# Connection formulas around a double turning point via Borel summability of transformation series

By

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**Abstract**

In the exact WKB analysis, in a neighborhood of a double turning point the equation is transformed to its canonical equation (the degenerate Weber equation) by a formal change of coordinate. Borel summability of the formal transformation was given in my preceding paper [6]. Based on the Borel summability, we complete derivation of the connection formula around a double turning point, which was formally derived in [8]

## § 1. Introduction

We consider a connection problem in the exact WKB analysis around a double turning point. The equation we deal with is the following:

$$\left[\frac{d^2}{dx^2} - \eta^2 Q(x)\right]\psi = 0, \tag{1.1}$$

where $\eta > 0$ is a large parameter and $Q(x)$ is a rational function. We assume that the origin $x = 0$ is a double turning point of the equation, i.e., a double zero of $Q(x)$. Connection problem around a double turning point was studied in [2], [3] in the framework of the exact WKB analysis.

We consider this problem from the viewpoint of transformation theory. In a complex neighborhood of $x = 0$, by a formal change of coordinate

$$z = z(x, \eta) = z_0(x) + \eta^{-1} z_1(x) + \eta^{-2} z_2(x) + \cdots, \tag{1.2}$$

the equation is transformed to the canonical equation (the degenerate Weber equation)

$$\left[\frac{d^2}{dz^2} - \eta^2\left(\frac{z^2}{4} - \eta^{-1}\kappa(\eta)\right)\right]\phi = 0, \tag{1.3}$$

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where $\kappa(\eta) = \kappa_0 + \eta^{-1}\kappa_1 + \eta^{-2}\kappa_2 + \cdots$ is a certain formal power series with complex constant coefficients([1, §1], [8], [6]). Here $z_j$'s are holomorphic in a common neighborhood of $x = 0$. $z(x, \eta)$ and $E(\eta)$ are unique once we fix a choice of $z_0$, and we call $z(x, \eta)$ (and $E(\eta)$) the transformation series. Under the assumption that any of the four Stokes lines generated by the turning point $x = 0$ flows into a singular point of the equation, Borel summability of $E(\eta)$ follows from that of WKB solutions ([5], [6]). Also, in my preceding paper [6], Borel summability of $z(x, \eta)$ was established.

In this paper, using the Borel summability of the transformation series, we derive connection formula for suitably normalized WKB solutions of (1.1) around the double turning point. This gives an analytic basis of the formal derivation employed in [8].

## § 2. Derivation of the connection formula

### § 2.1. WKB solutions and Borel resummation

The equation (1.1) has WKB solutions

$$\psi = \exp \int^x S\, dx, \tag{2.1}$$

where $S = S(x, \eta) = \eta S_{-1}(x) + S_0(x) + \eta^{-1} S_1(x) + \cdots$ is a formal power series. The top term $S_{-1}$ satisfies the characteristic equation

$$\xi^2 - Q(x) = 0. \tag{2.2}$$

Once we choose $S_{-1} = \xi^{(\pm)}(x)$, then the succeeding terms $S_0,\ S_1, \ldots$ are determined uniquely, and we have two WKB solutions $\exp \int^x S^{(\pm)}\, dx = \exp \int^x \left(\xi^{(\pm)} + \cdots\right) dx$. Letting $S^{(\pm)} = S_{\text{even}} \pm S_{\text{odd}}$ i.e. $S_{\text{even}} = \frac{1}{2}(S^{(+)} + S^{(-)})$ and $S_{\text{odd}} = \frac{1}{2}(S^{(+)} - S^{(-)})$, then $S_{\text{even}} = -\frac{1}{2}\frac{d}{dx}\log S_{\text{odd}}$ holds (cf. [4]) and two WKB solutions can be written as follows:

$$\psi^{(\pm)}(x, \eta) = \frac{1}{\sqrt{S_{\text{odd}}}} \exp\left(\pm \int^x S_{\text{odd}}\, dx\right). \tag{2.3}$$

Letting $\pm\zeta(x) = \int^x \xi^{(\pm)}(x)\, dx = \pm \int^x \sqrt{Q(x)}\, dx$ and using the Taylor expansion, we have

$$\psi^{(\pm)}(x, \eta) = e^{\pm\zeta(x)} \sum_{n=0}^{\infty} \psi_n^{(\pm)}(x) \eta^{-n-1/2}. \tag{2.4}$$

