K(x)$. Thus

we simplify (4.1) to (4.4) by $Y = K(x) + xP(x)U$
(3) Set $\Lambda' = \{\lambda_i; 1 \le i \le n'\}$ and assume Conditions 1 and 2 hold. Consider an $n' \times n'$ system of linear equations

$$\begin{cases} Z =& {}^t(z_1.z_2, \cdots, z_{n'}) \\ x^{1+\gamma}\dfrac{dz_i}{dx} =& b_i(x)z_i,\ 1 \le i \le n' \end{cases} \quad U(x) = \Phi(x, Z(x)) \tag{4.5}$$

By applying Theorem 2.5, we have

Theorem 4.2. *There exists* $\Phi(x,z) = {}^t(\phi_1(x,z), \cdots, \phi_n(x,z))$, $\phi_i(x,z) \in \mathscr{O}_{\{1/\gamma\}}(S_{\{0\}}(I_\epsilon) \times \{z \in \mathbb{C}^{n'}; |z| < r_\epsilon\})$ *with the following properties.*

(1) $\phi_i(x,z) = z_i + O(|z|^2)$ *for* $1 \le i \le n'$ *and* $\phi_i(x,z) = O(|z|^2)$ *for* $i > n'$.

(2) Let $\mathcal{S}$ *be an open set in* $S_{\{0\}}(I_\epsilon)$ *and* $Z(x) = (z_1(x), \cdots, z_{n'}(x))$ $(x \in \mathcal{S}, |Z(x)| < r_\epsilon)$ *be a solution of* (4.5)

Then $Y(x) = K(x) + xP(x)\Phi(x, Z(x))$ *satisfies* (4.1) *in* $\mathcal{S}$.

5 Examples

We remark that if equation (4.1) is 2×2 system with eigenvalue $\lambda_1\lambda_2 \neq 0$ and $\arg\lambda_1 = \omega$, $\arg\lambda_2 = \omega + \pi$, then we can apply Theorem 4.2 (Remark 2.2). We give examples. We apply Theorem 4.2 to *Painlevé 2*.

$$(P)_2 : y'' = 2y^3 + ty + a. \tag{P_2}$$

By $t = 1/s$

$$s(s^2\frac{d}{ds})^2 y = 2sy^3 + y + as \tag{5.1}$$

($\mathbf{P_2.1}$) There exists a unique formal power series solution $\widetilde{k}(s) = -as + 2(a^3 - a)s^4 + \cdots$ to (5.1). Let $y = -as + z$. Then $s(s^2\frac{d}{ds})^2 z = (1 + 6a^2s^3)z + 2(a - a^3)s^4 - 6as^2z^2 + 2sz^3$ and

$$(s^{5/2}\frac{d}{ds})^2 z - \frac{1}{2}s^4\frac{d}{ds}z = (1 + 6a^2s^3)z + a_0(s) + a_2(s)z^2 + a_3(s)z^3.$$

Put $u = z$, $v = s^{5/2}\frac{d}{ds}z$.

$$\begin{aligned} s^{5/2}\frac{d}{ds}\begin{bmatrix} u \\ v \end{bmatrix} = & \left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + s^{3/2}\begin{bmatrix} 0 & 0 \\ 0 & \frac{1}{2} \end{bmatrix} + O(s^3) \right) \begin{bmatrix} u \\ v \end{bmatrix} \\ & + \begin{bmatrix} 0 \\ a_0(s) + a_2(s)u^2 + a_3(s)u^3 \end{bmatrix} \end{aligned}. \tag{5.2}$$

We have by changing $x = s^{1/2}$

$$x^4 \frac{d}{dx}\begin{bmatrix} u \\ v \end{bmatrix} = \left(\begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix} + x^3 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} + O(x^6) \right) \begin{bmatrix} u \\ v \end{bmatrix}_{..} \\ + 2 \begin{bmatrix} 0 \\ a_0(x^2) + a_2(x^2)u^2 + a_3(x^2)u^3 \end{bmatrix} \tag{5.3}$$

Hence we can apply Theorem 4.2 to (5.3). $\gamma = 3$, $\lambda_1 = 2, \lambda_2 = -2$ and $\mathbf{\Theta}_0 \cup \mathbf{\Theta}_1 = \{\frac{\pi \ell}{3}; \ell \in \mathbb{Z}\}$ for (5.3) and $\tilde{k}(s) \in \mathbb{C}[[s]]$ is $\frac{3}{2}$ *Borel summable.*
($\mathbf{P_2.2}$). Let $2c^2 + 1 = 0$ and $y = s^{-1/2}(c + z)$. Then

$$(s^{5/2}\frac{d}{ds})^2 z - \frac{3}{2}s^4 \frac{d}{ds} z = (6c^2 + 1 + \frac{s^3}{4})z + 6cz^2 + 2z^3 + as^{3/2} + \frac{cs^3}{4}.$$

Set $u = z$, $v = s^{5/2}\frac{d}{ds}z$. Then

$$s^{5/2}\frac{d}{ds}\begin{bmatrix} u \\ v \end{bmatrix} = \left(\begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix} + s^{3/2} \begin{bmatrix} 0 & 0 \\ 0 & \frac{3}{2} \end{bmatrix} + O(s^3) \right) \begin{bmatrix} u \\ v \end{bmatrix}. \\ + \begin{bmatrix} 0 \\ a_0(s) + 6cu^2 + 2u^3 \end{bmatrix} \tag{5.4}$$

The situation of this case is the same as $(P_2.1)$ and we can also apply Theorem 4.2.

Another example is *Noumi-Yamada system*. This system is symmetric and it is knowun that it is almost equivalent to *Painlevé 4* (see Noumi [12]).

$$\begin{cases} f_0' = f_0(f_1 - f_2) + \alpha_0 \\ f_1' = f_1(f_2 - f_0) + \alpha_1 \qquad \alpha_0 + \alpha_1 + \alpha_2 = 1 \\ f_2' = f_2(f_0 - f_1) + \alpha_2 \end{cases} \tag{NY}$$

There is a constant k such that $f_0 + f_1 + f_2 = t + k$. We may assume $k = 0$. Hence $f_0 + f_1 + f_2 = t$ and get 2×2 system

$$\begin{cases} f_1' = -tf_1 + f_1^2 + 2f_1f_2 + \alpha_1 \\ f_2' = tf_2 - 2f_1f_2 - f_2^2 + \alpha_2. \end{cases} \tag{5.5}$$

Let $f_1(t) = a_1 t + g_1(t), f_2(t) = a_2 t + g_2(t),\ g_1(t), g_2(t) = O(1)\ t \to \infty$. Then $(a_1, a_2) = (0.0),\ (0, 1),\ (1, 0),\ (1/3, 1/3)$ and

$$\frac{d}{dt}\begin{pmatrix} g_1 \\ g_2 \end{pmatrix} = tA \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} + \begin{pmatrix} g_1^2 + 2g_1g_2 + \alpha_1 - a_1 \\ -g_2^2 - 2g_1g_2 + \alpha_2 - a_2 \end{pmatrix} \tag{5.6}$$

where

$$A = \begin{pmatrix} -1+2a_1+2a_2 & 2a_1 \\ -2a_2 & 1-2a_1-2a_2 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ -2 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 1/3 & 2/3 \\ -2/3 & -1/3 \end{pmatrix}$$

The eigenvalues of A of is $\pm\lambda \neq 0$ for each case, so conditions of Theorem 4.2 are satisfied and we can apply it to each case.

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