Also, the degenerate Weber equation (1.3) with a genuine constant $\kappa \in \mathbb{C}$ has WKB solutions

$$\phi^{(\pm)}(z, \kappa, \eta) = \frac{1}{\sqrt{S_{\text{odd}}^{(\text{can})}}} \exp\left(\pm \int^z S_{\text{odd}}^{(\text{can})}\, dz\right), \tag{2.5}$$

and especially these can be normalized as follows:

(2.6)
$$\phi^{(\pm)}(z,\kappa,\eta) = \frac{1}{\sqrt{S^{(\mathrm{can})}_{\mathrm{odd}}}} z^{\mp\kappa} \exp\left(\pm\frac{z^2}{4}\eta\right) \exp\left(\pm\int_\infty^z \left(S^{(\mathrm{can})}_{\mathrm{odd}} - \eta S^{(\mathrm{can})}_{-1} - S^{(\mathrm{can})}_0\right) dz\right).$$

This normalization is equivalent to giving the WKB solutions the following homogeneity:

$$\phi^{(\pm)}(z,\kappa,\eta) = \sqrt{2} z^{\mp\kappa-\frac{1}{2}} \exp\left(\pm\frac{z^2}{4}\eta\right) \sum_{n=0}^{\infty} c_n^{(\pm)} z^{-2n} \eta^{-n-1/2}, \tag{2.7}$$

where $c_n^{(\pm)}$'s are certain constants (depending on $\kappa$).

For the equation (1.1), we take the following WKB solutions

$$\psi^{(\pm)}(x,\eta) = \left(\frac{\partial z}{\partial x}(x,\eta)\right)^{-\frac{1}{2}} \phi(z(x,\eta),\kappa(\eta),\eta). \tag{2.8}$$

Here, $\kappa(\eta) = \kappa_0 + \eta^{-1}\kappa_1 + \cdots$ and $z(x,\eta) = z_0(x) + \eta^{-1}z_1(x) + \cdots$ are transformation series, which are determined from the equation (1.1).

To avoid confusion, we also clarify the definition of Borel resummation here: for a formal power series with an exponent $f(\eta) = e^{\eta\zeta}\sum_{n=0}^{\infty} f_n\eta^{-n-\alpha}$ $(\zeta,\alpha,f_n \in \mathbb{C}$, $\alpha \neq 0,-1,-2,\dots)$, the Borel transform $\mathcal{B}[f](y) = f_{\mathrm{B}}(y)$ is

$$\mathcal{B}[f](y) = f_{\mathrm{B}}(y) := \sum_{n=0}^{\infty} \frac{f_n}{\Gamma(n+\alpha)} (y+\zeta)^{n+\alpha-1}. \tag{2.9}$$

The Borel sum $F(\eta)$ of $f(\eta)$ is (if defined)

$$F(\eta) := \int_{-\zeta}^{\infty} f_{\mathrm{B}}(y) e^{-\eta y}\, dy. \tag{2.10}$$

## § 2.2. Connection formula

First we name regions; the turning point $z = 0$ of the canonical equation (1.3) generates four Stokes lines, and they divide $z$-plane into four regions. We call them region I, …, region IV as in Figure 1. Likewise, by the equation (1.1), we have four regions $x$-plane around $x = 0$, and we give numbers to them I, …, IV so that region J in $x$-plane is mapped to region J in $z$-plane (J = I, II, III, IV) by $z_0(x)$, the principal part of the transformation series (Figure 2).

We consider connection problem as $x$ moves from region IV to I. Following [8], we let

$$u^{(\pm)}(z,\kappa,\eta) := \eta^{\mp\frac{\kappa}{2}} \phi^{(\pm)}(z,\kappa,\eta), \tag{2.11}$$

$$w^{(\pm)}(x,\eta) := \eta^{\mp\frac{\kappa(\eta)}{2}} \psi^{(\pm)}(x,\eta). \tag{2.12}$$

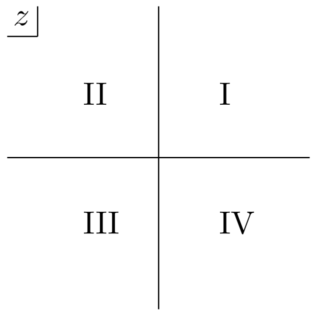


Figure 1. Stokes regions in $z$-plane

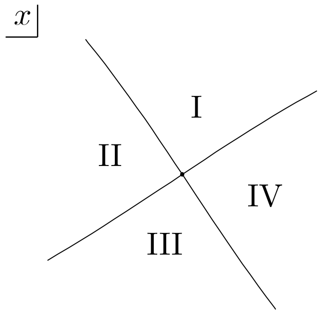


Figure 2. Stokes regions in $x$-plane

Then the Borel transform $u_{\mathrm{B}}^{(\pm)}(z,\kappa,y)$ can be explicitly written in terms of the Gauss hypergeometric function, and we see singularities in the Borel plane ([8, Appendix A.1]); $u_{\mathrm{B}}^{(\pm)}(z,\kappa,y)$ has singularities only at $y=\pm\frac{z^2}{4}$, and the discontinuity $\Delta_{y=+\frac{z^2}{4}}u_{\mathrm{B}}^{(+)}(z,\kappa,y)$ of $u_{\mathrm{B}}^{(+)}(z,\kappa,y)$ at $y=+\frac{z^2}{4}$ is

$$\Delta_{y=+\frac{z^2}{4}}u_{\mathrm{B}}^{(+)}(z,\kappa,y)=\frac{i\sqrt{2\pi}}{\Gamma(\frac{1}{2}+\kappa)}u_{\mathrm{B}}^{(-)}(z,\kappa,y). \tag{2.13}$$

This immediately leads to the connection formula

$$\begin{cases}\phi^{(+)} & \rightsquigarrow \phi^{(+)}+\frac{i\sqrt{2\pi}}{\Gamma(\frac{1}{2}+\kappa)}\eta^{\kappa}\phi^{(-)}\\ \phi^{(-)} & \rightsquigarrow \phi^{(-)}\end{cases}. \tag{2.14}$$

as we move from region IV to I in $z$-plane.

Now we translate this for $\psi^{(+)}(x,\eta)=\left(\frac{\partial z}{\partial x}(x,\eta)\right)^{-\frac{1}{2}}\phi^{(+)}(z(x,\eta),\kappa(\eta),\eta)$, the WKB solution of (1.1). We note that for two-turning-point problem (i.e., connection problem for the equation (1.1) with a pair of simple turning point connected by a Stokes line), similar argument is given in [7]. (cf. also [1], where the problem is discussed in a slightly different setting; the equation has a merging parameter $t$, and the two turning points merge in the limit $t\to 0$.) First, we note

$$w^{(\pm)}(x,\eta):=\eta^{\mp\frac{\kappa(\eta)}{2}}\psi^{(\pm)}(x,\eta) \tag{2.15}$$

$$=\left(\frac{\partial z}{\partial x}(x,\eta)\right)^{-\frac{1}{2}}u^{(\pm)}(z(x,\eta),\kappa(\eta),\eta). \tag{2.16}$$

We remark here that $u^{(\pm)}(z,\kappa,\eta)$ (or $\phi^{(\pm)}(z,\kappa,\eta)$) depends on the parameter $\kappa$ holomorphically and that the location of the singularity of the Borel plane is independent of $\kappa$. Then, by letting $z(x,\eta)=z_0(x)+Z(x,\eta)$ and $\kappa(\eta)=\kappa_0+K(\eta)$, we have

(2.17)
$$w^{(\pm)}(x,\eta)=\left(\frac{\partial z}{\partial x}(x,\eta)\right)^{-\frac{1}{2}}\sum_{n=0}^{\infty}\frac{Z^n(x,\eta)}{n!}\left(\sum_{m=0}^{\infty}\frac{K^m(\eta)}{m!}\frac{\partial^{n+m}u^{(\pm)}}{\partial\kappa^m\partial z^n}(z_0(x),\kappa_0,\eta)\right).$$

Through Borel transform, this turns into

$$(2.18)\quad w_{\mathrm{B}}^{(\pm)}(x,y)=\left(\left(\frac{\partial z}{\partial x}\right)^{-\frac{1}{2}}\right)_{\mathrm{B}}(x,y) * \sum_{n=0}^{\infty}\frac{Z_{\mathrm{B}}^{*n}(x,y)}{n!} * \left(\sum_{m=0}^{\infty}\frac{K_{\mathrm{B}}^{*m}(\eta)}{m!} * \frac{\partial^{n+m}u_{\mathrm{B}}^{(\pm)}}{\partial\kappa^m\partial z^n}(z_0(x),\kappa_0,y)\right),$$

where $*$ stands for the convolution product. Now we pick up a single term

$$(2.19)\qquad \left(\left(\frac{\partial z}{\partial x}\right)^{-\frac{1}{2}}\right)_{\mathrm{B}}(x,y) * \frac{Z_{\mathrm{B}}^{*n}(x,y)}{n!} * \frac{K_{\mathrm{B}}^{*m}(\eta)}{m!} * \frac{\partial^{n+m}u_{\mathrm{B}}^{(\pm)}}{\partial\kappa^m\partial z^n}(z_0(x),\kappa_0,y).$$

Since $K(\eta)$ is Borel summable and $Z(x,\eta)$ is Borel summable uniformly with respect to $x$ in a neighborhood of the four Stokes lines, the front factor

$$(2.20)\qquad \left(\left(\frac{\partial z}{\partial x}\right)^{-\frac{1}{2}}\right)_{\mathrm{B}}(x,y) * \frac{Z_{\mathrm{B}}^{*n}(x,y)}{n!} * \frac{K_{\mathrm{B}}^{*m}(\eta)}{m!}$$

is holomorphic in a strip of the Borel plane containing the positive real axis for $x$ in the neighborhood. Thus in some strip region in the Borel plane, the term (2.19) has singularities only at $y=\pm\frac{z_0(x)^2}{4}$, which comes from the last factor

$$(2.21)\qquad \frac{\partial^{n+m}u_{\mathrm{B}}^{(\pm)}}{\partial\kappa^m\partial z^n}(z_0(x),\kappa_0,y).$$

Also, the discontinuity at $y=+\frac{z_0(x)^2}{4}$ is

(2.22)
$$\Delta_{y=+\frac{z_0(x)^2}{4}}\frac{\partial^{n+m}u_{\mathrm{B}}^{(+)}}{\partial\kappa^m\partial z^n}(z_0(x),\kappa_0,y)=\left.\left(\frac{\partial^{n+m}}{\partial\kappa^m\partial z^n}\Delta_{y=+\frac{z^2}{4}}u_{\mathrm{B}}^{(+)}(z,\kappa,y)\right)\right|_{z=z_0(x),\,\kappa=\kappa_0}$$

$$(2.23)\qquad =\left.\left(\frac{\partial^{n+m}}{\partial\kappa^m\partial z^n}\frac{i\sqrt{2\pi}}{\Gamma(\frac{1}{2}+\kappa)}u_{\mathrm{B}}^{(-)}(z,\kappa,y)\right)\right|_{z=z_0(x),\,\kappa=\kappa_0}.$$

Then taking convolution with the factor (2.20) which does not affect the discontinuity and summing up all the terms with respect to $m$ and $n$, we have

$$(2.24)\qquad \Delta_{y=+\frac{z_0(x)^2}{4}}w_{\mathrm{B}}^{(+)}(x,y)=\left(\frac{i\sqrt{2\pi}}{\Gamma(\frac{1}{2}+\kappa(\eta))}w^{(-)}\right)_{\mathrm{B}}(x,y).$$

(It is easy to see the convergence.) Thus we have the following connection formula.

**Theorem 2.1.** *Assume that any of the four Stokes lines generated by the turning point $x = 0$ flows into a singular point. Let $\Psi^{(\pm),\mathrm{I}}(x,\eta)$ be the Borel sum of $\psi^{(\pm)}(x,\eta)$ resummed in region I and $\Psi^{(\pm),\mathrm{II}}(x,\eta)$ resummed in region II. Then the following relation holds:*

$$
\begin{cases}
\Psi^{(+),\mathrm{I}}(x,\eta) = \Psi^{(+),\mathrm{II}}(x,\eta) + \frac{i\sqrt{2\pi}}{\Gamma(\frac{1}{2}+\kappa(\eta))}\eta^{\kappa(\eta)}\Psi^{(-),\mathrm{II}}(x,\eta) \\
\Psi^{(-),\mathrm{I}}(x,\eta) = \Psi^{(-),\mathrm{II}}(x,\eta)
\end{cases} . \tag{2.25}
$$

*Here $\kappa(\eta)$ is also the Borel sum.*

Likewise, other connection formulas concerning other pairs of adjacent regions, which are given formally in [8], are also justified via Borel summability of the transformation series.

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